A gradient flow model in the space of signed measures

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Evolution model for ginzburg-landau vortices

Mean field model for the evolution of the vortex densities in a superconductor, derived by W. E (1994), Lin and Zhang (2000):

$$\frac{d}{dt}\mu(t) - \mathrm{div}(\nabla h_{\mu(t)}\, \mu(t)) = 0 \qquad \text{in } \mathbb{R}^2,$$

$$-\Delta h_{\mu(t)} = \mu(t) \quad \text{in } \mathbb{R}^2.$$

Model proposed by Chapman, Rubinstein and Schatzman (1996)

$$\frac{d}{dt}\mu(t) - \mathrm{div}(\nabla h_{\mu(t)} |\mu(t)|) = 0, \quad \textit{in } \Omega \qquad \text{(CRS)}$$

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases}$$

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The positive measure setting

Suppose first that the vortex density μ is a positive measure. In this case we search for solutions in $H^{-1}(\Omega) \cap \mathscr{P}_2(\overline{\Omega})$, where $\mathscr{P}_2(\overline{\Omega})$ is the space of probability measures over $\overline{\Omega}$ with finite second moment.

The basic idea is to view a solution to (CRS) as a steepest descent curve in $\mathscr{P}_2(\overline{\Omega})$ of the related energy:

$$\Phi_\lambda(\mu) := \frac{\lambda}{2}|\mu|(\Omega) + \frac{1}{2}\int_\Omega \left(|\nabla h_\mu|^2 + |h_\mu - 1|^2\right), \qquad \lambda \geq 0.$$

Formally find a curve $t \mapsto \mu(t)$ such that

$$\dot{\mu}(t) = -\nabla\Phi_\lambda(\mu(t)).$$

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Optimal transportation distance

Kantorovich optimal transportation problem

$$\inf\left\{\int_{X\times X}|x-y|^2\,d\gamma(x,y):\gamma\in\Gamma(\mu,\nu)\right\}.$$

Transport plans: $\gamma\in\Gamma(\mu,\nu)$ (i.e. $\gamma\in\mathscr{P}(X\times X),\pi^1_\#\gamma=\mu,\pi^2_\#\gamma=\nu$).

Optimal plans set: $\Gamma_0(\mu,\nu)$.
Plan induced by a map: $\gamma=(\mathbf{I},\mathbf{t})_\#\mu$.

Optimal transport distance (**Wasserstein distance**)

$$W_2(\mu,\nu):=\left(\inf\left\{\int_{X\times X}|x-y|^2\,d\gamma(x,y):\gamma\in\Gamma(\mu,\nu)\right\}\right)^{\frac{1}{2}}$$

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Gradient flow in the space $(\mathscr{P}_2(X), W_2)$

Jordan-Kinderlehrer-Otto (1998) framework:
Consider a functional $\Phi : \mathscr{P}_2(X) \to \mathbb{R}$ and a PDE of the form

$$\partial_t \mu_t + \operatorname{div}(\mathbf{v}_t \mu_t) = 0.$$

- Given $\mu^0 \in \mathscr{P}_2(X)$ and a time step $\tau > 0$, find recursively μ_τ^k among solutions of

$$\min_{\nu \in \mathscr{P}_2(X)} \Phi(\nu) + \frac{1}{2\tau} W_2^2(\nu, \mu_\tau^{k-1}), \qquad \mu_\tau^0 = \mu^0.$$

- Construct a curve $t \in [0, T] \mapsto \mu(t) \in \mathscr{P}_2(X)$ interpolating the discrete values and passing to the limit as $\tau \to 0$.
- Show that the obtained limit curve satisfies the continuity equation.

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Existence and regularity result

In the $\mathscr{P}_2(\overline{\Omega})$ framework, we have the following

Theorem (L. Ambrosio, S. Serfaty, 2008)

Let $\mu^0 \in H^{-1}(\Omega) \cap \mathscr{P}_2(\overline{\Omega})$. Then there exists a curve $t \mapsto \mu(t) \in H^{-1}(\Omega) \cap \mathscr{P}_2(\overline{\Omega})$ such that:

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(iii) If moreover $\chi_\Omega \mu^0 \in L^p(\Omega)$, then $\|\chi_\Omega \mu(t)\|_p \leq C$;

Task

The actual (CRS) model involves signed measures. Can we extend the above framework and results to the signed case?

