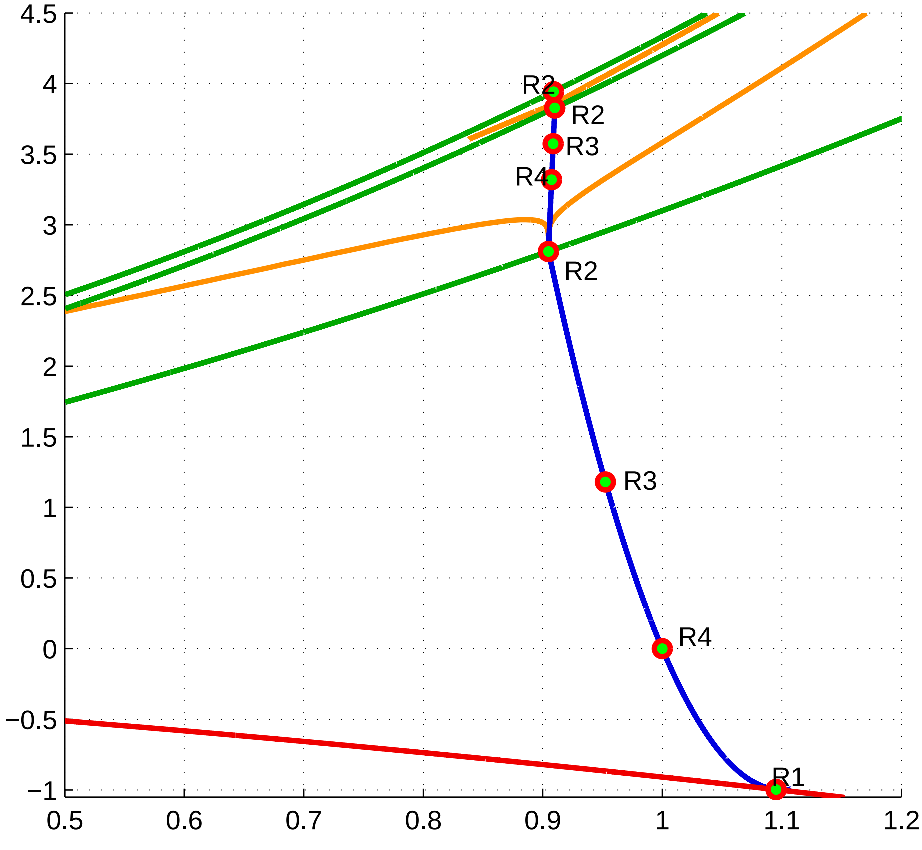
Universiteit Utrecht

MRI Master Class "Numerical bifurcation analysis of dynamical systems"

Numerical analysis of the accumulation of $1:2$ resonances in Generalized Hénon Map.



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1 Introduction

In the mid seventies of the last century a truly remarkable discovery was made independently by M. Feigenbaum [14] and by P. Coullet and C. Tresser [39]. They found that infinite sequences of period-doubling (flip) bifurcations in one-parameter families of unimodal maps demonstrate universal properties. Namely, it was found that a sequence of bifurcation parameter values converges exponentially and the rate of convergence is universal, i. e. independent of a particular family under consideration.

The first computer-assisted proof of the Feigenbaum universality conjecture has been given by O. Lanford [27]. Much later a "conceptual" proof has been found, with major contributions by D. Sullivan [38], M. Lyubich [28, 29], and C. McMullen [30].

The proof lead to huge development of the renormalization theory and holomorphic dynamics tools (although the problem considers real numbers).

Maps corresponding to period-doubling parameter value form a codimension 1 submanifold in the space of smooth maps. The aim of this research is to extend the Feigenbaum universality theory to bifurcations of codimension 2 known as strong resonances [7]. These bifurcations appear in general two-parameter families.

1.1 Representative planar family

A natural extension of the one-dimensional quadratic family to planar maps is a two-parameter family of maps introduced by M. Hénon [18] as an example of a planar map with a strange attractor[1]. However, this family is not suitable for our goals, since all strong resonances belong to a straight line and bifurcation structure is degenerate.

In generic dissipative maps, accumulating flip curves can be connected via curves of bifurcations with two eigenvalues on the unit circle[2]. If eigenvalues are not roots of unity of a small degree (< 5) and an extra nondegeneracy conditions holds, a closed invariant curve appears around the fixed point.

For every parameter values on period-doubling curve, Jacobian evaluated at the corresponding fixed point has eigenvalue equals to -1. Sometimes each period doubling curve intersects a Neimark-Sacker bifurcations curve at a 1:2 resonance point, which means the appearance of double eigenvalue -1 of the corresponding fixed point.

This is a bifurcation of codimension two and it is possible to have a cascade of bifurcations of codimension two (an infinite convergent sequence of such bifurcation points).

The first question that naturally arises is "How often does this phenomenon occur?" This structure has been first discovered numerically in a seasonally forced predator-prey model [23], and then observed many times, for instance in stock market dynamics [15], laser dynamics [41], meteorology [40], a driven Van der Pol-Duffing oscillator [26], and Chua's circuit [2]. All these observations strongly indicated a certain universality (already briefly mentioned in [8]). However, this phenomenon didn't receive much attention from mathematical community although it was studied by physicists. In [41] a treatment of the dynamics has been attempted, but the global picture is far from being understood. For instance, each 1:2 resonance point is the origin of two curves of homoclinic tangencies whose behaviour remained a mystery.

