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Useful Calculus Formulae

Using examples from the lectures, try the following problems. They are not compulsory. If you like any help or comments, please email *your code and question* to Polina Vytnova (P.Vytnova@warwick.ac.uk). I am happy to help when you get stuck and nothing comes to mind after some thinking.

Q0. First of all, set the output format to long: run `format long`.

Q1. Put yourself in Euler's place, looking for a way to extend $n!$ to non-integer values of n. Observe that

$$\frac{(n+\frac{1}{2})!}{n!}\cdot\frac{((n+\frac{1}{2})+\frac{1}{2})!}{(n+\frac{1}{2})!}=\frac{(n+1)!}{n!}=n+1,$$

therefore $\frac{(n+\frac{1}{2})!}{n!}\approx\sqrt{n}$. Using a similar identity check that $\frac{(n+\frac{1}{3})!}{n!}\approx\sqrt[3]{n}$. Make a conjecture about the ratio $\frac{(n+x)!}{n!}$ for large $n\gg 1$. Is your conjecture correct for integer x? Does it say anything about the value of $x!$ for non-integer x?

Q2. Prove rigorously the following inequality for integer $n\geq 1$:

$$\frac{n^n}{e^{n-1}}\leq n!\leq\frac{n^{n+1}}{e^{n-1}}.$$

(Hint: for any $x\in\mathbb{R}$ we have $1+x<e^x$.) Using `Matlab` find the smallest n such that

$$\frac{1}{n!}\cdot\Big(\frac{n^n\cdot(n-1)}{e^{n-1}}\Big)\leq 10^{-5}$$

How accurate is the approximation for $n!$ given by the Stirling formula?

Q3. Consider the harmonic sums $H_k\colon=\sum\limits_{j=1}^{k}\frac{1}{j}$. Calculate the difference

$$f(n)=\sum_{k=1}^{n}H_k-\sum_{k=1}^{n}\ln k$$

What can you say about the ratio $\frac{f(n)}{n}$ for large $n\gg 1$?

Q4. Monte Carlo numerical integration method suggests to take n random numbers (use `rand` function to generate random vectors) from the interval $[a;b]$ and calculate the limit

$$I_n=\lim_{n\to\infty}\frac{b-a}{n}\sum_{j=1}^{n}f(x_j)$$

Let $f(x)=\sqrt{4-x^2}$. Calculate I_n for $n=10,\ldots,100$ and compare it with the real value, for example, plot the difference $\delta_n:=\int_{-2}^{2}f(x)\mathrm{d}x-I_n$ against n. Does there exist N such that for $n>N$ the difference δ_n is less than 10^{-4}? 10^{-5}? 10^{-6}? Make an observation.

Q5. How many lineary independent vectors in n-dimensional space with k entries equal to 1 are there? Plot a graph (quantity versus k) for $n=50$.