Infinite minimum-weight perfect matching on the line

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Monge–Kantorovich optimal transportation problem, transport metrics, and their applications

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Joint work with

Julie Delon (Télécom ParisTech)
Julien Salomon (CEREMADE, Dauphine)

Zapiski nauch. sem. POMI **390** (2011) 52–68 and arXiv:1102.1558
SIAM J. Discr. Math. **26** (2012) 801–825 and arXiv:1102.1795

Sergei Nechaev (LPTMS, Orsay)
Olga Valba (PhysTech, Moscow & LPTMS, Orsay)

arXiv:1203.3248 submitted to *Phys. Rev. E*

Relevant prior work

A. Aggarwal *et al*
“Efficient minimum cost matching using quadrangle inequality”
Foundations of Computer Science, 33rd Annual Symposium (1992) 583–592

Robert J. McCann
“Exact solutions to the transportation problem on the line”
Proc. R. Soc. Lond. A **455** (1999) 1341–1380

D. Cordero-Erausquin
“Sur le transport de mesures périodiques”
C. R. Acad. Sci. Sér. I Math. **329** (1999) 199–202

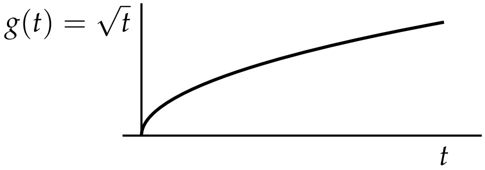
J. Delon, J. Salomon, A. Sobolevski
“Fast transport optimization for Monge costs on the circle”
SIAM J. Appl. Math. **70** (2010) 2239–2258

Motivation: metric on $\mathbf{R}^1$

Consider a metric on $\mathbf{R}^1$ given by

$$\rho(x, y) = g(|x - y|)$$

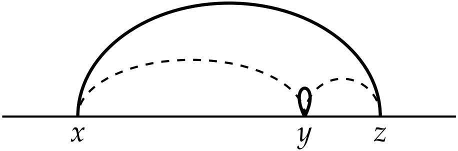
with $g(\cdot)\colon \mathbf{R}_+ \to \mathbf{R}_+$ a strictly growing and strictly concave function such that $g(0) = 0$.



Motivation: metric on $\mathbf{R}^1$

Subadditivity of g implies the triangle inequality for ρ

$$\rho(x,z)\ [+\ \rho(y,y)] \leq \rho(x,y) + \rho(y,z)$$

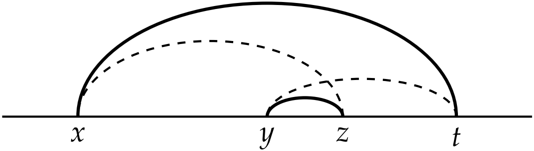


It is more efficient to leave, if possible, a mass element in place

Motivation: metric on $\mathbf{R}^1$

A stronger property also holds: **submodularity** or **Monge property**

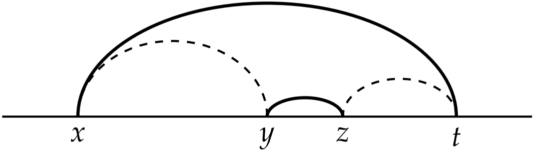
$$\rho(x,t) + \rho(y,z) \le \rho(x,z) + \rho(y,t)$$



It is more efficient to nest particle paths than to cross them

Motivation: metric on $\mathbf{R}^1$

$\rho(x,y) + \rho(z,t)$ vs $\rho(x,t) + \rho(y,z)$?



