Law invariant subsets

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Strongly exposed points

Let V be a Banach space, with dual V^*, and $C \subset X$ a closed set.

Definition

Let $v \in C$ and $v^* \in V^*$. We shall say that v^* strongly exposes v in V if every minimizing sequence of v^* in C norm-converges to v

$$\left[w_n \in C, \quad \lim_n \langle v^*, w_n \rangle = \inf \left\{ \langle v^*, w \rangle \mid w \in V \right\} \right] \Longrightarrow \lim_n \|w_n - v\| = 0$$

Denote by $\mathcal{E}(C)$ the set of extreme points.

Theorem (Phelps, 1974)

If C is convex and weakly compact, it is the closed convex hull of the set of its strongly exposed points:

$$C = \overline{\text{co}}\left(\mathcal{E}(C)\right)$$

Laws

We fix a probability space $(\Omega, \mathcal{A}, P)$. The *law* μ_X of a random vector X is a probability on $\mathbb{R}^d$ defined by:

$$\forall f \in C^0_+\left(\mathbb{R}^d\right), \; E_\mu\left[f\right] = \int_\Omega f\left(X\left(\omega\right)\right) dP = \int_{\mathbb{R}^d} f\left(x\right) d\mu_X$$

Denote by $\mathcal{P}\left(\mathbb{R}^d\right)$ the set of probabilities on R^d and by $\mathcal{P}_1\left(\mathbb{R}^d\right)$ the set of $\mu \in \mathcal{P}\left(\mathbb{R}^d\right)$ such that $E_\mu\left[||x||\right] < \infty$. We are interested in the space $L^1\left(\Omega, A, P; \mathbb{R}^d\right)$. Clearly $X \in L^1$ iff $\mu_X \in \mathcal{P}_1\left(\mathbb{R}^d\right)$

Measure-preserving transformations.

We shall write $X_1 \sim X_2$ to mean that X_1 and X_2 have the same law. Denote by Σ the set of all measure-preserving transformations from Ω into itself: $\sigma \in \Sigma$ iff σ is measurable, invertible, and $\forall A \in \mathcal{A}, \quad P(\sigma(A)) = P(A)$. It is a group, which operates on random variables and preserves the law:

$$X \sim X \circ \sigma \qquad \forall \sigma \in \Sigma$$

The set of all random vectors with given law

For $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$, we set:

$$M(\mu) := \{X \mid \mathrm{law}(X) = \mu\}$$

Lemma

$M(\mu) = \overline{\{X \circ \sigma \mid \sigma \in \Sigma\}}$, *the closure being taken in the* L^1 *norm.*

So $M(\mu)$ is closed, but it is not convex, nor weakly closed. We introduce its closed convex hull $C(\mu)$:

$$C(\mu) = \overline{\mathrm{co}}\,(M(\mu))$$

It is easily checked that $M(\mu)$ is equiintegrable, so $C(\mu)$ is weakly compact (Dunford-Pettis)

The structure of C

Definition

Take $\mu, \nu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$. We shall say that $\nu \preccurlyeq \mu$ if, for every convex function f, we have:

$$\int_{R^d} f(x)\, d\nu \preccurlyeq \int_{R^d} f(x)\, d\mu$$

This is an (incomplete) order relation.It is known in potential theory as sweeping (μ est une balayee de ν), or Choquet ordering.

- If $\nu \preccurlyeq \mu$, then ν and μ have the same barycenter.
- (certainty equivalence) if $x_0 = E[x]$, then $\delta_{x_0} \preccurlyeq \mu$
- (diversification) if $X_1 \sim X_2$ have law μ, and $Y = \frac{1}{2}(X_1 + X_2)$ has law ν, then $\nu \preccurlyeq \mu$

Theorem

$C(\mu) = \left\{ Y \in L^1 \mid \mathrm{law}(Y) \preccurlyeq \mathrm{law}(X) \right\}$

Weak convergence

Theorem

If $X_n \to Y$ *weakly,* $\mathrm{law}(X_n) = \mu$ *and* $\mathrm{law}(Y) = \nu$*, then* $\nu \preccurlyeq \mu$*, with equality iff* $\|X_n - Y\|_1 \to 0$

Proof.

