Kantorovich metric in noncommutative geometry

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Monge-Kantorovich optimal transportation problem, transport metrics and their applications,
International conference dedicated to the centenary of L. V. Kantorovich,

Euler International Mathematical Institute
St. Petersburg, 6th June 2012

Introduction

Any commutative C^*-algebra $\mathcal{A}$ (i.e. Banach $*$-algebra with $\|a\|^2 = \|a^* a\|$) is isomorphic to an algebra of continuous functions vanishing at infinity on some topological space $\mathcal{P}(\mathcal{A})$, Gelfand duality

$$\mathcal{A} \simeq C_0(\mathcal{P}(\mathcal{A})), \quad \mathcal{P}(C_0(\mathcal{X})) \simeq \mathcal{X}.$$

$\mathcal{P}(\mathcal{A})$ is the set of pure states of $\mathcal{A}$, i.e. the extremal points of the set $\mathcal{S}(\mathcal{A})$ of normalized ($\mathbb{I} \rightarrow 1$), positive ($a^* a \rightarrow \mathbb{R}^+$) linear map $\mathcal{A} \rightarrow \mathbb{C}$:

$$S(C_0(\mathcal{X})) \ni \varphi : f \rightarrow \int_{\mathcal{X}} f \, \mathrm{d}\mu, \qquad \mathcal{P}(C_0(\mathcal{X})) \ni \delta_x : x \rightarrow f(x).$$

Connes' theory of spectral triples $(\mathcal{A}, \mathcal{H}, D)$ extends Gelfand duality beyond topology, so that to encompass differential, homological, metric (spin) aspects,

commutative spectral triple	$\rightarrow$	noncommutative spectral triple
$\updownarrow$		$\downarrow$
Riemannian geometry		non-commutative geometry

- Geometry without points, but the latter are retrieved as pure states of $\mathcal{A}$.
- How does one retrieve the Riemannian distance on $\mathcal{P}(C_0(\mathcal{M})) \simeq \mathcal{M}$ a Riemannian manifold, and extend it to $\mathcal{P}(\mathcal{A})$ for noncommutative $\mathcal{A}$?

Outline:

I. The metric aspect of noncommutative geometry

- Monge-Kantorovich distance in optimal transport theory
- spectral distance in noncommutative geometry
- commutative case: $\mathcal{A} = C_0(\mathcal{M})$

II. Towards a theory of optimal transport in noncommutative geometry?

- spectral distance on pure states as a cost function
- finite dimensional example: $\mathcal{A} = M_n(\mathbb{C})$
- product of the continuum by the discrete: $\mathcal{A} = C_0(\mathcal{M}) \otimes \mathbb{C}^2$

III. Translated states in the Euclidean and the Moyal planes: $\mathcal{A} = \mathbb{K}$

- optimal element

IV. Carnot-Carathéodory distance: $\mathcal{A} = C_0(S^1) \otimes M_2(\mathbb{C})$

I. The metric aspect of noncommutative geometry

Optimal transport

Let $\mathcal{X}$ be a locally compact Polish space, $c(x, y)$ a positive real function - the "cost" - representing the work needed to move from x to y. The minimal work W required to transport the probability measure μ_1 to μ_2 is

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X}\times\mathcal{X}} c(x, y)\, \mathrm{d}\pi$$

where the infimum is over all transportation plans, i.e. measures π on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1, μ_2.

When the cost function c is a distance d, then

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X}\times\mathcal{X}} d(x, y)\, \mathrm{d}\pi$$

is a distance (possibly infinite) on the space of probability measures on $\mathcal{X}$, called the Monge-Kantorovich or Wasserstein distance of order 1.

Spectral triple

An involutive algebra $\mathcal{A}$, a faithful representation on $\mathcal{H}$, an operator D on $\mathcal{H}$ such that $[D, a]$ is bounded and $a[D - \lambda\mathbb{I}]^{-1}$ is compact for any $a \in \mathcal{A}$ and $\lambda \notin \text{Sp } D$.

Furthermore, when a set of conditions (dimension, regularity, finitude, first order, orientability, reality, Poincaré duality) is satisfied, then

Theorem — Connes 1996-2008

$\mathcal{M}$ a compact Riemann manifold, then $(C^\infty(\mathcal{M}), \Omega^\bullet(\mathcal{M}), d + d^\dagger)$ is a spectral triple.