Optimal transportation may involve nonlocal rearrangements

Hypotheses

Transport μ_0 to μ_1 with a **submodular** cost function:

$$c(x,t) + c(y,z) \leq c(x,z) + c(y,t)$$

whenever $x < y < z < t$

Suppose also suitable **monotonicity** of $c(x,\cdot)$ and $c(\cdot,y)$

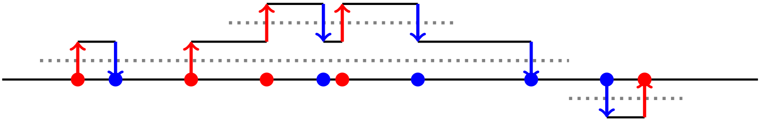
$c(x,y) = \rho(x,y)$,
$c(x,y) = \ln|x-y|$, etc
$c(x,y) = c(|x-y|)$ **not** necessary

Implications

1. Transport $\bar{\mu}_0 = (\mu_0 - \mu_1)_+$ to $\bar{\mu}_1 = (\mu_1 - \mu_0)_+$ with $\operatorname{spt}\bar{\mu}_0 \cap \operatorname{spt}\bar{\mu}_1 = \varnothing$

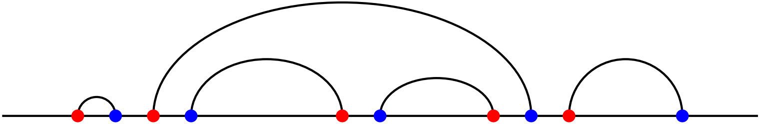
2. Particle paths in an optimal transport plan are nested

3. If $\bar{\mu}_0, \bar{\mu}_1$ are atomic, the problem can be split into transport in **alternating chains**



Reduction to unipartite matching

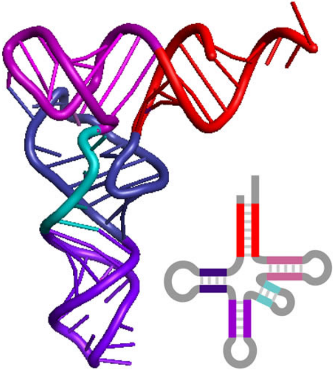
(Bipartite) optimal transport in an alternating chain of n pairs $\equiv$ (unipartite) **perfect minimum-weight matching** on $2n$ points:

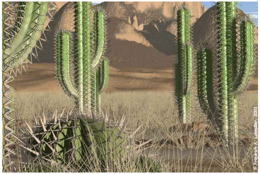


Given a set $x_1 < x_2 < \cdots < x_{2n}$ find n-element subset $\{(i', j')\}$ of $\{(i,j) \colon 1 \leq i < j \leq 2n\}$ such that $\sum_{(i',j')} c(x_{i'}, x_{j'})$ is minimal

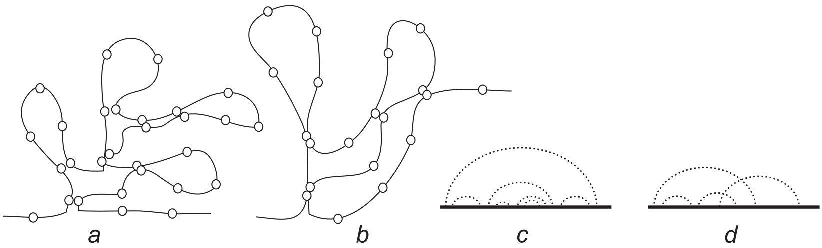
Physics perspective: RNA folding

Secondary structure of RNA

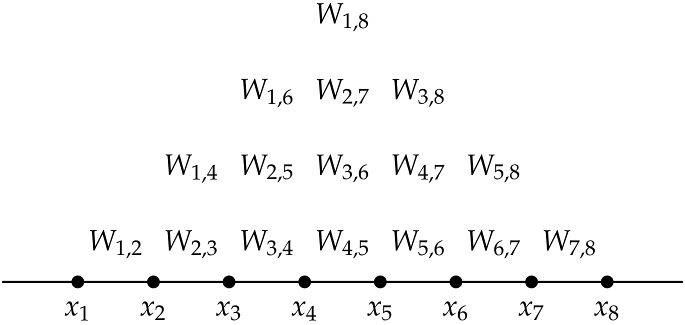




Physics perspective: RNA folding



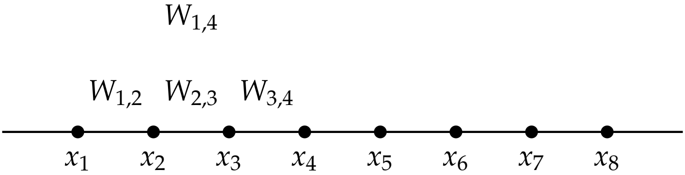
Localization principle



$W_{i,j}$ is the minimal weight of partial matching
on $x_i < x_{i+1} < \cdots < x_j$