If f is convex and continuous:

$$\int_{\mathbb{R}^d} f(x)\, d\mu = \int_\Omega f(X_n)\, dP \geq \int_\Omega f(Y)\, dP = \int_{\mathbb{R}^d} f(x)\, d\nu$$

so $\nu \preccurlyeq \mu$. Suppose $\mu = \nu$. Taking $f = |x|^p$, we find $\|X_n\|_p \to \|Y\|_p$. □

Corollary

$\overline{\mathrm{co}} M(\mu) = \overline{M(\mu)}^w = C(\mu)$

Compare with relaxation theory and chattering solutions in the calculus of variations (see Ekeland-Temam)

Optimal transportation

Take some $Z \in L^{\infty}\left(R^{d}\right)$ with law ν. The three following problems yield the same value (Kantorovitch) called maximal correlation between ν and μ

$$\max\left\{\langle Z, X\rangle \ \mid X \in M\left(\mu\right)\right\}$$

$$\max\left\{\int_{R^{d}\times R^{d}} z\cdot x \ d\pi \mid \pi \in \Pi\left(\nu,\mu\right)\right\}$$

$$\max\left\{\int_{R^{d}} z\cdot T\left(z\right) \ d\nu \mid \mu = T\sharp\nu\right\}$$

If ν is ac wrt Lebesgue. We get a unique solution T so the last problem.

Strongly exposing functionals

Setting $X = T \circ Z$, we find the following:

Theorem

Let $Z \in L^{\infty}\left(\mathbb{R}^d\right)$ with law ν, ac wrt Lebesgue measure. Then Z strongly exposes some point X in $C\left(\mu\right)$, and in fact $X \in M\left(\mu\right)$.

Theorem

Suppose μ and ν both are ac wrt Lebesgue measure, with $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$ and ν compact support. Then every point $X \in M\left(\mu\right)$ can be strongly exposed in $C\left(\mu\right)$ by some Z with law ν.

Minimal subsets

Given a convex, law invariant subset A of L^1, we consider:

$$K := \{(X, X \circ \sigma) \mid X \in A,\ \sigma \in \Sigma\} \subset A \times A$$

Definition

We shall say that A is minimal if:

$$A \times A = \overline{\mathrm{co}}\,(K)$$

Example

If $A = C(\mu)$, then $A = \overline{\{X \circ \sigma \mid \sigma \in \Sigma\}}^{w}$ for any X with law μ, so:

$$A \times A = \overline{\{(X,\ X \circ \sigma) \mid \sigma \in \Sigma\}}$$

Is there anything else ?

Structure of minimal subsets

Theorem

If a convex law-invariant set A is minimal and weakly compact, then there is some $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$ such that $A = C\left(\mu\right)$

Since A is weakly compact, it is the closed convex hull of its set of strongly exposed points. Let $X \in A$ be such a point and μ its law. We claim that $A = C\left(\mu\right)$.

Proof (cont'd)

By definition, there is some $Z \in L^{\infty}\left(R^d\right)$ such that $\sup_{Y \in A}$ $-\langle Y, Z\rangle = -\langle X, Z\rangle$ and if $X_n \in A$ is a maximizing sequence, then $X_n \to X$ strongly in $L^1\left(R^d\right)$. Since A is minimal we have, for every $Z' \in L^{\infty}\left(R^d\right)$:

$$\sup_{Y \in A} \langle -Y, Z\rangle + \sup_{Y \in A} \langle -Y, Z'\rangle = \sup \left\{ \langle -Y, Z\rangle + \langle -Y \circ \sigma, Z'\rangle \mid Y \in A, \ \sigma \in$$

Let (Y_n, σ_n) be a maximizing sequence. Since we have $=$ instead of $\geq$, the sequence Y_n must be maximizing for Z and the sequence $Y_n \circ \sigma_n$ for Z'. So $Y_n \to X$ in L^1 and:

$$\sup_{Y \in A} \langle -Y, Z'\rangle = \sup_{\tau} \langle -X \circ \sigma, Z'\rangle = \sup \left\{ \langle -\widetilde{X}, Z'\rangle \mid \widetilde{X} \in C(\mu) \right\}$$

