Optimal transportation, curvature flows, and related a priori estimates

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Curvature flow

Shrinking family of (convex) surfaces ∂A_t, where every point $x \in \partial A_t$ is moving in the direction of the normal n with the speed depending on the curvature of ∂A at x

$$\dot{x} = -K(x) \cdot n(x).$$

Here $K(x)$ is the (Gauss) curvature of ∂A_t at x.

Approaches to the curvature flows

- 1) Solving a parabolic nonlinear equation with smooth data
  (R.S. Hamilton, R. Huisken ...)
- 2) Consider surfaces as level sets of a potential function φ, satisfying a nonlinear parabolic equation in viscosity sence
  (L.C. Evans, J. Spruck, Y.G. Chen, Yo. Giga, S. Goto ...)
- 3) Singular limits
  (H.M. Soner, L. Ambrosio ...)

Transportational approach

The Gauss flows can be obtained from the optimal transportation by a certain scaling procedure. One has to construct a "parabolic" version of the optimal transportation.

(**V. Bogachev, A. Kolesnikov**) Let

$\mu = \rho_0 \, dx$ be a probability measure on convex set A,

$\nu = \rho_1 \, dx$ be a probability measure on $B_R = \{x\colon \ |x| \le R\}$.

There exist a function φ with convex sublevel sets $\{\varphi \le t\}$ and a mapping $T\colon \ A \to B_R$ such that $\nu = \mu \circ T^{-1}$ and T has the form

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}.$$

The level sets of φ are moving according to a (generalized) Gauss curvature flow

$$\dot{x} = -t^{d-1} \, \frac{\rho_1(T)}{\rho_0(x)} K(x) \cdot \mathrm{n}(x), \tag{1}$$

where $t = \varphi(x)$.

Scaling:

For every n consider another measure

$$\nu_n = \nu \circ S_n^{-1} \ \text{ with } \ S_n(x) = x|x|^n.$$

Let ∇W_n be the optimal transportation pushing forward μ to ν_n. Set $T_n = S_n^{-1} \circ \nabla W_n$. Define a new potential function φ_n by

$$W_n = \frac{1}{n+2} \varphi_n^{n+2}.$$

Then T is the limit of T_n, where

$$T_n = \varphi_n \frac{\nabla \varphi_n}{|\nabla \varphi_n|^{\frac{n}{n+1}}}.$$

and T_n pushes forward μ to ν.

Remark: There exist a unique mapping of this type.

Motivation for this study:

Two different ways of proving the classical isoperimetric inequality

1) Transportation proof (M. Gromov)

$$A \subset \mathbb{R}^d$$

$$B_r = \{x : |x| \le r\} \text{ — ball in } \mathbb{R}^d \text{ with } \mathrm{vol}(A) = \mathrm{vol}(B_r)$$

$$T = \nabla V : A \to B_r \text{ — optimal transportation of } \mathcal{H}^d|_A \text{ to } \mathcal{H}^d|_{B_r}$$

Change of variables formula

$$\det D^2 V = 1$$

Arithmetic-geometric inequality:

$$1 \le \frac{\Delta V}{d}.$$

$$\mathrm{vol}(A) = \int_A \det D^2 V \; dx \le \frac{1}{d} \int_A \Delta V \; dx$$

$$= \frac{1}{d} \int_{\partial A} \langle n_A, \nabla V \rangle d\mathcal{H}^{d-1} \le \frac{r}{d} \mathcal{H}^{d-1}(\partial A).$$

The isoperimetric inequality follows from

$$\mathrm{vol}(A) = \mathrm{vol}(B_r) = c_d r^d.$$

2) Geometric flows (P. Topping)

Let A_t be a family of convex sets such that ∂A_t evolve according to the Gauss curvature flow:

$$\dot{x}(t) = -K(x(t)) \cdot n(x(t))$$

Here $x(t) \in \partial A_{r-t}$, n — outer normal, K — Gauss curvature of ∂A_{r-t}, $A_{s_1} \subset A_{s_2}$ for $s_1 \geq s_2$.

Existence: K. Tso (Chou), 1975.

a) Evolution of the volume:

$$\frac{\partial}{\partial t}\text{vol}(A_t) = -\int_{\partial A_t} K \, d\mathcal{H}^{d-1} = -\mathcal{H}^{d-1}(S^{d-1}) = -\kappa_d \qquad (2)$$

(by Gauss-Bonnet theorem). Volume decreases with a constant speed.

b) Evolution of the surface measure:

$$\frac{\partial}{\partial t}\mathcal{H}^{d-1}(\partial A_t) = -\int_{\partial A_t} KH \, d\mathcal{H}^{d-1}, \ H \text{ — mean curvature}$$

Arithmetic-geometric inequality: $\sqrt[d-1]{K} \leq \frac{H}{d-1}$.

$$\frac{\partial}{\partial t}\mathcal{H}^{d-1}(\partial A_t) \leq -(d-1)\int_{\partial A_t} K^{\frac{d}{d-1}} \, d\mathcal{H}^{d-1}.$$

Hölder inequality:

$$\kappa_d = \int_{\partial A_t} K \, d\mathcal{H}^{d-1} \leq \Big(\int_{\partial A_t} K^{\frac{d}{d-1}} \, d\mathcal{H}^{d-1}\Big)^{\frac{d-1}{d}} \Big(\mathcal{H}^{d-1}(\partial A_t)\Big)^{\frac{1}{d}}.$$

$$\frac{\partial}{\partial t}\mathcal{H}^{d-1}(\partial A_t) \leq -\frac{(d-1)}{\kappa_d^{\frac{d}{d-1}}}\Big(\mathcal{H}^{d-1}(\partial A_t)\Big)^{\frac{1}{d-1}}. \qquad (3)$$

The isoperimetric inequality follows by comparison arguments from (2) and (3).

