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MA4K3

Example Sheet 2
Properties of Hardy Space Functions

Hand in solutions to at least five P-Problems of your choice. The mark will be a sum of the best five Solutions.
Deadline: 2pm, Tuesday 14th of November.

Notation. $\mathbb{D}$ stands for the open unit disk in $\mathbb{C}$. $\mathcal{O}(\mathbb{D})$ stands for holomorphic functions on $\mathbb{D}$. $\mathbb{H}^2$ stands for the Hardy space of functions $f \in \mathcal{O}(\mathbb{D})$. For any $f \in \mathcal{O}(\mathbb{D})$ a shorthand notation $f \rightsquigarrow \{a_k\}_{k=0}^{\infty}$ stands for $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$. Given $z = x + iy \in \mathbb{C}$, we denote $\bar{z} \colon = x - iy$. Given a Hilbert space H, $\mathcal{M}_H$ stands for the algebra of multipliers in H.

E1. Show that if H is a reproducing kernel Hilbert space (=: RKHS) on $\mathbb{C}$, with reproducing kernel $K(x, y)$, then $K(y, x) = \overline{K(x, y)}$.

E2. Show that if H is a RKHS and $H_0 \subset H$ is a closed subspace, then H_0 is also a RKHS. Prove that the reproducing kernel for H_0 for a point y is the function $P(k_y)$ where k_y is the reproducing kernel function for H and $P \colon H \to H_0$ denotes the orthogonal projection of H onto H_0 .

E3. Define inner product in the Dirichlet space $\mathcal{D}$ that is compatible with the norm (i.e. $\|f\|_2^2 = \langle f, f \rangle$ for any $f \in \mathcal{D}$). Show that the reproducing kernel of the Dirichlet space is given by

$$k_\omega(z) = \sum_{k=0}^{\infty} \frac{\bar{\omega}^k z^k}{k+1}.$$

E4. Establish that $\mathcal{M}_{\mathbb{H}^2} \subset \mathcal{M}_{\mathcal{A}}$.

E5. Show that if f and $\frac{1}{f}$ belong to the Hardy space, then f is an outer function.

E6. Find the Blaschke part of the following functions, show that their singular factors are trivial and find their outer parts.

$$(1)\ \frac{2z^2 + 5z + 3}{2 - z} \qquad (2)\ z^5 - \frac{1}{32}$$

P1. Let $\mathcal{H}_N \subset \mathbb{H}^2(\mathbb{D})$ denote the subspace consisting of all functions of the form $f(z) = \sum_{n=N}^{\infty} a_n z^n$. Find the reproducing kernel for $\mathcal{H}_N$.

P2. Define inner product compatible with the norm and find reproducing kernels in

1. the Bergman space $\mathcal{A}$;
2. a weighted Hardy space $\mathbb{H}^2_\omega$ for some $\{\omega_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \frac{\omega_k}{\omega_{k+1}} = 1$.

P3. Show that for any $k \in N$, the operator of multiplication by z^k given by $f(z) \mapsto z^k f(z)$ is an isometry of the Hardy space. Is it true for multiplication by $1 - z$?

P4. Show that the following functions are multipliers and find the multiplier norms.

$$(1) \exp\left(\frac{1+z}{z-1}\right) \qquad (2) \frac{1}{2-z} \qquad (3) \sqrt{1-z}$$

P5. Let $f \in \mathcal{O}(\mathbb{D})$ and assume that $f \rightsquigarrow \{a_k\}_{k=0}^{\infty} \in \ell^p$.

1. Show that $f \in \mathcal{M}_{\mathbb{H}^2}$, if $p = 1$.
2. Is it true that $f \in \mathcal{M}_{\mathbb{H}^2}$ for $1 < p \le 2$?

P6. Show that:

1. the product of two inner functions is an inner function;
2. the composition of two nonconstant inner functions is an inner function. Is it true without additional condition $f \not\equiv const$?

P7. Let $f \in \mathbb{H}^2$. Assume that there exists a constant $C > 0$ such that $\Re f(z) > C$. Show that f is an outer function.

P8. Show that a singular inner function $f \not\equiv const$ cannot extend continuously to the closed unit disc. Construct a function that extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ and yet has a nontrivial singular inner factor.

P9. Does there exist $f \in \mathbb{H}^2$ such that its zero set is

(1) $\{re^{i\pi/4} \mid r \in [1/3, 1)\}$

(2) $\{r_k e^{i\varphi_k} \mid r_k = 1 - 2^{-k},\ \varphi_k = 2^k\}_{k\in\mathbb{N}}$

(3) $\{r_k e^{i\varphi_k} \mid r_k = 1 - (k+1)^{-2},\ \varphi_k = 2^k\}_{k\in\mathbb{N}}$

(4) $\{r_k e^{i\varphi_k} \mid r_k = (k+1)^{-2},\ \varphi_k = 2^k\}_{k\in\mathbb{N}}$

(5) $\{x_k + iy_k \mid x_k = 1 - (k+1)^{-3},\ y_k = (k+1)^{-2}\}$

P10. Find the canonical factorization of the following functions on the unit disc.

$$(1)\ (2z-1)e^z \qquad (2)\ \exp\left(\frac{1}{z^2-1}\right) \qquad (3)\ \sin z$$

P11. Construct a sequence of points $z_n \in \mathbb{D}$, such that there exist a Blaschke product $f \in \mathbb{H}^2$ satisfying $f(z_n) = 0$ for all $n \in \mathbb{N}$ and for almost any point $e^{i\varphi} \in \partial\mathbb{D}$ there is a subsequence $z_{n_k} \to e^{i\varphi}$ as $k \to \infty$.