Since A is convex and closed, $A = C(\mu)$

Risk measures

A *d-dimensional risk measure* (d-r.m.) is a function $\rho : L^p\left(\Omega, \mathcal{F}, P; R^d\right) \to \mathbb{R}$, $1 \leq p \leq \infty$, such that:

- $\rho(0) = 0$
- $X \geq Y \implies \rho(X) \leq \rho(Y)$
- $\rho(X + me) = \rho(X) - m$ for $m \in \mathbb{R}$ and $e = (1, ..., 1)$

It is

- *law-invariant* if $X \sim Y$ implies that $\rho(X) = \rho(Y)$
- *convex* if ρ is a convex function
- *strongly coherent* if positively homogeneous and:

$$\rho(X) + \rho(Y) = \sup\left\{\rho\left(X + \tilde{Y}\right) \mid Y \sim \tilde{Y}\right\}$$

Examples of strongly coherent risk measures

1. Let $F = (F_1, ..., F_d) \in L^1(R^d)$ satisfy $F_i \geq 0, 1 \leq i \leq d$, and $\sum_i E[F_i] = 1$. Define

$$\rho_F := \sup\left\{\mathbb{E}\left[-F \cdot \widetilde{X}\right] \mid \widetilde{X} \sim X\right\} = \sup\left\{\sum_i \mathbb{E}\left[-F_i \widetilde{X}_i\right] \mid \widetilde{X} \sim X\right\}$$

2. $(p = \infty)$ Denote by S^d the unit simplex in R^d and let $\xi \in S^d$, so that $\xi_i \geq 0$ and $\sum \xi_i = 1$. Define

$$\rho_\xi(X) := \operatorname{ess\,sup} -X \cdot \xi = \operatorname{ess\,sup} -\sum X_i \xi_i$$

3. $(p = \infty)$ Let π be a probability on S^d. Set:

$$\rho_\pi(X) := \int_{S^d} \rho_\xi(X)\, d\pi(\xi)$$

Theorem

ρ is a strongly coherent d-r.m. if and only if there is some $F \in L^1_+\left(R^d\right)$ with $\sum_i E\left[F_i\right] = 1$, a probability π on S^d and a number s with $0 \leq s \leq 1$ such that:

$$\forall X \in L^\infty, \quad \rho\left(X\right) = s\rho_\pi\left(X\right) + \left(1 - s\right)\rho_F\left(X\right)$$

This result is due to:

- Kusuoka (2001) in the case $d = 1$
- IE, Galichon and Henry in the case $d > 1$, $p < \infty$
- IE and Schachermayer in the case $d > 1$, $p < \infty$

Duality

If $\rho : L^{\infty}\left(R^{d}\right) \rightarrow R$ is a convex and law-invariant, it is lower semi-continuous wrt $\sigma\left(L^{\infty}, L^{1}\right)$ (Jouini, Schachermayer, Touzi, 2006). It follows that ρ is coherent (homogeneous) if and only if there exists a closed, convex, law-invariant subset C of $L^{1}\left(R^{d}\right)$ such that.

$$\rho(X) = \sup_{F \in C} \langle F, X \rangle$$

Lemma

ρ is strongly coherent iff C is minimal

Suppose $\rho = \rho_C$ and $C \times C$ is the closed convex hull of K. Then:

$$\begin{aligned}
\rho_C(X) + \rho_C(Y) &= \sup\{-\langle F, X\rangle - \langle G, Y\rangle \mid (F, G) \in C \times C\} \\
&= \sup\{-\langle F, X\rangle - \langle F \circ \tau, Y\rangle \mid F \in C,\ \tau \in \mathcal{T}\} \\
&= \sup\left\{-\langle F, X + Y \circ \tau^{-1}\rangle \mid F \in C,\ \tau \in \mathcal{T}\right\} \\
&\leq \sup\left\{-\langle F, X + \widetilde{Y}\rangle \mid \widetilde{Y} \sim Y\right\} = \rho_C\left(X + \widetilde{Y}\right)
\end{aligned}$$

and the reverse inequality is always true