When $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

Whatever $\mathcal{A}$, commutative or not, one defines on its state space $\mathcal{S}(\mathcal{A})$ the spectral distance (possibly infinite)

$$d_D(\varphi, \tilde{\varphi}) = \sup_{a \in \mathcal{A}} \{ |\varphi(a) - \tilde{\varphi}(a)| \,/\, \|[D, a]\| \leq 1 \}.$$

Proposition

Rieffel 99, then D'Andrea, P.M. 2009

Let $\mathcal{X} = \mathcal{M}$ be a complete, connected, without boundary, Riemannian manifold. For any $\varphi, \tilde{\varphi} \in \mathcal{S}(C_0(\mathcal{M}))$,

$$W(\varphi, \tilde{\varphi}) = d_D(\varphi, \tilde{\varphi})$$

where W is the Monge-Kantorovich distance associated to the cost d_{geo}, while d_D is the spectral distance associated to $\left(C_0^\infty(\mathcal{M}), \Omega^\bullet(\mathcal{M}), D = d + d^\dagger\right)$.

i. Kantorovich duality:

$$W(\varphi, \tilde{\varphi}) = \sup_{\|f\|_{\mathrm{Lip}} \leq 1} \left(\int_{\mathcal{X}} f \, \mathrm{d}\mu - \int_{\mathcal{X}} f \, \mathrm{d}\tilde{\mu} \right) \qquad (1)$$

with supremum on all real 1-Lipschitz $f \in C(\mathcal{X})$: $|f(x) - f(y)| \leq d_{\text{geo}}(x, y)$.

ii. For $f = f^*$, $\left\| [d + d^\dagger, f]^2 \right\| = ||[\partial\!\!\!/, f]||^2 = \frac{1}{2} \|[[\Delta, f], f]\| = \|f\|^2_{\mathrm{Lip}}$.

iii. Any 1-Lip. f non-vanishing at infinity can be approximated by the 1-Lip.

$$f_n(x) \doteq f(x) e^{-d_{\text{geo}}(x_0, x)/n} \in C_0(\mathcal{M});$$

and any f_n is the uniform limit of a sequence of smooth 1-Lip. functions.

On the importance of being complete

Unknown to the speaker whether Kantorovich duality holds for non-complete space, so one takes (1) as a definition of Kantorovich distance.
Let $\mathcal{N}$ be compact and $\mathcal{M} = \mathcal{N} \setminus \{x_0\}$.

$$\left.\begin{array}{l}\mathcal{N} = S^1 = [0,1] \\ \mathcal{M} = (0,1)\end{array}\right\} W_{\mathcal{N}}(x,y) = \min\{|x-y|, 1-|x-y|\} \neq W_{\mathcal{M}}(x,y) = |x-y|.$$

$$\mathcal{N} = S^2,\, \mathcal{M} = S^2 \setminus \{x_0\} \quad \text{then} \quad W_{\mathcal{N}} = W_{\mathcal{M}}.$$

- Removing a point from a complete compact manifold may change or not W. But it does not modify the spectral distance:

$$d_D^{\mathcal{N}}(\varphi_1, \varphi_2) = \sup_{f \in C^\infty(\mathcal{N})} \left\{ |(\varphi_1 - \varphi_2)(f)|;\ ||f||_{\mathrm{Lip}} \leq 1 \right\}$$
$$= \sup_{f \in C^\infty(\mathcal{N}), f(x_0)=0} \left\{ |(\varphi_1 - \varphi_2)(f)|;\ ||f||_{\mathrm{Lip}} \leq 1 \right\} = d_D^{\mathcal{M}}(\varphi_1, \varphi_2)$$

since $C_0^\infty(\mathcal{N}) = C^\infty(\mathcal{N})$ has a unit and $(C^\infty(\mathcal{N}),$ vanishing at $x_0) = C_0^\infty(\mathcal{M})$.

$$\mathcal{N} = S^1, \mathcal{M} = (0,1): \qquad d_D^{\mathcal{M}} = d_{S^1} \neq W_{\mathcal{M}}.$$
$$\mathcal{N} = S^2, \mathcal{M} = S^2 \setminus \{x_0\}: \qquad d_D^{\mathcal{M}} = d_{S^2} = W_{\mathcal{M}}.$$

Connected components

Proposition

For any $x \in \mathcal{M}$ and any state φ of $C_0(\mathcal{M})$,

$$d_D(\varphi, \delta_x) = \mathbb{E}\big(d(x, \dot{}); \mu\big) = \int_{\mathcal{M}} d_{\text{geo}}(x, y)\mathrm{d}\mu(y)\ .$$

In particular for two pure states δ_x, δ_y, one retrieves $d_D(\delta_x, \delta_y) = d_{\text{geo}}(x, y)$.

