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Example Sheet 2
Advanced Matrix Manipulations

Using examples from the lectures, try the following problems. They are not compulsory. If you like any help or comments, please email *your code and question* to Polina Vytnova (P.Vytnova@warwick.ac.uk). I am happy to help when you get stuck and nothing comes to mind after some thinking.

Q0. First of all, set the output format to long: run `format long`.

Q1. Solve the linear equation $Ax = b$, for a 30-by-30 matrix A given by

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{pmatrix}$$

and a vector $b \in \mathbb{R}^{30}$ with coefficients $b_i = \cos\left(\frac{\pi(i-1)}{31}\right) - \cos\left(\frac{\pi(i+1)}{31}\right)$. Plot the coordinates of the vector x versus the coordinate number: (i, x_i).

Q2. Find the determinant of the $n \times n$ matrix

$$F_n = \begin{pmatrix} 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

for $n = 3, 4, 5, 6$. Can you suggest a general formula for $\det(F_n)$ as a function of n?

**Prove your conjecture rigorously.

Q3. The Hilbert H_n matrix is a $n \times n$ matrix with elements $h_{ij} = 1/(i + j - 1)$. Find the inverse of the Hilbert H_5 and H_8 matrices and make sure that every element of the inverse matrix is an integer, and check that the sum of elements of the inverse matrix is n^2 in each case. Then compare the sum of elements of the matrices H_{10} and H_{1000} with 100 and 10^6, respectively. Make an observation.

Q4. The Vandermonde matrix is

$$V_n(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \ldots & x_{n-2} & x_{n-1} & x_n \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & \ldots & x_{n-2}^2 & x_{n-1}^2 & x_n^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & \ldots & x_{n-2}^3 & x_{n-1}^3 & x_n^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & x_4^{n-1} & \ldots & x_{n-2}^{n-1} & x_{n-1}^{n-1} & x_n^{n-1} \\ x_1^n & x_2^n & x_3^n & x_4^n & \ldots & x_{n-2}^n & x_{n-1}^n & x_n^n \end{pmatrix}$$

Calculate the determinant of the following Vandermonde matrices $V_3(1, 2, 3)$, $V_4(1, 2, 3, 4)$, $V_4(2, 3, 4, 5)$, $V_5(1, 2, 3, 4, 5)$. State a conjecture about the determinant $\det(V_n(x_1, x_2, \ldots, x_n))$.

**Prove your conjecture rigorously.

Q5. The Combinatorial matrix is a $n \times n$ matrix given by

$$C_n(x, y) = \begin{pmatrix} x+y & y & y & \ldots & y & y \\ y & x+y & y & \ldots & y & y \\ y & y & x+y & \ldots & y & y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y & y & y & \ldots & x+y & y \\ y & y & y & \ldots & y & x+y \end{pmatrix}$$

Calculate the determinant of a few combinatorial matrices, suggest a general formula for the determinant, and prove your conjecture rigorously.