Dynamic Routing; Configuration of Overloaded Interacting Servers

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We consider a symmetrical network with k servers and l Poisson input flows. The protocol uses dynamic routing: each flow is assigned to a subgroup of m servers, upon its arrival a message is directed to the least busy of these servers.

Under the condition that at least m servers are overloaded the number of overloaded servers depends on the rate of input flows.

For a circle of interacting servers this effect is described in:

N.D. Vvedenskaya, E.A. Pechersky, *Circle of Interacting Servers: Spontanious Collective Fluctuations in case of large Fluctuation*, Probl. Inform. Transmission, 2008, V. 44, No 4, P. 370-384.

Some history.

Consider a system where the number of servers $k \to \infty$. In case of dynamic routing where the shortest of m *queues* is selected the stationary distribution of queue lengths decreases

superexponentially:

$$\Pr[queue \quad length \geq n] = \lambda^{\frac{n^m - 1}{m-1}}.$$

For example, as $m = 2$

$$\Pr[queue \quad length \geq n] = \lambda^{n^2 - 1}.$$

N.D. Vvedenskaya, R.L. Dobrushin, F.I. Karpelevich, *A Queueing System with the Selection of the Shortest of Two Queues, Asymptotical Approach*, Probl. Inform. Transmission, 1996, V. 32, No 1, 15-27.

If k is finite the situation is quite different.

Consider a symmetrical system $\mathcal{S} = \mathcal{S}(k, m)$ formed by k identical servers $S = (s_1, ..., s_k)$ and $l = \binom{k}{m}$ independent Poisson flows $F = (f_{A_1}, ..., f_{A_l})$, each of rate λ.

Here $A_j = (j_1, ..., j_m)$ are the numbers of servers $S_{A_j} = (s_{j_1}, ..., s_{j_m})$ assigned to f_{A_j}.

The servers have infinite buffers and operate with equal rate 1.

Upon its arrival with f_{A_j} a message is directed to a server from S_{A_j} that at the time of its arrival has the smallest workload.

The flows are described by the sequences of independent pairs

$(\xi_n^{(A_j)}, \tau_n^{(A_j)}), \quad n = ..., -1, 0, 1, ..., \quad j = 1, ..., l,$

$\tau_n^{(A_j)}$ – the intervals between arrivals of two messages, $\Pr(\tau_n^{(A_j)} > t) = e^{-\lambda t}$.

$\xi_n^{(A_j)}$ – the message lengths.

The distributions of $\xi_n^{(A_j)}$ are identical,

$$\varphi(\theta) = \mathbf{E}e^{\theta \xi_n^{(A_j)}} < \infty, \ \ \theta < \theta_+, \ \ \lim_{\theta \uparrow \theta_+} \varphi(\theta) = \infty.$$

All variables are iid.

The mean intensity of the sum of Poisson flows upon one server is $\Lambda = \frac{\lambda\binom{k}{m}}{k}$. The system is in stationary state,

$$\Lambda\varphi'(0) < 1, \quad \lambda < \widehat{\lambda} = k\left(l\varphi'(0)\right)^{-1}.$$

If during some time period the flow intensity is large the flow is said to be *overheated*, if there is a lot of unserved messages in a buffer of a server the server is said to be *overloaded*.

Let $\boldsymbol{w}(t) = (w_1(t), ..., w_k(t)$ be the load vector that indicates the load of buffers.

Virtual message arrive upon $\mathcal{S}$ at time moment $t = 0$ with flow f_{A_1}, has zero length and is directed to the servers according to the dynamic routing protocol. The delay (waiting time) of virtual message is denoted by ω_1. The event $\{\omega_1 \geq n\}$ is denoted by $\Gamma_1(n)$ We are interested in probability of $\Gamma_1(n)$, that is exponential:

$$\lim_{n\to\infty} \frac{-1}{n} \ln \Pr[\omega_1 \geq nd].$$

Logarithm of needed probability can be expressed via rate function

$$I = \int_0^\infty \sup_{\theta<\theta_+} \{\theta\dot{x}(t) - \lambda[\varphi(\theta) - 1]\} dt,$$

For $x(t)$ that is a trajectory of flow f we call $\dot{x}$ the *flow speed*.

One server.

For "optimal"trajectory $\dot{x}(t) = a =$const,

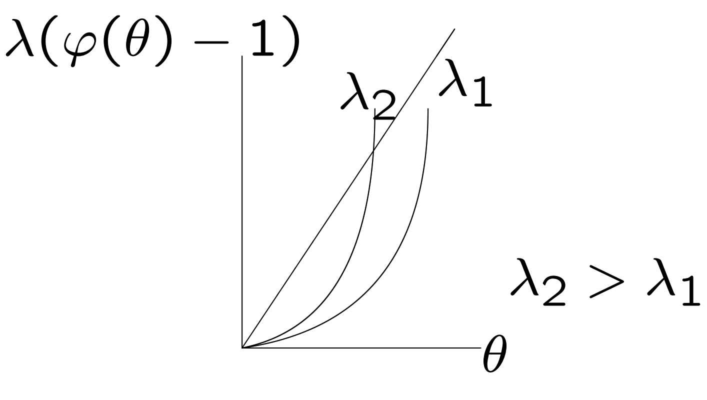
$\lim_{n\to\infty} \frac{-1}{n} \ln \Pr[\omega > nd] = Id$,

$I = t \sup_{\theta<\theta_+}\{\theta a - \lambda[\varphi(\theta) - 1]\}$

Optimization for a server of speed m

gives $I = m\theta$ where

$m\theta = \lambda[\varphi(\theta) - 1]$.