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Transport cost for signed measures

Extension of the 2-Wasserstein distance to the space

$$\mathcal{M}_{\kappa,M}(\overline{\Omega}) := \{\mu \in \mathcal{M}(\overline{\Omega}) : \mu(\overline{\Omega}) = \kappa, |\mu|(\overline{\Omega}) \le M\}, \quad \kappa \in \mathbb{R}, M \ge 0.$$

First attempt: given $\mu, \nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$, let $\mu = \mu^+ - \mu^-, \nu = \nu^+ - \nu^-$ (Hahn decomposition), and let

$$\mathbb{W}_2(\mu,\nu) := W_2^2(\mu^+ + \nu^-, \nu^+ + \mu^-).$$

- $\mathbb{W}_2$ does not satisfy the triangle inequality:
  Let $\mu = \delta_0, \nu = \delta_4$. Let $\sigma = \delta_1 - \delta_2 + \delta_3$. We get

$$\mathbb{W}_2(\mu,\nu) = 4 \quad \text{and} \quad \mathbb{W}_2(\mu,\sigma) + \mathbb{W}_2(\sigma,\nu) = 2\sqrt{2}.$$

- $\mu \mapsto \mathbb{W}_2(\cdot,\mu)$ is not weakly l.s.c.

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$$\left(\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2 \, d\gamma \right)^{1/2} \geq \sqrt{\frac{1}{2M}} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\gamma \geq \sqrt{\frac{1}{2M}} \mathbb{W}_1(\mu,\nu),$$

where

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The new object $\mathbb{W}_1$ is a distance, as clearly seen from the duality formula

$$W_1(\mu^+ + \nu^-, \nu^+ + \mu^-) = \sup_{\varphi\in\mathrm{Lip}(\Omega), \|\varphi\|_{\mathrm{Lip}}\leq 1} \int_\Omega \varphi \, d(\mu - \nu).$$

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Relaxed l.s.c. version: for $|\nu|(\overline{\Omega}) \leq |\mu|(\overline{\Omega})$,

$$\mathcal{W}_2^2(\nu,\mu) := \inf_{\sigma^+ - \sigma^- = \nu} \left\{ W_2^2(\sigma^+,\mu^+) + W_2^2(\sigma^-,\mu^-) \right\}.$$

We have

$$\mathcal{W}_2(\nu,\mu) \geq \sqrt{\frac{1}{2M}} \mathbb{W}_1(\mu,\nu).$$

The discrete scheme: given $\mu^0 \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$, find μ_τ^{k+1} by

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Transport cost for signed measures

Relaxed l.s.c. version: for $|\nu|(\overline{\Omega}) \leq |\mu|(\overline{\Omega})$,

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Convergence of the discrete scheme

Existence of a limit curve in $\mathcal{M}_{\kappa,M}(\overline{\Omega})$.

From

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Hence, for $n, m \in \mathbb{N}$, $n > m$, using $\left(\sum_{i=1}^N a_i\right)^2 \le N\sum_{i=1}^N a_i^2$, we get

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We are left with

$$\mathbb{W}_1(\mu_\tau^m, \mu_\tau^n) \le \sqrt{4M\Phi_\lambda(\mu)^0(n-m)\tau}.$$

Interpolation: for $t > 0$, let

$$\overline{\mu}_\tau(t) := \mu_\tau^k \quad \text{if } t \in ((k-1)\tau, k\tau], k > 0.$$

We find the $C^{0,1/2}$ estimate

$$\mathbb{W}_1(\overline{\mu}_\tau(t), \overline{\mu}_\tau(s)) \le \sqrt{2\Phi_\lambda(\overline{\mu})}\sqrt{|t-s|+\tau} \qquad \forall s, t > 0.$$

A limit exists by compactness:

$$\overline{\mu}_{\tau_n}(t) \rightharpoonup \mu(t) \quad \text{in the sense of measures, for a.e. } t.$$

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Variational result

For simplicity let $\Omega = \mathbb{R}^2$.

