The sharp constants in critical Sobolev embedding theorems via optimal transport

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This commercial reads:

Permanently sober loaders;

...

We select

optimal transport

$$n \geqslant 2; \quad 1 < p < n.$$

$\dot{W}_p^1(\mathbb{R}^n)$ stands for the completion of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla v\|_p$.

1. Critical Sobolev embedding:

$$K(n,p) = \inf_{v \in \dot{W}_p^1(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla v\|_{p,\mathbb{R}^n}}{\|v\|_{p^*,\mathbb{R}^n}} > 0.$$

2. Critical trace embedding:

$$K_1(n,p) = \inf_{v \in \dot{W}_p^1(\mathbb{R}^n_+)\setminus\{0\}} \frac{\|\nabla v\|_{p,\mathbb{R}^n_+}}{\|v\|_{p^{**},\partial\mathbb{R}^n_+}} > 0.$$

Here

$$p^* = \frac{np}{n-p}; \qquad p^{**} = \frac{(n-1)p}{n-p}.$$

NB: Without loss of generality, one can assume $v \geqslant 0$.

How to find the sharp constants in 1 and 2?

1. Critical Sobolev embedding. The classical approach (T. Aubin, 1976; G. Talenti, 1976):

Step 1: Symmetrization, i.e. the rearrangement mapping any level set to the ball of the same volume centered in origin.

It is well known (G. Pólya, G. Szegö, 1940s) that this rearrangement diminishes our functional.

Step 2: Thus, the problem is reduced to a one-dimensional inequality which was considered by G.A. Bliss (1930).

Alternative approach based on the optimal transport (D. Cordero-Erausquin, B. Nazaret, C. Villani, 2004).

Consider two probability measures in $\mathbb{R}^n$ with smooth densities F and G and bounded supports. Then there exists the *Brenier map* $T = \nabla\varphi$ such that for all measurable functions ψ

$$\int_{\mathbb{R}^n} \psi(x)G(x)\,dx = \int_{\mathbb{R}^n} \psi(T(x))F(x)\,dx. \quad (1)$$

Moreover, the function φ is convex and satisfies the Monge–Ampère equation

$$F(x) = G(\nabla\varphi(x)) \cdot \mathrm{det}(D^2\varphi(x)). \quad (2)$$

almost everywhere w.r.t. the measure Fdx. Here $D^2\varphi$ is a.e.-Hessian matrix of φ which exists by A.D. Aleksandrov's theorem.

By (1),

$$\int\limits_{\mathbb{R}^n} G^{1-\frac{1}{n}}(x)\,dx = \int\limits_{\mathbb{R}^n} G^{-\frac{1}{n}}(\nabla\varphi(x))F(x)\,dx.$$

Using (2) and the Hadamard inequality, we obtain

$$\int\limits_{\mathbb{R}^n} G^{1-\frac{1}{n}}(x)\,dx = \\ = \int\limits_{\mathbb{R}^n} \det^{\frac{1}{n}}(D^2\varphi(x))F^{1-\frac{1}{n}}(x)\,dx \leqslant \\ \leqslant \frac{1}{n}\int\limits_{\mathbb{R}^n} \Delta\varphi(x)F^{1-\frac{1}{n}}(x)\,dx. \quad (3)$$

Since φ is convex, we can change $\Delta\varphi$, understood as a.e.-Laplacian in the right-hand side of (3), to the full distributional Laplacian.

Integrating by parts, we get

$$\int\limits_{\mathbb{R}^n} G^{1-\frac{1}{n}}(x)\,dx \leqslant$$

$$\leqslant -\frac{1}{n}\int\limits_{\mathbb{R}^n} \langle \nabla\varphi(x), \nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx. \quad (4)$$

Put $F = v^{p^*}$, $G = u^{p^*}$. Then $\|v\|_{p^*,\Omega} = \|u\|_{p^*,\Omega} = 1$, and (4) becomes

$$\int\limits_{\mathbb{R}^n} u^{p^*(1-\frac{1}{n})}(x)\,dx \leqslant$$

$$\leqslant -\frac{p(n-1)}{n(n-p)}\int\limits_{\mathbb{R}^n} v^{\frac{n(p-1)}{n-p}}(x)\langle \nabla\varphi(x), \nabla v(x)\rangle\,dx.$$

(Note that the exponent in the last integral equals p^*/p').