A similar bifurcation diagram has been obtained in the dissipative family

$$G : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha - \beta y - x^2 + rxy \\ x \end{pmatrix} \tag{1}$$

called the generalized Hénon map (GHM). This family has been studied in great detail in the recent paper [16] and demonstrates the above described 1:2 resonance accumulation for

[1]The historical situation was converse, a quadratic family appeared as reduction of Hénon map

[2]Sometimes this bifurcation is called a Neimark-Sacker bifurcation, due to Ju. Neimark [33] and R. Sacker [36].

$|R| < 0.5$ and $0 < \alpha < 4$; $0.3 < \beta < 1.1$. Hence, we have a representative family of maps we are looking for.

1.2 The core research questions and answers

The aim of the current research project is to build a solid background for performing analysis of renormalization operator for strong resonances bifurcations in generic two-parameter families of planar maps.

We want to describe normal forms near resonance points of high-order iterations and to prove existence and convergence of an infinite sequence of 1:2 resonances.

Understanding of the behaviour of curves corresponding to global bifurcations, e.g. tangencies of invariant manifolds of saddle cycles, will be helpful for explanation of properties of the limit flip curve.

It is also interesting to find out dynamical properties of the limit map (e.g. entropy, attractor, and corresponding p-adic dynamics on the invariant set). The study of the two-dimensional renormalization is a part of M. Lyubich's program of proving the Palis conjecture [34, 9].

Here we perform numerical analysis and suggest unfolding for strong 1:2 resonances. The main results are computed phase portraits and invariant manifolds. It becomes clear, that invariant manifolds for different iterations looks very similar and they are responsible for dynamical picture. This suggest to guess how renormalization operator should be constructed. Checking different constructions of renormalization operator is the next step in our research program.

2 Differences with one-dimensional case and difficulties

The main idea of renormalization theory suggests that after a proper scaling second iteration of the map looks quite similar to the first iteration. That allows to conjecture that doubling map has similar properties as the map itself. In two-dimensional case it's nontrivial to understand how does the graph of the map looks like. Moreover, it's known that unlike the one-dimensional case, where there exists a global attractor for a limit map, called Feigenbaum attractor, two-dimensional map has no invariant domain, and almost all points go to infinity very fast. Thus direct generalization of the one-dimensional operator is impossible.

Note that for resonant parameter values the map is area-preserving, and since the determinant of Jacobian is a smooth function of the parameters, we should consider maps that are small perturbation of area-preserving. For area-preserving maps all strong resonances are degenerate (actually, all Neimark-Sacker bifurcations are degenerate). However, renormalization theory has been developed for area-preserving maps and existence and convergence of an infinite sequence of curves of period doubling bifurcations is proved with a help of a computer. The same fact is also proved for maps with a constant Jacobian that are small perturbations of area-preserving maps. However, in both cases the map could be reduced to one-dimensional, while bifurcations of codimension two are two-dimensional phenomenon.

From the very beginning we have some computation difficulties. Despite of existence of attracting periodic cycles, the attraction basins are very thin, almost all points of the smallest square containing all periodic points go to infinity very fast.

3 Analysis of the first iteration

Here we give som details on computations that has been done in [16]. We start with analysis of the first iteration. The Jacobian of the generalized Hénon map $F_r(x;y) = (y; a-bx-y^2+rxy)$ is

$$J_r = \begin{pmatrix} 0 & 1 \\ ry-b & rx-2y \end{pmatrix} \tag{2}$$

Out of the line $y = b/r$ the map is invertible. The inverse map is

$$F_r^{-1}(x;y) = \Big(\frac{x^2+y-a}{ry-b}; x\Big) \tag{3}$$

Generalized Hénon map has two fixed points, when $(b+1)^2 + 4a(1-r) \geq 0$. Given fixed r, $(b+1)^2 + 4a(1-r) = 0$ is a limit point curve.

The coordinates of two fixed points are

$$x^F_{1,2} = y^F_{1,2} = \frac{-(b+1) \pm \sqrt{(b+1)^2 + 4a(1-r)}}{2(1-r)} \tag{4}$$

3.1 Classical Hénon map

Here we analyze bifurcations of the Hénon map. Later we will show that strong resonances and Neimark-Sacker bifurcations are degenerate.

$$F_0(x;y) = (y; a-bx-y^2), \text{ Jacobian: } J_0 = \begin{pmatrix} 0 & 1 \\ -b & -2y \end{pmatrix}. \tag{5}$$

Limit point bifurcation. Limit point bifurcation condition is $\det(J_0-E)\mid_{x_{1,2},y_{1,2}}=0$. Given $r=0$ coordinates of the fixed points that undergoes limit point bifurcation are

$$x_{1,2}=y_{1,2}=\frac{-(b+1)\pm\sqrt{(b+1)^2+4a}}{2}.$$

After substitution, fixed point bifurcation condition implies

$$\det(J_0-E)\mid_{(x_{1,2},y_{1,2})}=0 \Longleftrightarrow 2y_{1,2}+b+1=0 \Longleftrightarrow (b+1)^2+4a=0$$

Thus, the limit point bifurcation curve for Hénon map is

$$a=\frac{-(b+1)^2}{4} \tag{6}$$

The corresponding fixed points are $x_{1,2}^{LP}=y_{1,2}^{LP}=(-(b+1)/2;-(b+1)/2)$. To check that bifurcation is nondegenerate, we compute restriction to a central manifold.