$W_{12} = c(x_1, x_2)$,
$W_{14} = \min\{c(x_1, x_2) + c(x_3, x_4);\ c(x_1, x_4) + c(x_2, x_3)\}$, etc

Localization principle



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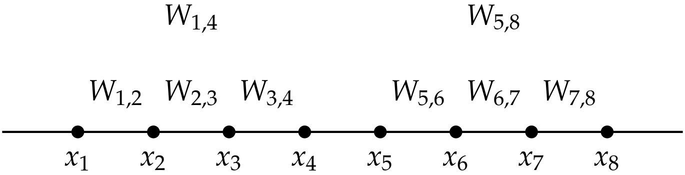

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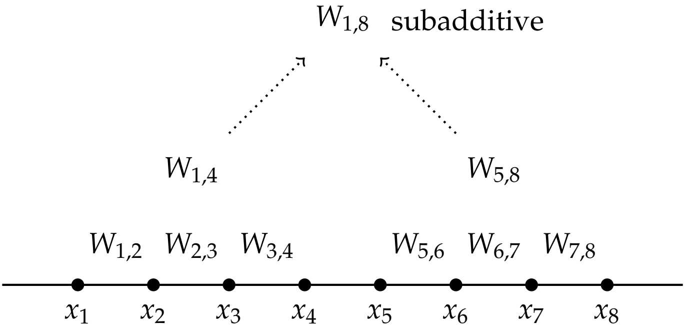

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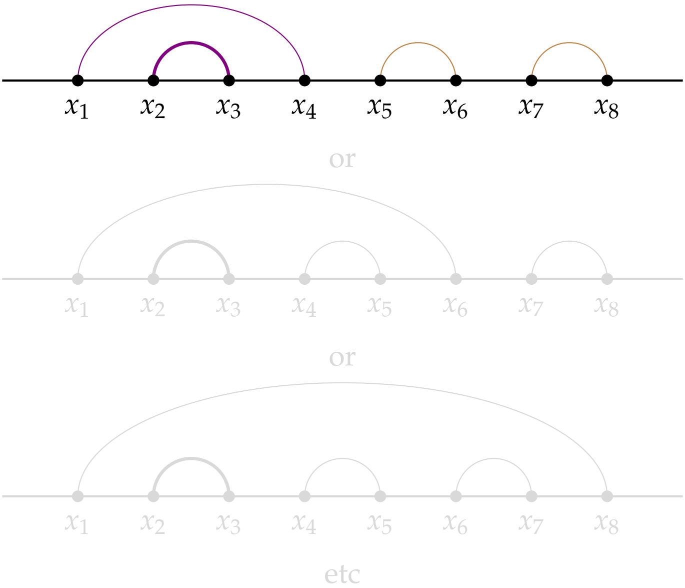

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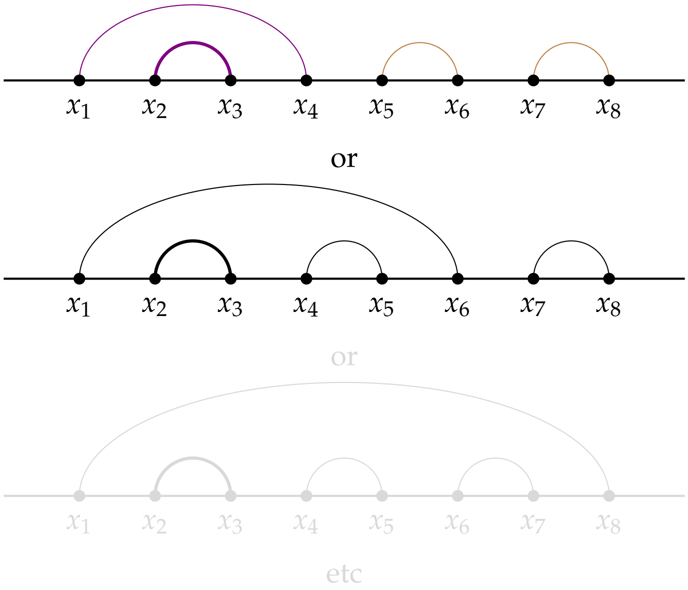