Let

$$S_1(C_0(\mathcal{M})) \doteq \{\varphi \in \mathcal{S}(C_0(\mathcal{M}))\ , \mathbb{E}\big(d(x, .); \mu\big) < \infty\}$$

and

$$\mathrm{Con}(\varphi) \doteq \{\varphi' \in \mathcal{S}(C_0(\mathcal{M})), d_D(\varphi, \varphi') \leq \infty\}.$$

Corollary

$\varphi \in S_1(C_0(\mathcal{M}))$ if and only if φ is at finite spectral distance from any pure state. Moreover for any $\varphi \in S_1(C_0(\mathcal{M}))$,

$$\mathrm{Con}(\varphi) = S_1(C_0(\mathcal{M})).$$

- Two states not in $S_1(C_0(\mathcal{M}))$ may be at finite distance from one another.

II. Towards a theory of optimal transport in noncommutative geometry ?

Let $\mathcal{A}$ be a separable C^*-algebra with unit and $\varphi \in \mathcal{S}(\mathcal{A})$. There exists a (non-necessarily unique) probability measure $\mu \in \mathrm{Prob}(\mathcal{P}(\mathcal{A}))$ such that

$$\varphi(a) = \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) \quad \text{where} \quad \hat{a}(\omega) \doteq \omega(a).$$

Define the Kantorovich distance on $\mathcal{S}(\mathcal{A})$,

$$W_D(\varphi, \tilde{\varphi}) \doteq \sup_{a \in \mathrm{Lip}_D(\mathcal{A})} \left\{ \left| \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) - \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\tilde{\mu}(\omega) \right| \right\},$$

with cost function the spectral distance on $\mathcal{P}(\mathcal{A})$,

$$\mathrm{Lip}_D(\mathcal{A}) \doteq \{a \in \mathcal{A} \text{ such that } |\omega_1(a) - \omega_2(a)| \leq d_D(\omega_1, \omega_2)\ \forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A})\}.$$

Proposition

P.M. 2011

For any $\varphi, \tilde{\varphi} \in \mathcal{S}(\mathcal{A})$, $d_D(\varphi, \tilde{\varphi}) \leq W_D(\varphi, \tilde{\varphi})$.

- Obvious because $\{a \in \mathcal{A}, \|[D, a]\| \leq 1\} \subset \mathrm{Lip}_D(\mathcal{A})$.
- If $d_D = W_D$, then Connes spectral distance is a problem of optimal transport, and noncommutative geometry provides examples of cost functions.

A two-point space

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$$

where $m \in \mathbb{C}$ and representation

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

This is a two-point space

$$\delta_1(z_1, z_2) \doteq z_1, \quad \delta_2(z_1, z_2) \doteq z_2$$

with distance

$$d_D(\delta_1, \delta_2) = \frac{1}{|m|}.$$

- Discrete space (i.e. no geodesic) but finite distance.
- For non pure states, $d_D = W_D$ since

$$\mathrm{Lip}_D(\mathbb{C}^2) = \left\{ a \in \mathbb{C}^2,\, |z_1 - z_2| \leq \frac{1}{|m|} \right\} = \left\{ a \in \mathbb{C}^2,\, \|[D, a]\| \leq 1 \right\}.$$

The sphere

$$\mathcal{A} = M_2(\mathbb{C}), \quad \mathcal{H} = \mathbb{C}^2, \quad D = D^* \in M_2(\mathbb{C}).$$

Diagonalization of D fixes a base in $\mathcal{H}$. Pure states space of $M_2(\mathbb{C})$ is $\mathbb{C}P^1 = S^2$:

$$\omega_\psi(a) = (\psi, a\,\psi) \qquad \forall a \in \mathcal{A}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}P^1 \leftrightarrow \begin{cases} x_\psi &= \mathrm{Re}(\overline{\psi_1}\psi_2) \\ y_\psi &= \mathrm{Im}(\overline{\psi_1}\psi_2) \\ z_\psi &= |\psi_1|^2 - |\psi_2|^2 \end{cases} \in S^2.$$

North $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and south $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ poles are eigenvectors of D with eigenvalues D_1, D_2.