First, note that for $b=\det(J_0)=1$ another eigenvalue is equal to 1, so strong 1:1 resonance bifurcation takes place; for $b=\det(J_0)=-1$ the second eigenvalue is -1, so limit point – period doubling bifurcation takes place. Thus we'll consider cases $b=\pm1$ separately. Let's begin with computations of eigenvector and adjoint eigenvector. The eigenvector corresponding to critical eigenvalue $\lambda=1$ is $v=(1;1)^t$. The adjoint eigenvector is $w=\frac{1}{b-1}(b;1)^t$. (Recall that $b\neq1$.) The vector w is a projection to a $\langle v\rangle$. By the Fredholm alternative theorem, $\forall x\in\mathbb{R}^2$ we have decomposition

$$(x_1;x_2)^t=\langle(x_1;x_2)^t,w\rangle v+\big((x_1;x_2)^t-\langle(x_1;x_2)^t,w\rangle v\big)=\frac{x_1b-x_2}{b-1}(1;1)^t+\frac{x_1-x_2}{b-1}(1;b)^t$$

We introduce new coordinate vectors $(1;1)^t$; $(1;b)^t$ and coordinate transformation is

$$G_2\colon (x_1;x_2)^t\mapsto\left(\frac{x_1b-x_2}{b-1};\frac{x_2-x_1}{b-1}\right)^t$$

and apply a coordinate shift, moving $(-(b+1)/2;-(b+1)/2)$ to the origin. Let G_1 be a shift map. Then in new coordinates Hénon map (5) takes form

$$\begin{aligned}&G_2G_1HG_1^{-1}G_2^{-1}\colon\\&(x;y)^t\mapsto\left(x+\frac{(x+by)^2}{b-1}-\frac{1}{b-1}\left(a+\frac{(b+1)^2}{4}\right);by-\frac{(x+by)^2}{b-1}+\frac{1}{b-1}\left(a+\frac{(b+1)^2}{4}\right)\right)^t\end{aligned}$$

and restriction to a central manifold is

$$H^{LP}\colon x\mapsto x+\frac{1}{b-1}\left(x^2+2bxy+(by)^2-a-\frac{(b+1)^2}{4}\right)$$

Limit point bifurcation takes place at the origin and is nondegenerate since $H^{LP}_{xx} = \frac{2}{b-1} \neq 0$, $H^{LP}_a = \frac{-1}{(b-1)} \neq 0$, and $H^{LP}_b = \frac{b+1}{2(1-b)} \neq 0$, if $b \neq -1$.

One can check that limit point bifurcation curve is also nondegenerate. Namely, consider the system

$$\begin{cases} F_0(x;y) - (x;y) = 0 \\ \det(F_{0x}(x;y) - E) = 0 \end{cases}$$

Its Jacobian matrix is

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -b & -2y-1 & 1 & -x \\ 0 & -2 & 0 & 1 \end{pmatrix}$$

The rank of the Jacobian matrix is 3, i. e. maximal, thus the curve is non-degenerate.

Period-doubling bifurcation. Period-doubling bifurcation condition is $\det(J_0 \mid_{x_{1,2},y_{1,2}} + E) = 0$ which implies

$$\det \begin{pmatrix} 1 & 1 \\ -b & (b+2) \pm \sqrt{(b+1)^2 + 4a} \end{pmatrix} = 0$$

The period-doubling bifurcation curve is

$$a = \frac{3(b+1)^2}{4}$$

As in the limit point bifurcation case, we find that for $b = \det(J_0) = 1$ the fixed point has two multipliers equal to -1, it is a strong $1{:}2$ resonance bifurcation, and for $b = -1$ period-doubling and limit point bifurcation curves intersect at fold-flip bifurcation point $(a;b) = (0;-1)$.

Now let's compute restriction onto a central manifold. The critical eigenvector, corresponding to -1 is $v = (-1;1)^t$, the adjoint eigenvector is $w = \frac{1}{1-b}(b;1)^t$. The Fredholm alternative theorem implies the following decomposition $\forall x \in \mathbb{R}^2$:

$$(x_1;x_2)^t = \frac{x_1 b + x_2}{1-b}(-1;1)^t + \frac{x_1 + x_2}{1-b}(1;-b)^t$$

we introduce new coordinate vectors $(-1;1)^t$ and $(1;-b)^t$. The coordinate transformation is

$$G_2\colon (x_1;x_2)^t \mapsto \Big(\frac{x_1 b + x_2}{1-b}; \frac{x_1 + x_2}{1-b}\Big)$$

We apply a coordinate shift, moving $((b+1)/2;(b+1)/2)$ to the origin. Let G_1 be a shift map. Rewritten in new coordinates, the Hénon map(5) reads

$$G_2 G_1 F_0 G_1^{-1} G_2^{-1}\colon (x_1;x_2)^t \mapsto$$
$$\Bigg(-x_1 + \frac{1}{1-b}\Big(a - \frac{3(b+1)^2}{4} - (x_1 - bx_2)^2\Big); -bx_2 + \frac{1}{1-b}\Big(a - \frac{3(b+1)^2}{4} - (x_1 - bx_2)^2\Big)\Bigg)$$

The restriction to a central manifold is

$$H^{PD}\colon (x_1; x_2) \mapsto -x_1 + \frac{1}{1-b}\Big(a - \frac{3(b+1)^2}{4} - (x_1 - bx_2)^2\Big)$$

The period-doubling bifurcation is non-degenerate, since $H^{PD}_{x_1x_1} = \frac{2}{b-1} \neq 0$, $H^{PD}_a = \frac{1}{1-b} \neq 0$, $H^{PD}_b = \frac{3(b+1)}{2(b-1)} \neq 0$, if $b \neq \pm 1$. We can check that period-doubling bifurcation curve is nondegenerate. As for the limit point bifurcation, we have the system

$$\begin{cases} F_0(x;y) - (x;y) = 0 \\ \det(F_{0x}(x;y) + E) = 0 \end{cases}$$

The Jacobian matrix is

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -b & -2y-1 & -1 & -x \\ 0 & -2 & 0 & 1 \end{pmatrix}$$

The rank of the Jacobian matrix is maximal, hence the period-doubling bifurcation curve is nondegenerate.