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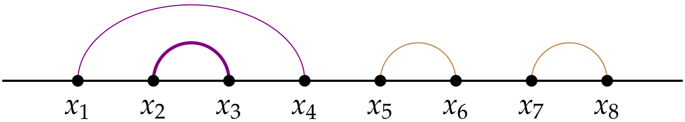
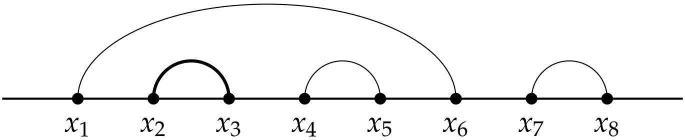


or



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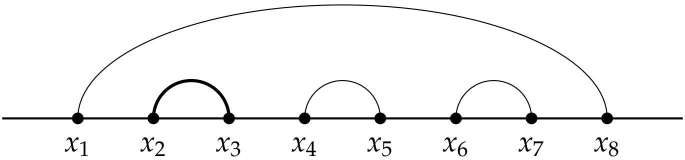
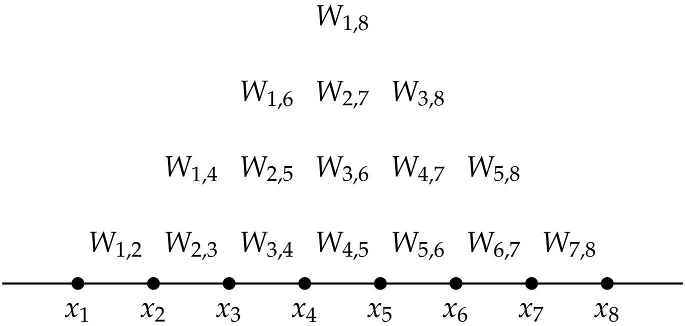


etc

Localization principle

After joining two minimum-weight matchings, in the joint matching **all the hidden arcs are preserved** while **exposed arcs may reconnect**

Bellman recurrence



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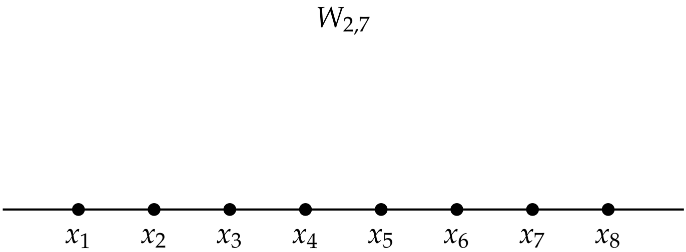

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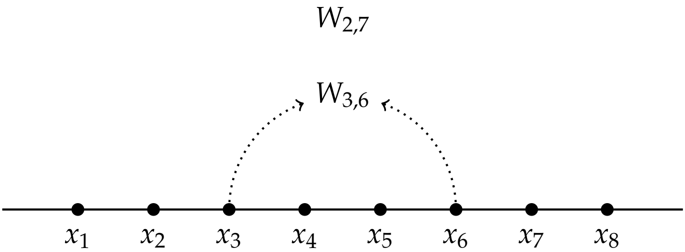
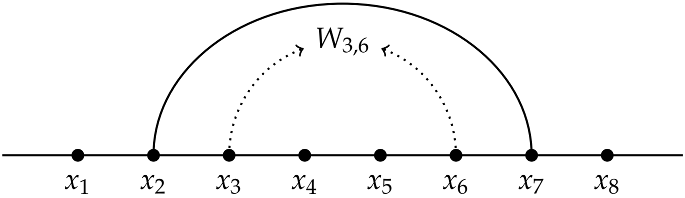