Proposition — Iochum, Krajewski, P.M. 2001

$$d_D(\omega_\psi, \omega_{\tilde\psi}) = \begin{cases} \frac{2}{|D_1 - D_2|}\sqrt{(x_\psi - x_{\tilde\psi})^2 + (y_\psi - y_{\tilde\psi})^2} & \text{if } z_\psi = z_{\tilde\psi}, \\ +\infty & \text{if } z_\psi \neq z_{\tilde\psi}. \end{cases}$$

Product of the continuum by the discrete

Product of a manifold $\mathcal{M}$ by $\left(\mathbb{C}^2, \mathbb{C}^2, D_I = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}\right)$, namely

$$\mathcal{A}' = C_0^\infty(\mathcal{M}) \otimes \mathbb{C}^2,\ \mathcal{H}' = \Omega^\bullet(\mathcal{M}) \otimes \mathbb{C}^2,\ D' = (d + d^\dagger) \otimes \mathbb{I}_2 + \Gamma \otimes D_I.$$

Proposition

P.M., Wulkenhaar 2001

The spectral distance $d_{D'}$ between pure states of $\overline{\mathcal{A}'} = C_0(\mathcal{M}) \otimes \mathbb{C}^2$,

$$\mathcal{P}(\overline{\mathcal{A}'}) \simeq \mathcal{M} \cup \mathcal{M} = \left\{x_i \doteq (x, \delta_i),\ x \in \mathcal{M}, \delta_i \in \mathcal{P}(\mathbb{C}^2)\right\},$$

coincides with the geodesic distance in $\mathcal{M}' = \mathcal{M} \times [0,1]$ with Riemannian metric

$$\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \frac{1}{|m|} \end{pmatrix}.$$

- Possible to make m a function on $\mathcal{M}$: Higgs field in the standard model of elementary particles.

$\mathcal{S}(\overline{\mathcal{A}'})$ is the set of couples of measures (μ, ν) on $\mathcal{M}$, normalized to

$$\int_{\mathcal{M}} d\mu + \int_{\mathcal{M}} d\nu = 1,$$

whose evaluation on $\mathcal{A}' \ni a = (f, g)$, with $f, g \in C_0^\infty(\mathcal{M})$, is

$$\varphi(a) = \int_{\mathcal{M}} f\, d\mu + \int_{\mathcal{M}} g\, d\nu.$$

- As before, $d_{D'}(\varphi, \tilde{\varphi}) \leq W_{D'}(\varphi, \tilde{\varphi})$ where $W_{D'}$ is the Kantorovich distance on $\mathcal{M} \cup \mathcal{M}$ associated to the cost $d_{D'}$.

- Equality holds - $d_{D'} = d_D = W_D$ - for states localized on the same copy:

$$\varphi = (0, \nu),\ \tilde{\varphi} = (0, \tilde{\nu}) \quad \text{or} \quad \varphi = (\mu, 0),\ \tilde{\varphi} = (\tilde{\mu}, 0).$$

- For two states localized on distinct copies, one may project back the problem on a single copy, using a cost function defined solely on $\mathcal{M}$,

$$c(x, y) \doteq d_{D'}(x_1, y_2) \doteq \sqrt{d(x, y)^2 + \frac{1}{|m|^2}}.$$

  The Higgs field would then represent the cost to stay at the same point of space-time, but jumping from one copy to the other: $c(x, x) = \frac{1}{|m|} \neq 0$.

III. Translated states: Euclidean and Moyal planes

$$\mathcal{A} = (\mathcal{S}(\mathbb{R}^2), \star), \quad \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, \quad D = -i \sum_{\mu=1}^{2} \sigma^\mu \partial_\mu$$

where

$$(f \star g)(x) = \frac{1}{(\pi\theta)^2} \int d^2s\, d^2t\ f(x+s)g(x+t)e^{-i2s\Theta^{-1}t}$$

with

$$s\Theta^{-1}t \equiv s^\mu \Theta^{-1}_{\mu\nu} t^\nu \ \text{ with } \Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \theta \in \mathbb{R}^{+*},$$

and

$$D = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} \quad \text{with} \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2).$$

The Moyal algebra $\mathcal{A}$ acts on $\mathcal{H}$ as $\pi(f)\psi = \begin{pmatrix} f \star \psi_1 \\ f \star \psi_2 \end{pmatrix}$.