Now we apply the Hölder inequality and arrive at

$$\int\limits_{\mathbb{R}^n} u^{p^*(1-\frac{1}{n})}(x)\,dx \leqslant \frac{p(n-1)}{n(n-p)}\|\nabla v\|_{p,\mathbb{R}^n} \cdot$$

$$\cdot \left[\int\limits_{\mathbb{R}^n} v^{p^*}(x)|\nabla\varphi(x)|^{p'}\,dx\right]^{1/p'}.$$

By (1),

$$\int\limits_{\mathbb{R}^n} v^{p^*}(x)|\nabla\varphi(x)|^{p'}\,dx = \int\limits_{\mathbb{R}^n} u^{p^*}(y)|y|^{p'}\,dy.$$

This gives

$$\frac{\int\limits_{\mathbb{R}^n} u^{p^*(1-\frac{1}{n})}(x)\,dx}{\left[\int\limits_{\mathbb{R}^n} u^{p^*}(x)|x|^{p'}\,dx\right]^{1/p'}} \leqslant \frac{p(n-1)}{n(n-p)}\|\nabla v\|_{p,\mathbb{R}^n}.$$

Since the Brenier map φ is not contained in the last inequality, it remains valid for all u and v normalized in $L_{p^*}(\mathbb{R}^n)$.

Now we observe that the equality in $\det^{\frac{1}{n}}(D^2\varphi) \leqslant \frac{1}{n}\Delta\varphi$ implies $D^2\varphi = C\mathbb{I}$, and thus, we can assume $\nabla\varphi(x) = Cx$.

Further, the equality in the Hölder inequality means $v^{p^*/p'}\nabla\varphi = C\nabla v$. This implies $v = v(|x|)$ and provides a 1st order ODE for v. Solving it, we obtain the Bliss function

$$h(x) = (a + b|x|^{p'})^{1-\frac{n}{p}}.$$

Direct calculation shows that we really have the equality for $u = v = Ch$. In particular, this means

$$\frac{\|\nabla v\|_{p,\mathbb{R}^n}}{\|v\|_{p^*,\mathbb{R}^n}} \geqslant \frac{\|\nabla h\|_{p,\mathbb{R}^n}}{\|h\|_{p^*,\mathbb{R}^n}},$$

and $K(n,p) = n^{\frac{1}{p}}\Big(\frac{n-p}{p-1}\Big)^{\frac{1}{p'}}\Big(\omega_{n-1}\cdot\mathcal{B}\Big(\frac{n}{p},\frac{n}{p'}+1\Big)\Big)^{\frac{1}{n}}$.

2. Critical trace embedding.

Escobar (1988) conjectured that the minimizer in the half-space is

$$w(x) = |x - \varepsilon \mathbf{e}|^{-\frac{n-p}{p-1}}, \tag{5}$$

with $\mathbf{e} = (0, \ldots, 0, -1)$, and proved it for $p = 2$ using the conformal invariance of the quotient $\|\nabla v\|_{2,\mathbb{R}^n_+} \big/ \|v\|_{2^{**},\partial\mathbb{R}^n_+}$. B. Nazaret (2006) proved this conjecture by the optimal transport approach.

Now we consider two probability measures in $\mathbb{R}^n_+$ with smooth densities F and G and bounded supports. Then the identity (1) becomes

$$\int\limits_{\mathbb{R}^n_+} \psi(x) G(x)\, dx = \int\limits_{\mathbb{R}^n_+} \psi(T(x)) F(x)\, dx. \tag{6}$$

Just as earlier, we obtain

$$\int_{\mathbb{R}^n_+} G^{1-\frac{1}{n}}(x)\,dx \leqslant \frac{1}{n}\int_{\mathbb{R}^n_+} \Delta\varphi(x)F^{1-\frac{1}{n}}(x)\,dx.$$

Integrating by parts, we get

$$n\int_{\mathbb{R}^n_+} G^{1-\frac{1}{n}}(x)\,dx \leqslant$$
$$\leqslant \int_{\partial\mathbb{R}^n_+} F^{1-\frac{1}{n}}(x)\langle\nabla\varphi(x),\mathbf{n}\rangle\,d\mathbf{\Sigma} -$$
$$-\int_{\mathbb{R}^n_+} \langle\nabla\varphi(x),\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx.$$

