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MA222

Example Sheet 6
Compactness and Uniform Continuity

Hand in solutions to the Problems P7, P10 and P11. Deadline: 2pm, Thursday 28th of February. We consider the space $\mathbb{R}^n$ with Euclidean topology, unless stated otherwise.

Problems P12–P15 are for *independent practice.*

P1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on the same space X. Establish the following facts.

1. The identity map $\mathrm{Id}: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_2 \subset \mathcal{T}_1$.
2. If $\mathcal{T}_1 \subset \mathcal{T}_2$ and $(X, \mathcal{T}_2)$ is compact, then so is $(X, \mathcal{T}_1)$.
3. If $\mathcal{T}_1 \subset \mathcal{T}_2$ and $(X, \mathcal{T}_1)$ is Hausdorff, then so is $(X, \mathcal{T}_2)$.
4. If $\mathcal{T}_1 \subset \mathcal{T}_2$, $(X, \mathcal{T}_2)$ is compact and $(X, \mathcal{T}_1)$ is Hausdorff, then $\mathcal{T}_1 = \mathcal{T}_2$.

P2. Let $(X, \mathcal{T}_X)$ be a compact topological space and $\sim$ be an equivalence relation on X. Show that the quotient topology on $X/\sim$ is compact.

P3. Find an example to demonstrate the following facts.

1. There exists a Hausdorff space $(X, \mathcal{T}_X)$, a compact Hausdorff space $(Y, \mathcal{T}_Y)$ and a continuous bijection $f: X \to Y$ which is not a homeomorphism.
2. There exists a compact Hausdorff space $(X, \mathcal{T}_X)$ a compact space $(Y, \mathcal{T}_Y)$ and a continuous bijection $f: X \to Y$ which is not a homeomorphism.

P4. Show that the set of integers with p-adic metric is a bounded metric space but there are infinite sequences which don't have a convergent subsequence.

P5. Decide whether or not a closed unit ball is compact in the following spaces.

1. The space of sequences $\{0,1\}^{\mathbb{N}}$ with the Hausdorff distance.
2. The space of bounded sequences $\ell_\infty(\mathbb{C})$.
3. The space of summable sequences $\ell_1(\mathbb{C})$.
4. The space of subsets of a finite set with the Hamming distance.
5. The space of continuous functions $C([0,1])$ with the maximum norm.

P6. Let $X = (0,1)$ and $G_n = (\frac{1}{n}, 1)$. Does a Lebesgue number exist for the open cover $\bigcup_{n=1}^{\infty} G_n$ of X?

P7. Give an example of a uniformly continuous function on $\mathbb{R}$ differentiable everywhere save a finite set and with unbounded derivative.

P8. Describe all uniformly continuous functions (1) $f\colon \mathbb{Z} \to \mathbb{R}$ and (2) $g\colon \mathbb{R} \to \mathbb{Z}$

P9. Consider a sequence in a compact metric space $\{x_n\}_{n=1}^{\infty} \subset X$ and assume that there exixts a unique point $\tilde{x}$ such that any neighbourhood of $\tilde{x}$ contains x_n for infinitely many n. Show that $\lim_{n\to\infty} x_n$ exists.

P10. Show that the image of a sequentially compact metric space under a continuous map is sequentially compact.

P11. Show directly from the definition that a product of countably many sequentially compact spaces is a sequentially compact space. (Hint: first show that a product of two (or any finite number) of sequentially compact spaces is sequentially compact).

☞ **P12.** Show that a space of continuous bounded functions $\mathbb{N} \to \mathbb{R}$ with the topology of pointwise convergence is not sequentially compact. Show that the space of all sequences $\ell([0,1]) = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in [0,1]\}$ is compact in the topology of pointwise convergence.

☞ **P13.** Consider a topology $\mathcal{T}$ on $\mathbb{R}$ consists of all sets of the form $U \cup S$ where U is an open set for the usual Euclidean topology and $S \subset \mathbb{R} \setminus \mathbb{Q}$.

1. Show that $\mathcal{T}$ is a topology. (It is called the "scattered topology".)
2. Show that $\mathcal{T}$ is Hausdorff.
3. Show that a one-point set $\{x\}$ is *open and compact* if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$.
4. Is $(\mathbb{R}, \mathcal{T})$ a normal topological space?

☞ **P14.** Let X be a compact topological space and let Y be Hausdorff topolgical space. Denote by $\sim$ an equivalence relation on X and assume that $f\colon X/{\sim} \to Y$ is a bijection. Let $\pi\colon X \to X/{\sim}$ be a projection $\pi(x) = [x]$.

1. Show that if $f \circ \pi\colon X \to Y$ is continuous then f is a homeomorphism.
2. Let D be a closed ball in $\mathbb{R}^2$ and let $S = \partial D$ be its boundary. Show that D/S is homeomorphic to a sphere (the boundary of an open ball) in $\mathbb{R}^3$.

☞ **P15.** Let $C \subset \mathbb{R}$ be the Cantor set. Show that the set $C \times \overline{[0,1] \setminus C} \subset \mathbb{R}^2$ is compact.