- The evaluation at x is not a state for $(f^* \star f)(x)$ may not be positive.
- The pure states of $\bar{\mathcal{A}} = \mathbb{K}$ are the vector states in an irreducible representation.

For $\kappa \in \mathbb{R}^2 \simeq \mathbb{C}$, we write $(\alpha_\kappa f)(x) = f(x+\kappa)$. The κ-translated of a state φ is

$$\alpha_\kappa \varphi(f) \doteq \varphi \circ \alpha_\kappa(f).$$

Theorem P.M., L. Tomassini 2011

$$d_D(\varphi, \alpha_\kappa \varphi) = |\kappa|.$$

Let us call optimal element an element in $\mathcal{A}$ that attains the supremum in the spectral distance formula or, in case the supremum is not attained, a sequence

$$a_n \in \mathcal{A}, \quad \|[D, a_n]\| \leq 1, \quad \lim \left(\tilde{\varphi}(a_n) - \varphi(a_n)\right) = d_D(\tilde{\varphi}, \varphi).$$

Then, given any state φ and $\kappa = |\kappa| e^{i\Xi}$,

$$f(z) = \frac{1}{\sqrt{2}} \left(z e^{-i\Xi} + \bar{z} e^{i\Xi}\right)$$

is an optimal element both in the commutative case through pointwise action

$$f\psi = \sqrt{2}(z.\frac{\kappa}{|\kappa|})\,\psi \qquad \psi \in L^2(\mathbb{R}^2),$$

and in the Moyal noncommutative case through the $\star$-action:

$$f \star \psi = \frac{1}{\sqrt{2}} \left(e^{-i\Xi} z + e^{i\Xi} \bar{z}\right) \star \psi \qquad \psi \in L^2(\mathbb{R}^2).$$

- In fact f is made cancel at infinity : $f\, e^{-\frac{|z|}{n}}$ or $f \star e_\star^{-\frac{\bar{z}\star z}{n}}$.

IV Carnot-Carathéodory distance

$$\mathcal{A}' = C_0^\infty(\mathcal{M}) \otimes M_n(\mathbb{C}),\ \mathcal{H}' = L^2(\mathcal{M}, S) \otimes M_n(\mathbb{C}),$$

$$D' = -i\sum_{\mu=1}^m \gamma^\mu \partial_\mu \otimes \mathbb{I}_n + \gamma^\mu \otimes A_\mu$$

$A_\mu dx^\mu$ is a field of 1-form on $\mathcal{M}$ with value in $\mathfrak{u}(n)$: a $U(n)$-connection 1-form.

$\mathcal{P}(\overline{\mathcal{A}'})$ is a trivial bundle $P \xrightarrow{\pi} \mathcal{M}$ with fiber $\mathbb{C}P^{n-1}$, that is

$$P \ni p = (x, \xi) = \xi_x \text{ with } x \in \mathcal{M},\ \xi \in \mathbb{C}P^{n-1}.$$

Evaluation of $\xi_x \in P$ on $a \in \mathcal{A}$,

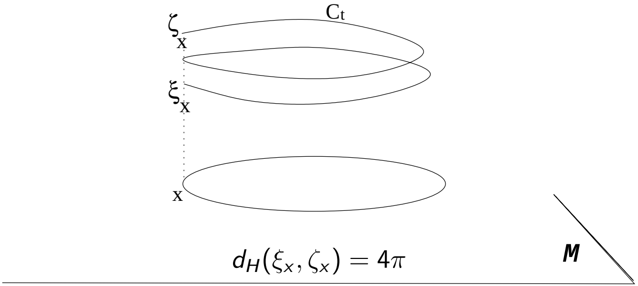
$$\xi_x(a) = (\xi, a(x)\,\xi) = \mathrm{Tr}(s_\xi\, a(x)).$$

- D' is the covariant Dirac operator $-i\gamma^\mu(\partial_\mu + A_\mu)$ on the bundle P, associated to the connection A_μ.