By definition of the Brenier map, for all $x \in \mathbb{R}^n_+$ one has $\nabla\varphi(x) \in \mathbb{R}^n_+$. Therefore, $\langle\nabla\varphi(x),\mathbf{n}\rangle \leqslant 0$ on $\partial\mathbb{R}^n_+$, and

$$n\int_{\mathbb{R}^n_+} G^{1-\frac{1}{n}}(x)\,dx \leqslant$$
$$\leqslant -\int_{\mathbb{R}^n_+} \langle\nabla\varphi(x),\nabla(F^{1-\frac{1}{n}})(x)\rangle\,dx. \quad (7)$$

Adding to both parts of (7) the integral

$$\int_{\mathbb{R}^n_+} \langle \mathbf{e}, \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx =$$
$$= \int_{\partial\mathbb{R}^n_+} F^{1-\frac{1}{n}}(x)\langle \mathbf{e}, \mathbf{n}\rangle\, d\boldsymbol{\Sigma} =$$
$$= \int_{\partial\mathbb{R}^n_+} F^{1-\frac{1}{n}}(x)\, d\boldsymbol{\Sigma},$$

we arrive at

$$\int_{\partial\mathbb{R}^n_+} F^{1-\frac{1}{n}}(x)\, d\boldsymbol{\Sigma} + n \int_{\mathbb{R}^n_+} G^{1-\frac{1}{n}}(x)\, dx \leqslant$$
$$\leqslant \int_{\mathbb{R}^n_+} \langle \mathbf{e} - \nabla\varphi(x), \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx. \quad (8)$$

Put $F = v^{p^*}$, $G = u^{p^*}$. Then $\|v\|_{p^*,\mathbb{R}^n_+} = \|u\|_{p^*,\mathbb{R}^n_+} = 1$, and (8) becomes

$$\|v\|^{p^{**}}_{p^{**},\partial\mathbb{R}^n_+} \leqslant \frac{(n-1)p}{n-p} \int\limits_{\mathbb{R}^n_+} v^{\frac{n(p-1)}{n-p}}(x) \cdot$$

$$\cdot \langle \mathbf{e} - \nabla\varphi(x), \nabla v(x)\rangle \, dx - n\|u\|^{p^{**}}_{p^{**},\mathbb{R}^n_+}.$$

By the Hölder inequality,

$$\|v\|^{p^{**}}_{p^{**},\partial\mathbb{R}^n_+} \leqslant \frac{(n-1)p}{n-p} \|\nabla v\|_{p,\mathbb{R}^n_+} \cdot$$

$$\cdot \left[\int\limits_{\mathbb{R}^n_+} v^{p^*}(x) |\mathbf{e} - \nabla\varphi(x)|^{p'} \, dx \right]^{\frac{1}{p'}} -$$

$$- n\|u\|^{p^{**}}_{p^{**},\mathbb{R}^n_+}.$$

By (6),

$$\int\limits_{\mathbb{R}^n_+} v^{p^*}(x) |\mathbf{e} - \nabla\varphi(x)|^{p'} \, dx =$$

$$= \int\limits_{\mathbb{R}^n_+} u^{p^*}(y) |\mathbf{e} - y|^{p'} \, dy.$$

This gives

$$\|v\|^{p^{**}}_{p^{**},\partial\mathbb{R}^n_+} \leqslant \frac{(n-1)p}{n-p}\|\nabla v\|_{p,\mathbb{R}^n_+}\cdot$$
$$\cdot\left[\int\limits_{\mathbb{R}^n_+} u^{p^*}(x)|\mathbf{e}-x|^{p'}\,dx\right]^{\frac{1}{p'}} -$$
$$- n\|u\|^{p^{**}}_{p^{**},\mathbb{R}^n_+}. \quad (9)$$

Note that both sides of (9) do not contain the Brenier map. Hence, by approximation, this inequality remains valid for all u and v normalized in $L_{p^*}(\mathbb{R}^n_+)$.

