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MA222

Example Sheet 1
Getting to know `Metric Spaces`

Hand in solutions to the Problems P7 and P9. Deadline: 2pm, Thursday 24th of January. By the unit ball we understand a ball centred at the origin with radius 1.

P1. Let's assume that $0 < p < 1$, let $\|\cdot\|$ stand for the Euclidean distance. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points. Consider functions $f_k \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

(1) $f_1(P_1, P_2) = \sin^2(|x_1 - x_2|) + \sin^2(|y_1 - y_2|)$, (2) $f_2(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

(3) $f_3(P_1, P_2) = \begin{cases} \|P_1 - P_2\|, & \text{if } P_1 = \alpha P_2 \text{ for some } \alpha \in \mathbb{R} \\ \|P_1\| + \|P_2\|, & \text{otherwise.} \end{cases}$

(4) $f_4(P_1, P_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{otherwise.} \end{cases}$

(5) $f_5(P_1, P_2) = |x_1 - x_2|^p + |y_1 - y_2|^p$, (6) $f_6(P_1, P_2) = \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p}$.

(a) Determine which of the f_k define a metric on $\mathbb{R}^2$.

(b) For every metric sketch the unit ball and another two balls of your choice. Determine whether or not they are convex.

(c)* For each pair of metrics f_k and f_j, decide whether there exist constants $A > 0$ and B such that for any $P_1, P_2 \in \mathbb{R}^2$

$$A \cdot f_k(P_1, P_2) \leq f_j(P_1, P_2) \leq B \cdot f_k(P_1, P_2).$$

P2. Let $\Sigma = \{0, 1\}^{\mathbb{N}}$ be the set of sequences of zeros and ones. Define $d \colon \Sigma \times \Sigma \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 2^{-n}, & \text{if } x_j = y_j \text{ for all } 0 \leq j < n \text{ and } x_n \neq y_n, \\ 0, & \text{if } x = y. \end{cases}$$

(a) Show that d is a metric on Σ. It is called *the Hausdorff metric* on the space of sequences.

(b) Describe the open unit ball and the closed ball of radius $\frac{1}{10}$ centred at $x_j = 1$ $\forall j$.

P3. Let p be a prime number. Define $d \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d_p(x, y) = \begin{cases} p^{-k}, & \text{if } p^k \mid |x - y| \text{ and } p^{k+1} \nmid |x - y| \text{ for some } k \in \mathbb{N}, \\ 0, & \text{if } x = y. \end{cases}$$

(a) Show that d is a metric on $\mathbb{Z}$. It is called *a p-adic metric* on $\mathbb{Z}$.

(b) Describe the open and closed unit balls and the open ball of radius p^{-2} centred at 1.

(c) In the case $p = 2$ find a connection between the Hausdorff metric on $\{0, 1\}^{\mathbb{N}}$ and d_2.

P4. Let E be a finite set and let $\mathcal{E}$ be the set of subsets of E.

(a) Show that $d(A, B) = \mathrm{Card}(A \triangle B)$ is a metric on $\mathcal{E}$. It is called *the Hamming distance.*

(b) Describe the open and closed unit balls centred at the empty set $A = \varnothing$.

P5. Let (M, d) be a metric space. Which of the following functions define a metric on M?

(1) $\rho_1(x, y) = \min(1, d(x, y))$; (2) $\rho_2(x, y) = d(x, y) + |f(x) - f(y)|$ where $f\colon M \to \mathbb{R}$;
(3) $\rho_3(x, y) = \rho_1(x, y) - \rho_2(x, y)$; (4) $\rho_4(x, y) = \alpha\rho_1(x, y) + (1 - \alpha)\rho_2(x, y)$, $0 < \alpha < 1$;

(5) $\rho_5(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$.

P6. Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces. Show that the following expressions define a metric on $M_1 \times M_2$.

(a) $d_1\left((x_1, y_1), (x_2, y_2)\right) = \rho_1(x_1, x_2) + \rho_2(y_1, y_2)$

(b) $d_1\left((x_1, y_1), (x_2, y_2)\right) = \max(\rho_1(x_1, x_2), \rho_2(y_1, y_2))$

$\boxed{\textbf{P7.}}$ Determine whether or not the following subsets are open or closed.

(1) $\mathbb{Q} \subset \mathbb{R}$, with respect to the discrete metric,
(2) $\{(x, y) \in \mathbb{R}^2 \mid xy < 7\} \subset \mathbb{R}^2$, with respect to the Manhattan metric,
(3) $\left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \dfrac{e^{x-y^2+z^3}}{2 + x - y - z} > 1 \right\} \subset \mathbb{R}^3$ with respect to the Euclidean metric.

P8. Let ρ be a metric on the product space $M_1 \times M_2$ which satisfies $\rho\left((x_1, y_1), (x_2, y_2)\right) = \rho_1(x_1, x_2) + \rho_2(y_1, y_2)$ Is it true that ρ_1 defines a metric on M_1?

$\boxed{\textbf{P9.}}$ Establish the following facts.

(a) A subset of a metric space is open if and only if it is a union of open balls.

(b) A closed ball is a closed set (directly from definition).

P10. Let ρ be the Euclidean metric on $\mathbb{R}$ and let d be the discrete metric on $\mathbb{R}$. Is the identity map $(\mathbb{R}, \rho) \to (\mathbb{R}, d)$ continuous? Is its inverse continuous?

P11. Define a metric d on $\mathbb{Q}$ such that the induced metric on $\mathbb{Z} \subset \mathbb{Q}$ agrees with the p-adic metric.

P12. Let M be a linear space over $\mathbb{C}$ and let (M, d) be a metric space. Is it true that $(M, (d(\cdot, 0))$ is a normed linear space?