The connection defines a spectral distance $d_{D'}$ and an horizontal distance d_H:

$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \operatorname*{Inf}_{\dot{c}_t \in H_{c_t}P} \int_0^1 \|\dot{c}_t\| \, dt.$$

For instance



$$d_H(\xi_x, \zeta_x) = 4\pi$$

Question: $d_{D'} = d_H$?

Connes 96

Proposition

P.M. 2006

$$d_D \leq d_H.$$

Denoting Acc(ξ_x) the set of points at finite horizontal distance from ξ_x, and Con(ξ_x) the set points at finite spectral distance,

$$\mathrm{Acc}(\xi_x) \subsetneq \mathrm{Con}(\xi_x).$$

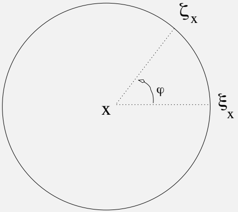
Example: $\mathcal{A} = C(S^1) \otimes M_2(\mathbb{C})$

$$A_1 = i \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \quad \omega \doteq \int_0^{2\pi} \frac{\theta_1(t) - \theta_2(t)\, dt}{2\pi}.$$

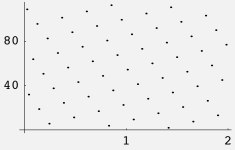
Proposition

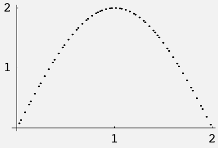
P.M. 2006

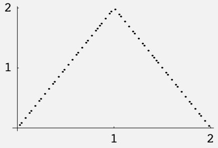
Fiberwise,



$$\begin{cases} d_H(0,\varphi) = 2k\pi & \text{if} \quad \varphi = k(2\pi\omega) \bmod [2\pi], \\ \qquad\quad\;\; = \infty & \text{otherwise;} \\ d_D(0,\varphi) = C \sin\frac{\varphi}{2} & \text{with} \quad C = \frac{4\pi|V_1||V_2|}{|\sin\omega\pi|}. \end{cases}$$







Conclusion

Connes spectral distance d_D in noncommutative geometry coincides with the Monge-Kantorovich metric in the commutative case, and offers a possible generalization in the noncommutative framework.

$$\begin{array}{rcl} d_D = W \text{ commutative case} & \rightarrow & d_D \text{ noncommutative case} \\ \uparrow & & | \\ \text{Kantorovich duality} & & d_D = W_D? \\ \downarrow & & \downarrow \\ \text{distance as shortest length} & & \text{noncommutative cost ?} \end{array}$$

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1. Dimension: D^{-1} is an infinitesimal of order $\frac{1}{m}$.

2. Regularity: for any $a \in \mathcal{A}$, a and $[D, a]$ belong to the intersection of the domains of all the powers δ^k of the derivation $\delta(b) \doteq [|D|, b]$, where b belongs to the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$.

3. Finitude: $\mathcal{A}$ is a pre-C^*-algebra and the set $\mathcal{H}^\infty \doteq \bigcap_{k \in \mathbb{N}} \text{Dom } D^k$ of smooth vectors of $\mathcal{H}$ is a finite projective module.

4. First order: the representation of $\mathcal{A}^\circ$ commutes with $[D, \mathcal{A}]$

$$[[D, a], Jb^* J^{-1}] = 0 \text{ for all } a, b \in \mathcal{A}.$$

5. Orientability: there exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ such that $\pi(c) = \Gamma$.

6. Reality $(\mathcal{A} \otimes \mathcal{A}^{\circ}, \mathcal{H}, D, \Gamma, J)$ is aKR^n-cycle with $[a, Jb^* J^{-1}] = 0$. J is called the *real structure*. That is

- J is a anti-unitary bijection on $\mathcal{H}$ that implements the involution, i.e. $JaJ^{-1} = a^*$ for all $a \in \mathcal{A}$;
- if n is even, there is a graduation Γ of $\mathcal{H}$ that commutes with $\mathcal{A}$ and anticommutes with D;
- the following table holds

n mod 8	0	1	2	3	4	5	6	7
$J^2 = \pm \mathbb{I}$	+	+	-	-	-	-	+	+
$JD = \pm DJ$	+	-	+	+	+	-	+	+
$J\Gamma = \pm \Gamma J$	+		-		+		-	

For odd n, one sets $\Gamma = \mathbb{I}$.

7. Poincaré duality: the additive coupling on $K_*(\mathcal{A})$ coming from the index of the Dirac operator is non-degenerated.