Now we specify (7) by setting $u = Cw$, with w defined in (5) and $C = \|w\|^{-1}_{p^*,\mathbb{R}^n_+}$. Then for any $v \in \dot{W}^1_p(\mathbb{R}^n_+)$ such that $\|v\|_{p^*,\mathbb{R}^n_+} = 1$, we have

$$\|v\|^{p^{**}}_{p^{**},\partial\mathbb{R}^n_+} \leqslant A\|\nabla v\|_{p,\mathbb{R}^n_+} - B, \qquad (10)$$

where

$$A = \tfrac{(n-1)p}{n-p} \cdot C^{\frac{n(p-1)}{n-p}} \cdot \mathcal{I}^{\frac{1}{p'}}, \qquad B = n\, C^{p^{**}} \cdot \mathcal{I};$$

$$\mathcal{I} = \|w\|_{p^{**},\mathbb{R}^n_+}^{p^{**}} = \int\limits_{\mathbb{R}^n_+} \frac{dx}{|x-\mathbf{e}|^{(n-1)p'}}.$$

For arbitrary $v \in \dot{W}_p^1(\Omega)$, without normalization, (9) can be rewritten as follows:

$$\left(\frac{K(v)}{J(v)}\right)^{p^{**}} \leqslant AK(v) - B,$$

i.e.

$$J^{p^{**}}(v) \geqslant \mathcal{F}(K(v)) \equiv \frac{K^{p^{**}}(v)}{AK(v) - B},$$

where

$$J(v) = \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^{**},\partial\Omega}}, \qquad K(v) = \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^{*},\Omega}}.$$

By elementary calculus, the function $\mathcal{F}$ achieves its minimum at the point

$$\frac{p(n-1)B}{n(p-1)A} = \frac{n-p}{p-1} C\mathcal{I}^{\frac{1}{p}} = K(w),$$

and therefore,

$$J^{p^{**}}(v) \geqslant \frac{K^{p^{**}}(w)}{AK(w)-B} = \left(\tfrac{n-p}{p-1}\right)^{\frac{n(p-1)}{n-p}} \mathcal{I}^{\frac{p-1}{n-p}}.$$

If $v = u = Cw$, then the Brenier map is the identity. Direct calculation shows that all the inequalities become equalities. This means

$$\frac{\|\nabla v\|_{p,\mathbb{R}^n_+}}{\|v\|_{p^{**},\partial\mathbb{R}^n_+}} \geqslant \frac{\|\nabla w\|_{p,\mathbb{R}^n_+}}{\|w\|_{p^{**},\partial\mathbb{R}^n_+}},$$

and

$$K_1(n,p) = \left(\tfrac{n-p}{p-1}\right)^{\frac{1}{p'}} \left(\tfrac{\omega_{n-2}}{2} \cdot \mathcal{B}\left(\tfrac{n-1}{2}, \tfrac{n-1}{2(p-1)}\right)\right)^{\frac{1}{(n-1)p'}}.$$

The following observation is by A. Nazarov (to appear in Algebra and Analysis, 2010, N5).

Theorem. Let Ω be a convex circular cone with aperture 2θ. Then the minimum for the critical trace embedding in Ω is provided by the function (5).

The proof runs almost without changes.

In particular, this implies

$$K_1(n,p;\Omega) = \left(\frac{n-p}{p-1}\right)^{\frac{n(p-1)}{(n-1)p}} \mathcal{I}^{\frac{p-1}{(n-1)p}} \sin^{\frac{1}{p^{**}}}(\theta),$$

with

$$\mathcal{I} = \|w\|_{p^{**},\Omega}^{p^{**}} = \int\limits_{\Omega} \frac{dx}{|x-\mathbf{e}|^{(n-1)p'}}.$$

Remark. The value of $\mathcal{I}$ for circular cones can be calculated explicitly:

$$\mathcal{I} = \pi^{\frac{n}{2}-1} 2^{a-\frac{3}{2}} \Gamma\big(a - \tfrac{n-1}{2}\big) \mathcal{B}\big(\tfrac{n}{2}, a - \tfrac{n}{2}\big) \cdot$$
$$\cdot \sin^{n-a-\frac{1}{2}}(\theta) \mathsf{P}^{\frac{1}{2}-a}_{n-a-\frac{3}{2}}\big(\cos(\theta)\big),$$

where $a = \frac{(n-1)p}{2(p-1)}$ while $\mathsf{P}^{\mu}_{\nu}(x)$ is the Legendre function.

Theorem remains valid for any convex cone Ω, if its supporting hyperplanes at almost every point have a constant angle θ with the axis x_n. The simplest example of such cone is a dyhedral angle less than half-space. Another interesting example is a cone supported by *arbitrary* simplex in $\mathbb{S}^{n-1}$.

It is worth to note that for nonconvex cone of such type ($\theta > \frac{\pi}{2}$), the function (5) does not provide minimum in the critical trace embedding, though it is a stationary point.