Neimark-Sacker bifurcation Neimark-Sacker bifurcation condition $J_0 = 1$ implies $b = 1$. The fixed points are $x = y = -1 \pm \sqrt{1+a}$, and for eigenvalue of the Jacobian matrix we have $\lambda = \cos\phi + i\sin\phi$ with $\cos\phi = 1 \pm \sqrt{1+a}$, so $a \in [-1; 3]$. We skip computations here and provide normal form coefficient:

$$d = \frac{1}{1-\bar{\lambda}^2}\left(2 + \frac{1+2\lambda^2}{\lambda^4+1}\right)$$

Thus $\Re(\bar{\lambda} d) = 0$ and the bifurcation is degenerate, the bifurcation curve is a straight line.

3.2 Bifurcations of generalized Hénon map

Recall that generalized Hénon map is

$$F_r\colon (x;y) = (y; a - bx - y^2 + rxy); \tag{1}$$

and its Jacobian is

$$J_r = \begin{pmatrix} 0 & 1 \\ ry - b & rx - 2y \end{pmatrix}. \tag{2}$$

To analyze bifurcations structure we use the same way as for Hénon map. The following formulas will be used. Let

$$B(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_r}{\partial x_j \partial y_k} x_j y_k = \begin{pmatrix} 0 \\ r(x_1y_2 + y_1x_2) - 2x_2y_2 \end{pmatrix}$$

be the bilinear form corresponding to F_r. Let v be an eigenvector corresponding to the critical eigenvalue and w be an adjoint eigenvector.

In the case of the limit point bifurcation, topological normal form of the restriction to a central manifold is defined by

$$H^{LP}\colon x \mapsto \gamma + x + ax^2 + O(x^3); \quad a = \frac{\langle w, B(v,v)\rangle}{2}. \tag{7}$$

Here a is a normal form coefficient responsible for a type of bifurcation.

For period-doubling bifurcation, the restriction to the central manifold is defined by

$$H^{PD}\colon x \mapsto -(1+\gamma)x + bx^3 + O(x^4); \quad b = \frac{\langle B(v,(E-J_r^{PD})^{-1}B(v,v)), w\rangle}{2} \tag{8}$$

Here b is a normal form coefficient responsible for bifurcation type.

In the case of Neimark-Sacker bifurcation, a central manifold is two-dimensional and restriction to the central manifold could be written in complex coordinate $z = x + iy$:

$$H^{NS}\colon z \mapsto e^{i\theta(\gamma)}(1+\gamma)z + dz|z|^2 + O(|z|^4)$$

here d is a normal form coefficient given by

$$d = \frac{\langle w, 2B(v,(E-J_0)^{-1}B(v,\bar{v})) + B(\bar{v},(e^{2i\theta_0}E - A)^{-1}B(v,v))\rangle}{2} \tag{9}$$

where v is a (complex) eigenvector corresponding to critica eigenvalue $\lambda = e^{i\theta_0}$ and $J_r^t w = e^{-i\theta_0}w$.

To prove that bifurcations of this three types are non-degenerate, it's enough to show that normal form coefficients are nonzero and to check that $\Re(e^{-i\theta_0}d) \neq 0$ in the Neimark-Sacker case.

Now we are proceed to computations.

Limit point bifurcation. As before, we compute bifurcation point from the bifurcation condition $\det(J_r - E)\mid_{x_{1,2},y_{1,2}} = 0$. Given the Jacobian (2) it follows that $x^{LP} = y^{LP} = \frac{b+1}{2(r-1)}$ and together with expression for the fixed point (4) we have a period-doubling curve

$$a = \frac{(b+1)^2}{4(r-1)}.$$

The Jacobian matrix evaluated at the fixed point has the critical eigenvector $v = (1;1)^t$ and the adjoint eigenvector is $w = \left(\frac{r(1-b)+2b}{2(r-1)};1\right)$. Using formula (7) for the normal form coefficient, we get

$$a = \frac{2(r-1)^2}{3r-2+b(2-r)};$$

the normal form coefficient for the fold bifurcation. Since $a \neq 0$ for $|r| < 1$, the limit point bifurcation is nondegenerate.

Period-doubling bifurcation. Period doubling bifurcation condition is $\det(J_r + E) = 0$. This implies $x^{PD} = y^{PD} = \frac{b+1}{2}$. Together with expression for the fixed point (4) it gives us period-doubling bifurcation curve

$$a = \frac{3(b+1)^2}{4(1-r)}$$

The Jacobian matrix evaluated at a period – doubling point given above has critical eigenvector $v = (-1;1)^t$ and adjoint eigenvector $w = \frac{2}{(r-b(2-r))+2}((b(2-r)-r)/2;1)^t$. The formula (8) gives us normal form coefficient

$$b = \frac{8(r+1)}{(b+1)(4-r)(2+r-b(2-r))}.$$

We see that for $b \neq 1$ period-doubling bifurcation is nondegenerate and $b = -1$ gives us a fold-flip bifurcation point of codimension two.