Theorem (L. Ambrosio, E. M., S. Serfaty, 2010)

Consider a single step of the minimization problem above, starting from $\mu \in L^p(\mathbb{R}^2)$, $p \geq 4$.

- There exists a minimizer $\mu_\tau \in L^p(\mathbb{R}^2)$ such that $\|\mu_\tau\|_p \leq \|\mu\|_p$. (uniform in τ estimate)
- There holds

$$-\nabla h_{\mu_\tau} \mu_\tau = \frac{1}{\tau} \pi^1_\#((x-y)\gamma_0^+) + \frac{1}{\tau} \pi^1_\#((x-y)\gamma_0^-),$$

where $\gamma_0^+ \in \Gamma_0(\mu_\tau^+, \mu_0^+)$ and $\gamma_0^- \in \Gamma_0(\mu_\tau^-, \mu_0^-)$.

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Sketch of the proof

- The regularity part: there exists functions $\varphi : \mathbb{R} \to \mathbb{R}$, with p-growth, such that

$$\int_{\mathbb{R}^2} \varphi(\mu_\tau) \le \int_{\mathbb{R}^2} \varphi(\mu).$$

(They are characterized characterized by the McCann (1997) displacement convexity inequality: $2x^2\varphi''(x) \ge x\varphi'(x) - \varphi(x)$)

- The Euler-Lagrange equation: suppose μ_τ is the minimizer. Consider a variation of the form

$$(\mu_\tau)_\varepsilon = (\mathbf{I} + \varepsilon\xi)_\# \mu_\tau,$$

where ξ is a C_0^∞ vector field. First order argument:

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Limit PDE

Theorem (L. Ambrosio, E. M., S. Serfaty, 2010)

Let $\mu^0 \in L^4(\mathbb{R}^2)$. There exists a minimizing movement $\mu(t)$ and it satisfies

$$\frac{d}{dt}\mu(t) - \mathrm{div}\left(\nabla h_{\mu(t)}\varrho(t)\right) = 0 \quad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2),$$

where $\varrho(t)$ is a suitable positive measure satisfying $\varrho(t) \geq |\mu(t)|$.

Idea of the proof: in the sense of distributions ($\phi \in C_0^2(\mathbb{R}^2)$),

$$\frac{d}{dt}\int_{\mathbb{R}^2}\phi\, d\overline{\mu}_\tau(t) = \sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^2}\phi\, d\mu_\tau^{k+1} - \int_{\mathbb{R}^2}\phi\, d\mu_\tau^{k}\right)\delta_{\{k\tau\}},$$

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Limit PDE

Theorem (L. Ambrosio, E. M., S. Serfaty, 2010)

Let $\mu^0 \in L^4(\mathbb{R}^2)$. There exists a minimizing movement $\mu(t)$ and it satisfies

$$\frac{d}{dt}\mu(t) - \mathrm{div}\,(\nabla h_{\mu(t)}\varrho(t)) = 0 \quad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^2),$$

where $\varrho(t)$ is a suitable positive measure satisfying $\varrho(t) \geq |\mu(t)|$.

Idea of the proof: in the sense of distributions ($\phi \in C_0^2(\mathbb{R}^2)$),

$$\frac{d}{dt}\int_{\mathbb{R}^2}\phi\, d\overline{\mu}_\tau(t) = \sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^2}\phi\, d\mu_\tau^{k+1} - \int_{\mathbb{R}^2}\phi\, d\mu_\tau^{k}\right)\delta_{\{k\tau\}},$$

$$\frac{d}{dt}\int_{\mathbb{R}^2}\phi\, d\overline{\mu}_\tau(t) = \sum_{k=0}^{\infty}\delta_{\{k\tau\}}\int_{\mathbb{R}^2\times\mathbb{R}^2}(\phi(x)-\phi(y))\, d\gamma_\tau^{k+1}(x,y).$$

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$$= \sum_{k=0}^{\infty}\delta_{\{k\tau\}}\left(\int_{\mathbb{R}^2\times\mathbb{R}^2}\langle\nabla\phi(x), x-y\rangle\, d\left((\gamma_0^+)_\tau^{k+1} - (\gamma_0^-)_\tau^{k+1}\right)(x,y)\right) + o(1).$$