System with k servers

For any k, $k \geq 3$ there exist

$$\lambda_{(k)},\ \lambda^{(k)},\ 0 < \lambda_{(k)} \leq \lambda^{(k)} < \widehat{\lambda},$$

that depend on ξ^{A_j} distribution such that

▶ If $\lambda < \lambda_{(k)}$, then the event Γ_1 is mainly defined by f_{A_1}, only this flow is overheated

$$\lim_{n\to\infty} \frac{-1}{n} \ln \Pr\left[\Gamma_1(n)\right] = m\theta_m d,$$

where θ_m is a positive root to equation

$$m\theta = \lambda[\varphi(\theta) - 1].$$

▶ If $\lambda > \lambda^{(k)}$, then all flows are overheated

$$\lim_{n\to\infty} \frac{-1}{n} \ln \Pr\left[\Gamma_1(n)\right] = k\theta_k d,$$

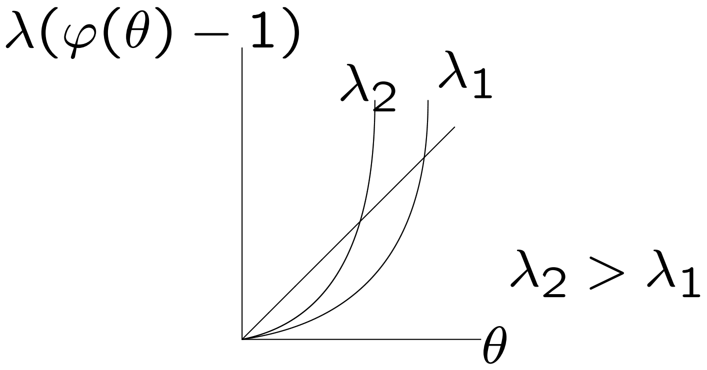
where θ_k is a positive root to equation

$$k\theta = \lambda\binom{k}{m}[\varphi(\theta) - 1].$$

The last statement is easy to explain formaly. Let

$$m\theta_m = \lambda[\varphi(\theta) - 1],$$
$$k\theta_k = \lambda\binom{k}{m}[\varphi(\theta) - 1].$$

It easy to see that
$m\theta_m < k\theta_k$ if λ is sufficiently small,
$m\theta_m > k\theta_k$ if λ is sufficiently close to $\widehat{\lambda} = k\left(l\varphi'(0)\right)^{-1}$.



$$\theta = \lambda[\varphi(\theta) - 1].$$

Consider an auxiliary system $\mathcal{S}^{(0)}$ with similar k servers and l flows. The realization of flows in both systems are identical. At $\mathcal{S}^{(0)}$ the routing is random: a message of flow f_{A_j} with given probability $\alpha(j,r)$ is directed to the server $s_r,\ s_r \in S_{A_j}$. The flows of $\mathcal{S}^{(0)}$ upon the servers are independent and Poisson.

Suppose the flows F have the speeds $a_{A_1}, ..., a_{A_l}$. We call these flows *balanced* with respect to servers S if for any j,r there exist such $\alpha(j,r)$ and such $b > 0$ that

$$\sum_j \alpha(j,r) a_{A_j} = \frac{\sum_{j=1}^{l} a_{A_j}}{k} = b.$$

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Now we compare the performance of systems $\mathcal{S}$ and system $\mathcal{S}^{(0)}$.

Let in both systems m', $m \leq m' \leq k$ server be equally overloaded.

In $\mathcal{S}^0$ the flows upon m' servers are balanced. For auxiliary system the difference $|w_i(t) - w_j(t)| = o\,(t)$.

It can be shown that the components of load vector in main system are even more concentrated in the neighborhood of bisectrix $w_1 = ... = w_{m'}$ than similar components in the auxiliary system.

Therefore the probability of events $\Gamma_1(n)$ in the systems are equal in case where k' server are equally overloaded.

It can be shown that if the event $\Gamma_1(n)$ takes place (and may be some other servers are also overloaded) the probability of this event is not greater than the probability of similar event where several servers are equally overloaded and the overheated flows have equal speeds.

Therefore it is sufficient to consider the equal overload of m', $k \geq m' \geq m$ servers caused by the equal overheat of several flows.

The formulas for probability of $\Gamma_1(n)$ presented above are the formulas for probability of equally overheated flows in auxiliary system

Comparison of large delay probability for different values of m' brings the statement and formulas presented above.

Example

The case of exponentially distributed ξ length, $\Pr[\xi \geq x] = e^{-x}$:

$$\lambda_{(k)} = \lambda^{(k)} = \frac{(k-m)}{\binom{k}{m} - 1},$$

$\lambda_{(k)} = \lambda^{(k)} \to \widehat{\lambda}$ as k increases.

Constant message length, circle system.

$\Pr(\xi = x) = \delta$.

$\varphi(\theta) = e^{\theta}$, $\lambda < 1$.

a) As $3 \leq k \leq 12$ $\lambda_k = \lambda^{(k)}$.

For example $\lambda_k \sim 0.311$ as $k = 3$;

$\lambda_k \sim 0.667$ as $k = 5$.

b) As $k > 12$: $\lambda_k \sim 0.888$;

$\lambda^{(k)} \sim 0.910$ as $k = 15$