Neimark-Sacker bifurcations. Neimark-Sacker condition implies that two eigenvalues are complex conjugated pair and belong to the unit circle, thus their product is equal to 1. Hence $\det(J_r) = b - ry = 1$ and $x^{NS} = y^{NS} = (b-1)/r$. Using general expression for the fixed points (4), we get neutral saddle curve:

$$a = \frac{(b-1)(b-1+2r)}{r^2} \tag{10}$$

We have to exclude values corresponding to $\lambda_1 > 1$, $\lambda_2 = 1/\lambda_1 < 1$ and vice versa. We rewrite $|\lambda| = 1$ in trigonometric form $\lambda = \cos\phi + i\sin\phi$ and substitute it to the characteristic polynomial $\det(J_r\mid_{x_{NS},y_{NS}} - \lambda E) = 0$.

$$\cos\phi = \frac{(1-b)(2-r)}{2r}.$$

This gives us the condition $\left|\frac{(1-b)(2-r)}{2r}\right| < 1$. So the neutral saddle curve (10) is a Neimark-Sacker curve onto the intervals

$$\begin{aligned} \frac{2-3r}{2-r} < b < \frac{r+2}{2-r}, \qquad & \text{if } r > 0 \\ \frac{r+2}{2-r} < b < \frac{2-3r}{2-r}, \qquad & \text{if } r < 0 \end{aligned}$$

To compute normal form coefficient, one needs critical eigenvector and an adjoint eigenvector. Let's compute eigenvalue and then compute coefficient in terms of eigenvalue.

$$\lambda = \frac{1}{2r}\left((1-b)(2-r) + i\sqrt{4r^2 - (1-b)^2(2-r)^2}\right) \tag{11}$$

The Jacobian matrix evaluated at a Neimark-Sacker point has critical eigenvector $v = (\lambda; 1)^t$ and an adjoint eigenvector is $\frac{1}{1-\lambda^2}(-\lambda; 1)^t$. The normal form coefficient is (9)

$$d = \frac{1}{1-\bar{\lambda}^2}\left(\frac{2(b(r-2)-r)}{b(2-r)+r+2}(r(\lambda-1)-2) + \frac{(r\lambda-1)(r(\lambda-1)-2\lambda^2)}{\lambda^4-(b-1)(r-2)\frac{\lambda^2}{2}+1}\right)$$

This formula looks dangerous, but we have to deal with $\Re(e^{-i\theta}d)$. Now we are lucky:

$$c_{NS} = \frac{(1-r)r^2}{2(b(2-r)-(2-3r))} \neq 0$$

We see that $c_{NS} \neq 0$ for $r \neq 1$ and $r \neq 0$ Neimark-Sacker bifurcation is nondegenerate.

Strong resonances. Recall the expression for eigenvalues (11) in the Neimark-Sacker case:

$$\lambda = \frac{1}{2r}\left((1-b)(2-r) + i\sqrt{4r^2-(1-b)^2(2-r)^2}\right)$$

Strong resonance 1 : 1: $\lambda_1 = \lambda_2 = 1$

$$a = \frac{4(r-1)}{(r-2)^2}; \qquad b = \frac{2-3r}{2-r}; \qquad x = y = \frac{2}{r-2}$$

Strong resonance 1 : 2: $\lambda_1 = \lambda_2 = -1$

$$a = \frac{4(3-r)}{(r-2)^2}; \qquad b = \frac{r+2}{2-r}; \qquad x = y = \frac{b-1}{r}$$

Note that b-values corresponding to 1 : 1 and 1 : 2 resonances is boundary points of the interval corresponding to Neimark-Sacker curve.
Strong resonance 1 : 3: $\lambda_{1,2} = (1 \pm i\sqrt{3})/2$

$$a = \frac{5-2r}{(r-2)^2}; \qquad b = \frac{2}{2-r}; \qquad x = y = \frac{b-1}{r}$$

Strong resonance 1 : 4: $\lambda_{1,3} = \pm 1; \qquad \lambda_{2,4} = \pm i.$

$$a = 0; \qquad b = 1; \qquad x = y = 0.$$

4 Numerical analysis

Now we fix $r = -0.1$ to get a family of slightly dissipative, that are better for numerical experiments. Extensive numerical analysis of the Generalized Hénon map (1) has been done in [16], but accumulation of 1:2 has not been considered.

4.1 Phase portraits

We start numerical analysis of the map (1) with computations of bifurcation curves and phase portraits. Then we proceed to computation of global bifurcation curves.

Since for almost all parameter values close to the strong $1:2$ resonances almost all points turn to infinity quadratically, to plot phase portraits near periodic points we perform as follows.

We consider a thin grid on the square $\Delta = \{(x,y) \mid \|x\| \leq 3,\ \|y\| \leq 3\}$ and apply $N > 1000$ iterations. Then we choose points that don't go to infinity too fast:

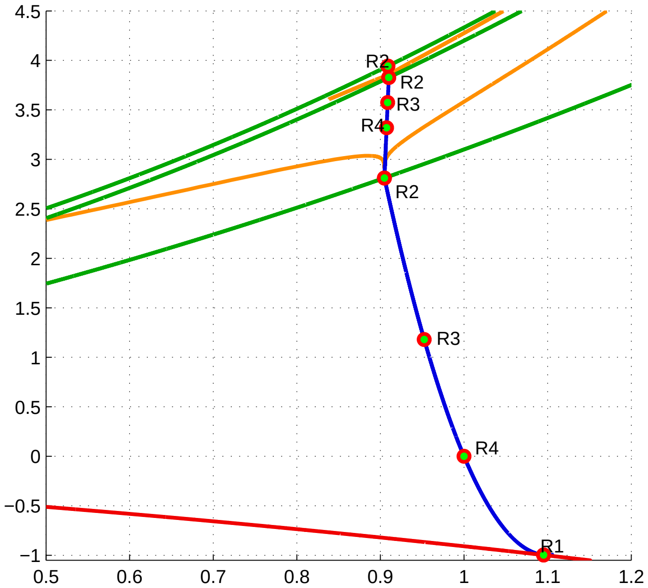
$$\Lambda = \{(x,y) \in \Delta \mid f^{\circ N}(x,y) < 10^{16}\}.$$

The choice of the constant 10^{16} is given by the default double float precision. Then we plot first 100 or 50 iterations of representative points of Λ with qualitatively different behaviour. We also plot first 100 iteration of all points of Λ. This gives us a domain of attraction in the case of an attracting forward invariant set (a fixed point, a periodic point, or an invariant curve).

