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MA222

Example Sheet 4
Continuity, Metrizability, and Hausdorff property

Hand in solutions to the Problems P9 and P10. Deadline: 2pm, Thursday 14th of February. We consider the space $\mathbb{R}^n$ with Euclidean topology, unless stated otherwise.

Problems P11–P14 are for *independent practice.*

P1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be a pair of topological spaces.

1. Show that $f : X \to Y$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$.
2. If $f : X \to Y$ is continuous, is it true that $f(\bar{A}) = \overline{f(A)}$?

P2. Consider the space of sequences of real numbers $\ell(\mathbb{R})$ as a countable infinite product of copies of $\mathbb{R}$. Let $A\colon = \{\{x_j\}_{j=1}^{\infty} \in \ell(\mathbb{R}) \mid \exists N \geq 1\ \forall j \geq N\ x_j = 0\}$. Find the closure of A in the product topology.

P3. Let X be infinite (for a specific example, take $X = \mathbb{Z}$ or $X = \mathbb{R}$). We say that $E \subset X$ lies in $\mathcal{T}_X$ if either $E = \varnothing$ or $X \setminus E$ is finite. Show that $\mathcal{T}_X$ is a topology and that every point set $\{x\}$ is closed, but that $(X, \mathcal{T}_X)$ is not Hausdorff. What happens if X is finite?

P4. Let $(X, \mathcal{T}_X)$ be a topological space and let $(Y, \mathcal{T}_Y)$ be a Hausdorff topological space. Show that for any pair of continuous functions $f, g\colon X \to Y$ the set $\{x \in X \mid f(x) = g(x)\}$ is closed.

P5. Let H_1 and H_2 be collections of subsets of X_1 and X_2, respectively. Let $\mathcal{T}_j$ be the smallest topology on X_j containing H_j, $j = 1, 2$. Show that if a function $f\colon X_1 \to X_2$ has the property that $f^{-1}(H) \in H_1$ for any $H \in H_2$ then $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous.

P6. Consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} 0, & \text{if } x = y = 0 \\ \frac{xy}{x^2+y^2}, & \text{otherwise.} \end{cases}$$

1. Show that for any $x \in \mathbb{R}$ the function $h_x(y) = f(x, y)$ is continuous.
2. Show that for any $y \in \mathbb{R}$ the function $g_y(x) = f(x, y)$ is continuous.
3. Show, however, that f is not continuous.

P7. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and consider $X \times Y$ with the product topology. Let $(Z, \mathcal{T}_Z)$ be a topological space. Show that the map $f\colon Z \to X \times Y$ is continuous if and only if $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f\colon Z \to Y$ are continuous.

P8. Let $(X, \mathcal{T})$ be a topological space and assume that $\mathcal{T}$ is derived from a metric. Show that, for any given $x \in X$, there exists open sets U_j such that $\{x\} = \cap_{j=1}^{\infty} U_j$.

P9. Consider the space of functions $f\colon [0,1] \to \mathbb{R}$. Define a collection of subsets $\mathcal{T}$ as follows. We say that $U \in \mathcal{T}$ if and only if for any $f_0 \in U$, there exists an $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n \in [0,1]$ such that

$$\{f\colon [0,1] \to \mathbb{R} \mid |f(x_j) - f_0(x_j)| < \varepsilon \text{ for } 1 \le j \le n\} \subseteq U.$$

1. Show that $\mathcal{T}$ is a topology.
2. Show that the topology $\mathcal{T}$ is Hausdorff but cannot be derived from a metric.

P10. Consider $\mathbb{R}$ with the Euclidean topology. Let $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Show that $\sim$ is an equivalence relation. Show that $\mathbb{R}/\sim$ uncountable and that the quotient topology on $\mathbb{R}/\sim$ is the indiscrete topology. *Hint: show that for any interval* (a,b) *we have that* $\cup_{x \in (a,b)} \{x + q \mid q \in \mathbb{Q}\} = \mathbb{R}$.

☞ **P11.** Sierpinski space is a topological space $S\colon = (\{0,1\}, \mathcal{T})$ where $\mathcal{T} = \{\varnothing, \{1\}, \{0,1\}\}$. Describe all continuous maps $f\colon S \to S$.

☞ **P12.** Let $(X, \mathcal{T})$ be a topological space.

1. Find a set $A \subset \mathbb{R}$ such that A, $\overline{A}$, $\mathrm{Int}(\overline{A})$, and $\overline{\mathrm{Int}(\overline{A})}$ are all distinct.
2. Show that for any $A \subset X$ we have that $\mathrm{Int}\left(\overline{\mathrm{Int}\left(\overline{A}\right)}\right) = \mathrm{Int}\left(\overline{A}\right)$.
3. Deduce that, starting from a set $A \subset X$, the operations of taking interior and closure in various orders can produce at most seven different sets (including A itself).
4. Find a set $A \subset \mathbb{R}$ with the standard topology such that the operations of taking closures and interiors in various orders produce exactly seven different sets.

☞ **P13.** Consider $E = \{(x,-1) \mid x \in \mathbb{R}\} \cup \{(x,1) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ with the subspace topology. Define a relation $\sim$ on E by

$$\begin{aligned} (x,y) &\sim (x,y) && \text{for all } (x,y) \in E, \\ (x,y) &\sim (x,-y) && \text{for all } (x,y) \in E \text{ with } x \neq 0. \end{aligned}$$

1. Show that that $\sim$ is an equivalence relation on E.
2. Consider with $E/\sim$ the quotient topology. Show for any $[(x,y)] \in E/\sim$ there exists an open neighbourhood U of $[(x,y)]$ which is homeomorphic to $\mathbb{R}$.
3. Show that $E/\sim$ is not Hausdorff.

☞ **P14.** Let $(Y, \mathcal{T}_Y)$ be a topological space. Show that the following are equivalent.

1. $(Y, \mathcal{T}_Y)$ is Hausdorff.
2. The diagonal $\{(y, y) \mid y \in Y\} \subset Y \times Y$ is closed with respect to the product topology.
3. For any topological space $(X, \mathcal{T}_X)$ and a pair of continuous functions $f, g \colon X \to Y$ the set $\{x \in X \mid f(x) = g(x)\}$ is closed.