Bellman recurrence

$$W_{2,7} = c(x_2, x_7) + W_{3,6} \; ?$$



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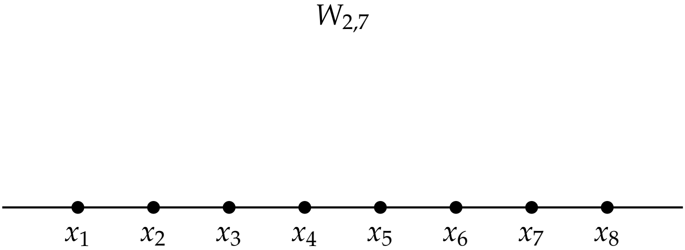

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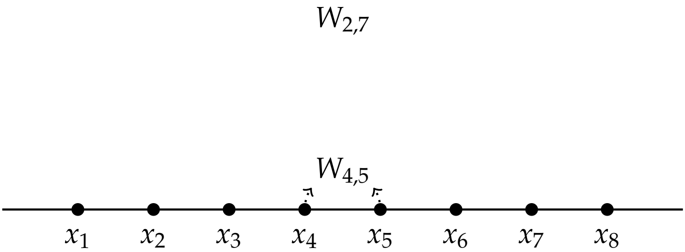

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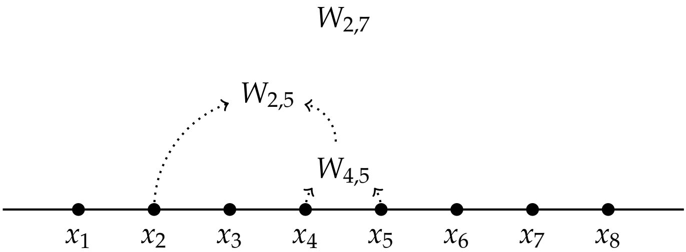

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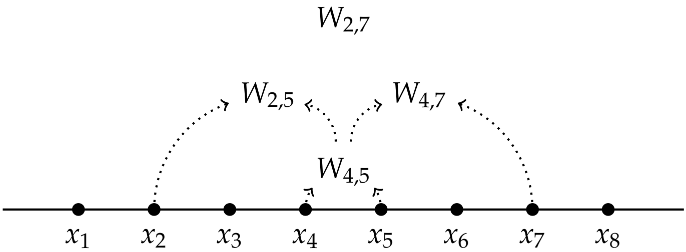
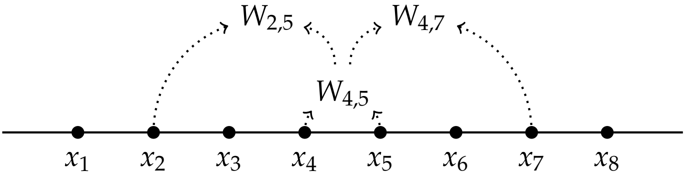


Bellman recurrence

$$W_{2,7} \quad = W_{2,5} + W_{4,7} - W_{4,5} \text{ ?}$$



Submodularity again

$$W_{i,j} = \min\left[c(x_i, x_j) + W_{i+1,j-1};\ W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right]$$

$W_{\cdot,\cdot}$ submodular: $W_{i,j} + W_{i+2,j-2} \leq W_{i,j-2} + W_{i+2,j} \quad \forall i < j-4$

$W_{\cdot,\cdot}$ is the **greatest submodular function such that**

$$W_{i,j} - W_{i+1,j-1} \leq c(x_i, x_j) \quad \forall i < j-2$$

As $j-i$ grows $W_{i,j} - W_{i+1,j-1}$ decrease while $c(x_i, x_j)$ increase

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Infinite configurations

Consider a locally finite point configuration on $\mathbf{R}^1$

$$\cdots < x_{-1} < x_0 < x_1 < x_2 < \ldots$$

$$W_{i,j} = \min\left[c(x_i, x_j) + W_{i+1,j-1};\ W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right]$$

Periodic case: $x_{i+K} = x_i + L$

— $W_{i,j}$ stabilize, optimal configuration exists

Non-periodic case: (x_i) random stationary

— intermittency, multiscale random growth possible

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