Besides local bifurcations, a lot of global bifurcations, such as homoclinic tangencies. The mechanism is the following. When a point lost stability via a period-doubling bifurcation, it becomes a saddle, since one of multipliers cross the unit circle. This saddle point has two invariant manifolds, stable and unstable one. For parameter values close to period-doubling curve, two manifolds are smooth curves and unstable manifold turns to the stable period two point in the backward time, and stable manifold "goes around" period two cycle. When the parameter a grows up while parameter b remains fixed, two manifolds become tangent to each other, and then a transversal intersections occur. (See figures in the part 4.2 below).

The are also many periodic points born via Arnold tongues mechanism. They also lost stability via limit point bifurcations and become saddles, giving a lot of homoclinic tangencies bifurcations. We compute invariant manifolds for several cycles, although we don't study the global bifurcations of high iterations that are not a degree of 2.

Here and below we keep the following notations: limit point curves are red, period-doubling curves are green, Neimark-Sacker curves are blue, homoclinic tangencies are orange. We abuse notations by put only numbers on bifurcation diagrams, but D_k refers to the region number k in the text. Note that since the GHM map is orientation reversing, each point is connected with its second iteration. Otherwise, the picture would be unreadable.

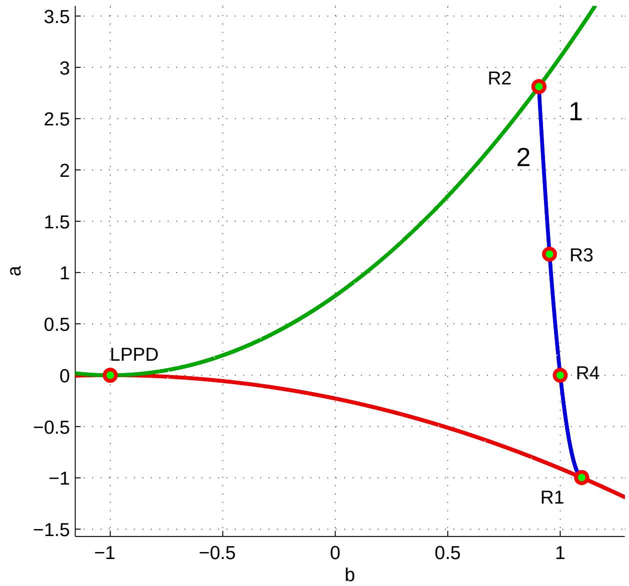


Partial bifurcation diagram for the GHM (1) on the (b, a)-plane.

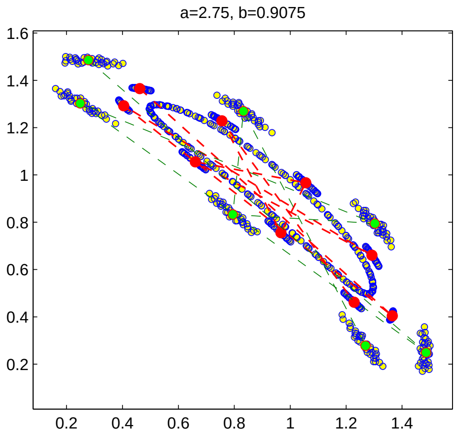
type	curves	coordinates	coefficients
First Iteration			
Limit point curve (LP)			
LPPD	$PD \cap LP$	0; -1	0.55 0.225 -0.06566
R1	$LP \cap NS$	-0.9977; 1.095	-1
Period Doubling (PD)			
R2	$PD \cap NS$	2.812; 0.9048	-0.9 -0.0275
LPPD	$PD \cap LP$	0; -1	0.55 0.225 -0.06566
Neimark Sacker (NS)			
R3	NS	1.179; 0.9524	-0.02015
R2	$PD \cap NS$	2.812; 0.9048	-0.9 -0.0275
R4	NS	-1.434e-09; 1	-0.05753 -1.198
R1	$LP \cap NS$	-0.9977; 1.095	-1
Second Iteration			
Period Doubling (PD^2)			
R2	$PD^2 \cap NS^2$	3.827; 0.91	-31.44 0.1158
R2	PD^2	1; -1.005	-8.067 5.774e-03
Neimark Sacker (NS^2)			
R4	NS^2	3.320 ; 0.9074	5.986e-03 -0.5873