But
$$-\nabla h_{\mu_\tau^k}(\mu_\tau^k)^+ = \tfrac{1}{\tau}\pi^1_\#((x-y)(\gamma_0^+)_\tau^k),$$
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We find

$$\frac{d}{dt}\int_{\mathbb{R}^2}\phi\, d\overline{\mu}_\tau(t) = -\sum_{k=0}^{\infty}\tau\delta_{\{k\tau\}}\int_{\mathbb{R}^2}\left\langle\nabla\phi(x), \nabla h_{\mu_\tau^k}(x)\right\rangle d|\mu_\tau^k|(x) + o(1).$$

Passing to the limit as τ goes to zero, we have $\overline{\mu}_\tau(t) \rightharpoonup \mu(t)$ for any t, but $|\overline{\mu}_\tau(t)| \rightharpoonup \varrho(t) \neq |\mu(t)|$. Then

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A more interesting formulation is

$$\begin{cases} \dfrac{d}{dt}\varrho^+(t) - \operatorname{div}\left(\nabla h_{\mu(t)}\varrho^+(t)\right) = -\sigma(t) \\ \dfrac{d}{dt}\varrho^-(t) + \operatorname{div}\left(\nabla h_{\mu(t)}\varrho^-(t)\right) = -\sigma(t). \end{cases}$$

The term $\sigma \geq 0$ is responsible of mass cancellation.

Remark: Orthogonality preserving solutions to the system above are solution to

$$\frac{d}{dt}\mu(t) - \operatorname{div}(\nabla h_{\mu(t)}\,|\mu(t)|) = 0. \qquad \text{(CRS)}$$

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The boundary

Task

Uniqueness of solutions up to the boundary.

We work with probability measures. We begin with a formulation that accounts for the boundary.

The actual formulation in Ambrosio, Serfaty (2008) is

$$\frac{d}{dt}\mu(t) - \mathrm{div}(\chi_\Omega \nabla h_{\mu(t)}\mu(t)) = 0 \quad \mathit{in}\ \mathbb{R}^2 \qquad \text{(CRS 2)}$$

In the sense of distributions this means

$$\int_0^T \int_{\overline{\Omega}} \partial_t \phi(x,t)\, d\mu_t + \int_0^T \int_\Omega \nabla h_{\mu_t}(x) \cdot \nabla \phi(x,t)\, d\mu_t = 0.$$

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Notation. If $\mu \in \mathscr{P}(\overline{\Omega})$, let $\widehat{\mu} = \chi_\Omega \mu$, $\widetilde{\mu} = \chi_{\partial\Omega}\mu$.

We try to be even more precise on the role of the boundary with the following

Definition (regular gradient flow)

Let $T > 0$. A solution of problem (CRS 2) is a regular gradient flow if

i) $\|\widehat{\mu}(t)\|_\infty \in L^\infty(0, T)$,

ii) $\langle \nabla h_{\mu(t)}(x), y - x \rangle \geq 0$ for all $(x, y) \in \mathrm{supp}(\widetilde{\mu}(t)) \times \overline{\Omega}$ and $t \in (0, T]$.

Condition (ii) means that, at least for convex Ω, the (limit of) velocity field at the boundary points away from Ω.

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Existence of a regular gradient flow

Theorem (E.M., 2009)

Let Ω be convex. Let $\widehat{\mu}^0 \in L^\infty(\Omega)$.
Then there exists a regular gradient flow $\mu(t)$ such that $\mu(0) = \widehat{\mu}^0$

The proof is based on a new variation, made on the boundary: let μ_τ be a discrete minimizer and

$$(\mu_\tau)_\varepsilon := \widehat{\mu}_\tau + \alpha^2 T_{\varepsilon\#}(\sigma) + (1-\alpha^2)\widetilde{\mu}_\tau.$$

where $\alpha = (1-\epsilon)^2$. Here $\sigma \ll \mathcal{L}^2 \llcorner \Omega$ and $T \in \Gamma_0(\widetilde{\mu}_\tau, \sigma)$.

Theorem (E.M., 2009)

Let Ω be convex. Let μ^1, μ^2 be regular gradient flows. Then $\mu^1(0) = \mu^2(0)$ implies $\mu^1(t) = \mu^2(t)$ for all $t \in [0, T]$.

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