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MA448
Example Sheet 2
Classification of Isometries

In what follows, FP_j stands for the set of fixed points of $T_j \in \mathrm{Aut}(\hat{\mathbb{C}})$; Id stands for the identity transformation and I for its matrix.

Given a circle centred at O with radius r the inverse of a point P with respect to the circle is the unique point P' on the ray OP such that $|OP| \cdot |OP'| = r^2$.

For any group G the commutator of $g, h \in G$ is $[g, h] = ghg^{-1}h^{-1} \in G$.

P1. Show that

1. $T_1, T_2 \in \mathrm{Aut}(\hat{\mathbb{C}})$ have a common fixed point if and only if $\mathrm{Tr}[T_1, T_2] = 2$.
2. If $T_1, T_2 \in \mathrm{Aut}(\hat{\mathbb{C}})$ have a common fixed point in $\hat{\mathbb{C}}$ then either
   (a) $[T_1, T_2] = Id$ and $FP_1 = FP_2$;
   (b) $[T_1, T_2]$ is parabolic, and $FP_1 \neq FP_2$.

P2. Let $T_1, T_2 \in \mathrm{Aut}(\hat{\mathbb{C}})$ be different from identity. Show that the following are equivalent

1. $T_1 T_2 = T_2 T_1$
2. $T_1(FP_2) = FP_2$ and $T_2(FP_1) = FP_1$.

P3. Let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ be parabolic with fixed point w. Show that for all $z \in \hat{\mathbb{C}}$ we have $T^n(z) \to w$ as $n \to \infty$ uniformly on compact subsets of $\hat{\mathbb{C}}$.

P4. Let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ be loxodromic with fixed point w_1 and w_2. Show that $T^n(z) \to w_1$ as $n \to \infty$ for all $z \neq w_2$ uniformly on compact subsets of $\hat{\mathbb{C}} \setminus \{w_2\}$ (up to the labeling of w_1 and w_2).

P5. Let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ be elliptic with fixed points $w_{1,2}$. Prove that any circle for which w_1 and w_2 are the inverse points (to each other) is invariant with respect to T.

P6. Show that if $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ is such that $T^k = Id$ for some $k > 1$ then T is elliptic and $\mathrm{Tr}^2(T) \leq 4\cos^2(\pi/k)$. Prove that the inequality becomes an equality if and only if T is the rotation by angle $\pm 2\pi/k$.

P7. Find two transformations $T_1, T_2 \in \mathrm{Aut}(\hat{\mathbb{C}})$ such that $\mathrm{Tr}[T_1, T_2] = -2$ and T_1 and T_2 have no common fixed points in $\hat{\mathbb{C}}$.

P8. Let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ be such that $T(\infty) \neq \infty$. Show that $T = T_1 T_2 T_3$, where T_1 and T_3 are parabolic transformations fixing ∞ and $T_2^2 = Id$.

P9. An nth root of an automorhism of $\mathbb{C}$ is any $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ such that $T^n = Id$. Show that

1. If $T = Id$ then T has infinitely many nth roots.
2. If T is parabolic then T has a unique nth root.
3. In all other cases, T has exactly n nth roots.

P10. Show that if $A, B \in SL(2, \mathbb{C})$ then (1) $\det(A - I) = 2 - \mathrm{Tr}(A)$ and $\det(AB - BA) = 2 - \mathrm{Tr}[A, B]$. Deduce that if Möbius transformations given by A and B don't have a common fixed point in $\hat{\mathbb{C}}$, then $(AB - BA)^2 = I$.

P11. Let T be a Möbius transformation. Show that if $T(z) \neq z$ then the cross-ratio $[z, Tz, T^2z, T^3z]$ is independent on z and compute it in terms of $\mathrm{Tr}^2\, T$.

P12. Let us define the norm on the space of matrices by $\| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \| = (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{1/2}$. Let $T_1, T_2, \ldots$ be Möbius transformations and suppose that $g_n(w) \to w$ for $w = 0, 1, \infty$. Prove that

1. There exist matrices A_n representing T_n which converge to I;
2. $T_n \to I$ uniformly in $\hat{\mathbb{C}}$.

P13. Show that a sequence T_n of loxodromic transformations can converge to an elliptic element (in the topology induced by the norm defined in P12) and if this is so then T_n is strictly loxodromic for almost all n.

P14. Show that a sequence of elliptic elements cannot converge to a loxodromic element (in the topology induced by the norm defined in P12).