Change of variables formula

Change of variables for the optimal transportation (R. McCann)

If ∇V is the optimal transportation of μ to ν, then μ-almost everywhere

$$\det D_a^2 V = \frac{\rho_0}{\rho_1(\nabla V)}.$$

Here $D_a^2 V$ is the second Alexandrov derivative of V.

Main difficulty: potential φ is not Sobolev, but only BV.

The second derivatives of φ do exist only in directions orthogonal to $\frac{\nabla\varphi}{|\nabla\varphi|}$.

Change of variables for T

Theorem

The following change of variables formula holds for μ-almost all x:

$$K|D_a\varphi|\varphi^{d-1} = \frac{\rho_0}{\rho_1(T)}.$$

Here K is the Gauss curvature of the corresponding level set and $D_a\varphi$ is the absolutely continuous component of $D\varphi$.

Reverse mapping

Take $x \in B_r$ with $|x| = t$. Let H be the support function of $A_t = \{\varphi \le t\}$.

$$H(v) = \sup_{x \in A_t} \langle x, v \rangle,$$

$$S(x) = T^{-1}(x) = H \cdot n + \nabla_{S^{d-1}} H$$

$n = \frac{x}{|x|}$, $\nabla_{S^{d-1}}$ — spherical gradient.

Variants of the parabolic maximum principle

1) Let f be a twice continuously differentiable function on a convex set $A \subset \mathbb{R}^d$. Then there exists a constant $C = C(d)$ depending only on d such that

$$\sup_{x \in A} f(x) \le \sup_{x \in \partial A} f(x) + C(d) \int_{\mathcal{C}_f} |\nabla f| K dx.$$

where $\mathcal{C}_f$ are contact points of the level sets $\{f = t\}$ with the convex envelopes of $\{-f \le t\}$.

2) **Maximum principle:** $(d = 2)$ For every smooth f defined on

$$\Omega : \{0 < R_0 \le r \le R_1, \alpha \le \theta \le \beta\}$$

with $|\beta - \alpha| < \pi$ one has:

$$\sup_{\Omega} f \le C_{1,\Omega} \cdot \sup_{\partial_p \Omega} f \; + C_{2,\Omega} \sqrt{\int_{\Gamma_f} \frac{|f_r(f + f_{\theta\theta})|}{r} \, dx},$$

where

$$\Gamma_f : \{f_r \le 0, \; f + f_{\theta\theta} \le 0\}$$

and $\partial_p \Omega$ is the *parabolic boundary* of Ω.

Regularity results

1) **Sobolev estimates for** φ

Theorem: Let $d = 2$, $\rho_\nu = \frac{C_R}{r} \cdot I_{B_R}$,

$$T = \varphi \frac{\nabla \varphi}{|\nabla \varphi|}.$$

Assume that T pushes forward μ to ν. Then

$$C_{p,R} \int_A |\nabla \varphi|^{p+1} \, d\mu \le \int_A \Big| \frac{\nabla \rho_\mu}{\rho_\mu} \Big|^{p+1} \, d\mu + \int_{\partial A} \frac{\rho_\mu^{1+p}}{K^p} \, d\mathcal{H}^1.$$

(Proof: change of variables formula, integration by parts).

2) **Uniform estimates for** φ

Theorem: There exists a universal constant $p > 0$ such that

$$\sup_A |\nabla \varphi| \le C_1(M) \sup_{\partial A} |\nabla \varphi| + C_2(M)$$

provided

$$M = \sup\Big(\|\rho_\mu\|_{L^p(\mu)}, \|\rho_\mu^{-1}\|_{L^p(\mu)}, \big\| |\nabla \rho_\mu| \rho_\mu^{-1} \big\|_{L^p(\mu)} \Big) < \infty.$$

(Proof: Sobolev estimates of φ + a parabolic analog of the Alexandrov maximum principle).

Problem: What kind of flows can be constructed by mass-transportational methods?

Assume for simplicity that $d = 2$. Let $F(r, \theta)$ be a smooth function. Consider a mapping of the type

$$T = F(\varphi, n) \cdot n,$$

where φ has level convex subsets and $n = \frac{\nabla\varphi}{|\nabla\varphi|}$. One has

$$\det DT = |\nabla\varphi| F F_r(\varphi, n) K.$$

In particular, assume that F depends on φ in the following way

$$F(r, n) = \sqrt{2\int_0^r g(H_r^{-1}(s, n)) H_r(s, n)\, ds},$$

where $H(t, n) = \sup_{x \in A_t} \langle x, n \rangle$ is the corresponding dual potential (support function).

Then, assuming that T pushes forward $\mu = \rho_\mu(x)dx$ to $\lambda|_{B_R}$, one has the following change of variables formula

$$\rho_\mu = g(|\nabla\varphi|)K.$$

Since $|\nabla\varphi|^{-1}$ is the speed of level sets A_t in the direction of the inward normal, one gets that A_t are moving according to the following curvature flow:

$$\dot{x} = -\frac{1}{g^{-1}(\rho_\mu/K)}.$$

Examples of flows of this type

1) Power-Gauss curvature flows

$$\dot{x} = -K^p \cdot n.$$

(geometry, computer vision)

2) Logarithmic Gauss curvature flows

$$\dot{x} = -\log K \cdot n$$

(Minkowsky-type problems)

Main difficulty: One needs more regularity of φ. In particular, it is natural to expect that φ is Sobolev (not only BV).