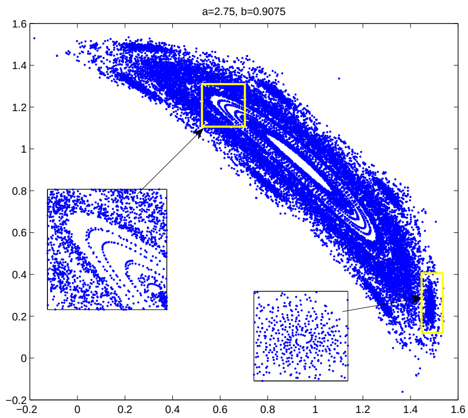
R3	NS^2	3.573 ; 0.9087e-01	2.182e-03
R2	$NS^2 \cap PD^2$	3.827; 0.91	-31.43 0.1158
Forth Iteration			
Period doubling (PD^4)			
R2	$NS^4 \cap PD^4$	3.943; 0.909	-459.8 -0.2282
Neimark Sacker (NS^4)			
R4	NS^4	3.886; 0.9095	-7.774e-04 -0.6239
R3	NS^4	3.915; 0.9092	-2.883e-04
R2	$NS^4 \cap PD^4$	3.943; 0.909	-459.8 -0.2282
Eigth Iteration			
Period Doubling (PD^8)			
R2	$PD^8 \cap NS^8$	3.957; 0.909	-8.746e+04 5.626
Neimark Sacker (NS^8)			
R4	NS^8	3.95; 0.909	1.029e-04 -0.6204
R3	NS^8	3.954 ; 0.909	3.816e-05
R2	$NS^8 \cap PD^8$	3.957; 0.909	-8.746e+04 5.626
Sixteength Iteration			
Period Doubling (PD^{16})			
R2	$PD^{16} \cap NS^{16}$	3.958 ; 0.909	-3.419e+07 -294.1
Neimark Sacker (NS^{16})			
R4	NS^{16}	3.958 ; 0.909	-1.365e-05 -0.6209
R3	NS^{16}	3.958 ; 0.909	-5.062e-06
R2	$NS^{16} \cap PD^{16}$	3.958 ; 0.909	-3.419e+07 -294.1
32th Iteration			
Period Doubling (PD^{32})			
R2	PD^{32}	3.959 ; 0.909	-1.893e+08 -1.162e+03

Table. 1. Codimension two bifurcations and normal form coefficients

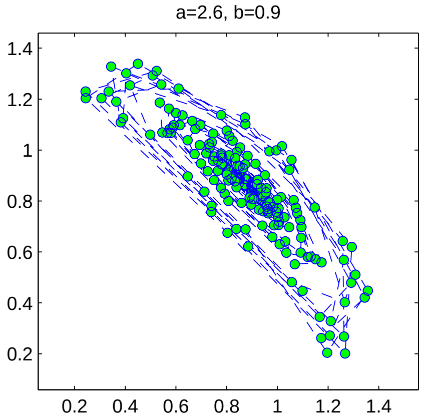


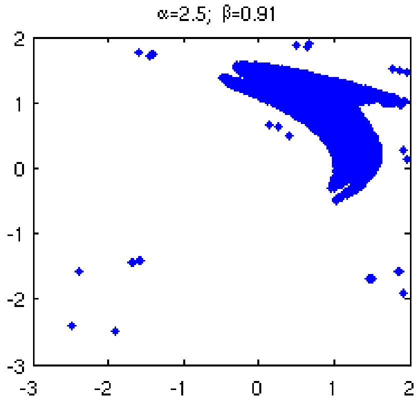
Zoomed partial bifurcation diagram, a part corresponding to the first iteration.



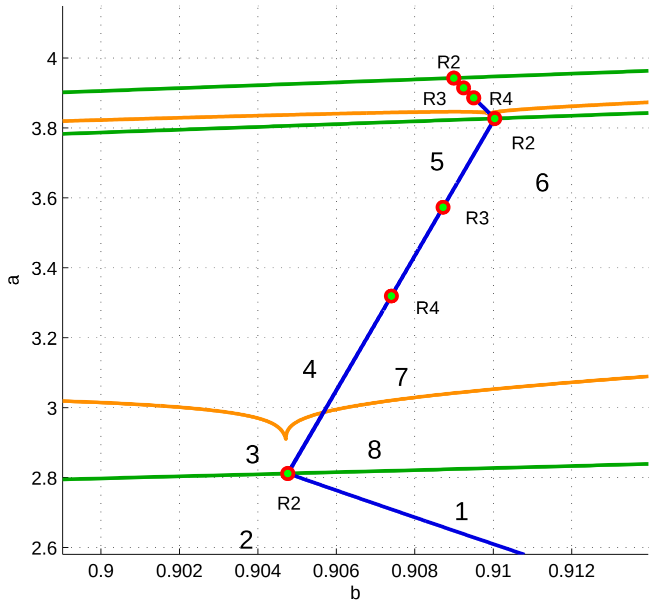


$(b, a) \in D_1$. Stable 7-cycle, stable 9-cycle, both of the focus-type, and stable invariant curve. For a slightly bigger b 9-cycle will approach invariant curve and destroy it. Fixed point is unstable and not shown. On the right figure we see a "hole" in the center, that means that fixed point is repelling, bottom subfigure show a domain of attraction of the 7-cycle, and upper subfigure show a small part of the invariant curve — a boundary of the "hole".

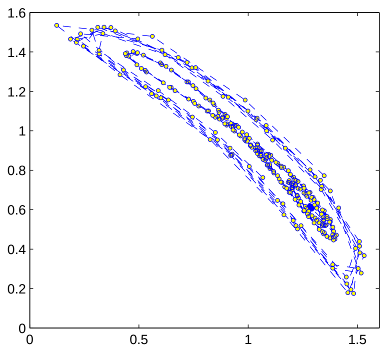


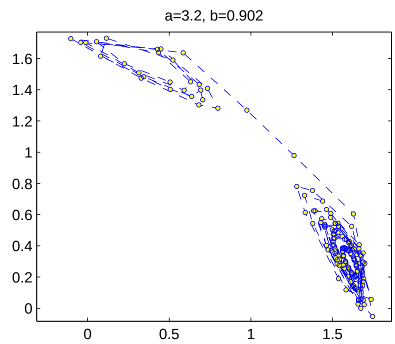


$(b, a) \in D_2$. Attracting fixed point (focus) and domain of attraction (right).

