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MA222

Example Sheet 7
Connectedness in Topological Spaces

Hand in solutions to the Problems P4, P9 and P10. Deadline: 2pm, Thursday 7th of March. We consider the space $\mathbb{R}^n$ with Euclidean topology, unless stated otherwise.

Problems P5, P12, P13, and P14 are for *independent practice*.

P1. Establish the following facts.

1. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, $E \subset X$ is a connected subset and $g\colon E \to Y$ is continuous, then $g(E)$ is connected. (More briefly, the continuous image of a connected set is connected.)
2. If $(X, \mathcal{T}_X)$ is a connected topological space and $\sim$ is an equivalence relation on X, then $X/{\sim}$ with the quotient topology is connected.
3. If $A \subset (X, \mathcal{T}_X) \times (Y, \mathcal{T}_Y)$ with the product topology is connected, then the projection $\pi_X(A) \subset (X, \mathcal{T}_X)$ is also connected.
4. If $(X, \mathcal{T}_X)$ is a connected topological space and $E \subset X$, then E with the subspace topology may not be connected.

P2. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. Does any of the following hold true?

1. If $(X, \mathcal{T}_1)$ is connected then $(X, \mathcal{T}_2)$ is connected.
2. If $(X, \mathcal{T}_2)$ is connected then $(X, \mathcal{T}_1)$ is connected.

P3. Deduce the Intermediate Value Theorem from the Heine–Borel Theorem. Let $\overline{B(0,1)} \subset \mathbb{R}^n$ be the closed unit ball. Suppose $g\colon \overline{B(0,1)} \to \mathbb{R}$ is a continuous function with $g(x) < c < g(y)$ for some $x, y \in \overline{B(0,1)}$. Consider $g^{-1}((-\infty, c))$ and $g^{-1}((c, +\infty))$ and show that there exists a $t \in \overline{B(0,1)}$ with $g(t) = c$.

P4. Let (X, d) be a metric space. Suppose that $E_1 \supset E_2 \supset \ldots \supset E_n \ldots$ are connected sets. Show that, if the E_j are compact, then $\bigcap_{j=1}^{\infty} E_j$ is compact and connected.

☞ **P5.** Does there exist a non-empty connected bounded set $X \subset \mathbb{R}^2$ such that for any $\alpha, \beta \subset \mathbb{R}$ the intersection $X \cap \{(x, y) \in \mathbb{R}^2 \mid \alpha x + \beta = y\}$ is either empty or disconnected?

P6. Assuming A contains at least two points, show that a topological space $(X, \mathcal{T}_X)$ is connected if and only if every continuous function $f\colon (X, \mathcal{T}_X) \to (A,$ discrete topology$)$ is constant.

P7. Let X, Y be topological spaces and $y \in Y$. Show that the map $f(x) = (x, y)$ from X to $X \times Y$ is a homeomorphism of X onto its image (i.e., onto $X \times \{y\}$). Deduce that if X is connected then $X \times \{y\}$ is connected for every y.

P8. Which of the following subsets of $\mathbb{R}^2$ is connected?

$$(1)\ \{(x,y) \mid x \in \mathbb{Q} \text{ or } y = 0\}, \qquad (2)\ \{(x,y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$$
$$(3)\ \{(x,y) \mid (x+1)^2 + y^2 < 1 \text{ or } (x-1)^2 + y^2 \leq 1\}.$$

P9. Let $X \subset \mathbb{R}^n$ be an n-dimensional[1] connected set, $n \geq 2$.

1. Let z be a point in the interior of X. Show that $X \setminus \{z\}$ is connected.
2. Let $\overline{B(x,r)} \subset V \subset \text{Int}(X)$ be a closed ball, where $V \subset \mathbb{R}^k$ is a k-dimensional subset for $k \leq n$. Show that $X \setminus \overline{B(x,r)}$ is connected.

P10. Let C be a subset of a topological space such that $\overline{C}$ is connected. Is it necessarily true that C is connected?

P11. Let $(X, \mathcal{T}_X)$ be a topological space. For any subset $A \subset X$ we consider $\overline{A}$ with respect to $\mathcal{T}_X$

1. Show that X is disconnected if and only if there are nonempty sets $A, B \subset X$ such that $X = A \cup B$ and $\overline{A} \cap \overline{B} = \varnothing$.
2. Give an example of a disconnected subspace $Y \subset X$ for which there are no nonempty sets $A, B \subset Y$ such that $Y = A \cup B$ and $\overline{A} \cap \overline{B} = \varnothing$.
3. Show that a subspace $Y \subset X$ is disconnected if and only if there are nonempty sets $A, B \subset Y$ such that $Y = A \cup B$ and $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

☞ **P12.** Show that there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

☞ **P13.** Divide the unit ball $B(0,1) \subset \mathbb{R}^n$ into two disjoint everywhere dense connected sets, considering the metrics induced by the following norms.

1. $n = 2$ and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.
2. $n = 2$ and with the Euclidean norm.
3. $n = 3$ with $\|x\|_1 = |x_1| + |x_2| + |x_3|$.

P14. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the graph of f' is connected. The first student sending me a correct solution to this problem will receive a chocolate bar.

[1] We say that a subset of $\mathbb{R}^n$ is n-dimensional, if it is not contained in any k-dimensional subset for $k < n$.