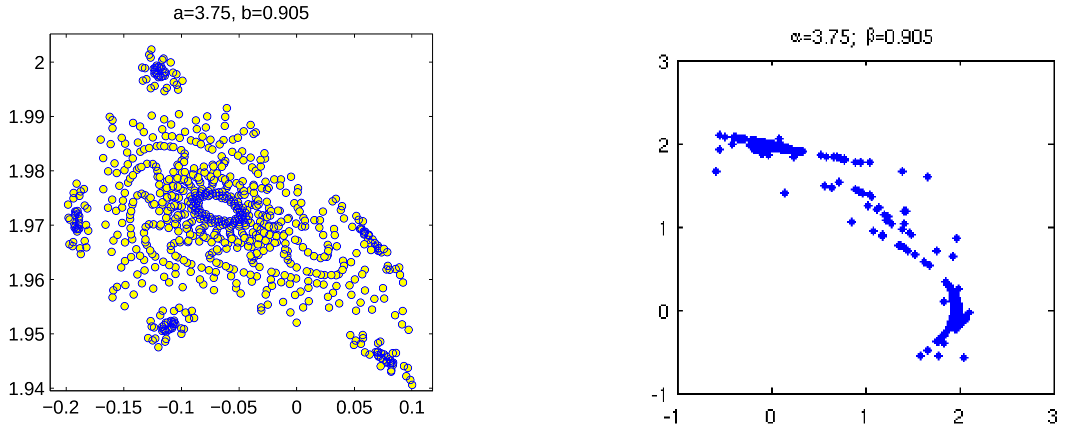


Zoomed partial bifurcation diagram, a part corresponding to the second iteration.

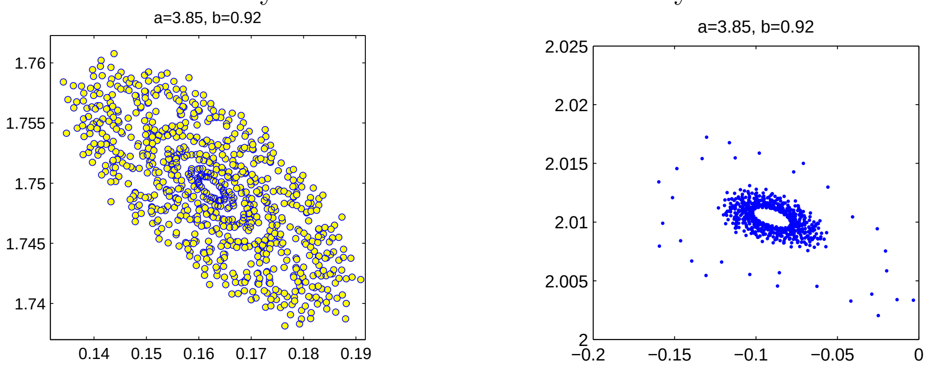




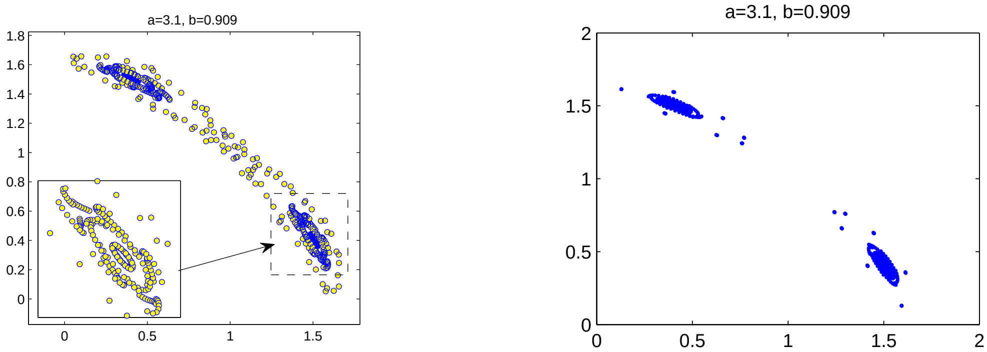
Stable 2-cycle. Dynamics before ($(b, a) \in D_3$, left) and after ($(b, a) \in D_4$, right) homoclinic tangency. On the left figure an orbit around 2-cycle belongs to the stable invariant manifold of unstable fixed point.



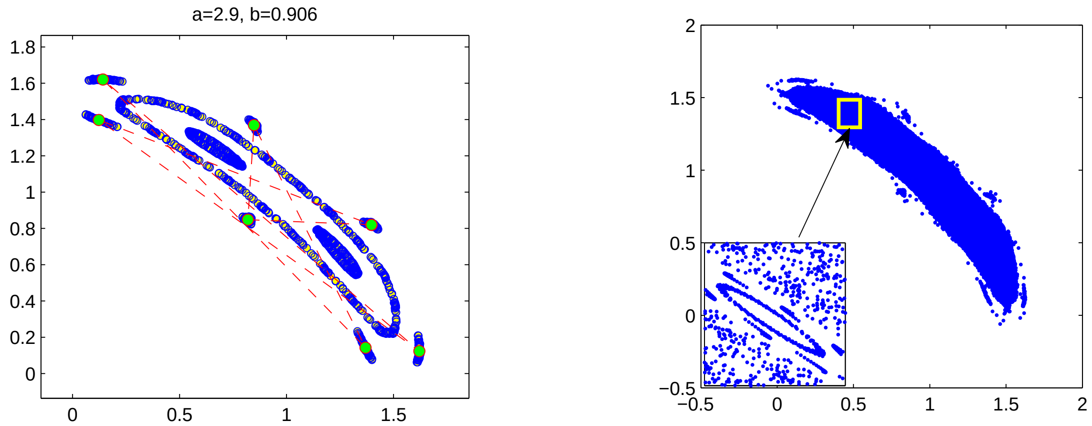
$(b, a) \in D_5$. One half of stable 2-cycle, unstable invariant curve (not shown) around, stable 5-cycle. Attraction domain of 2-cycle.



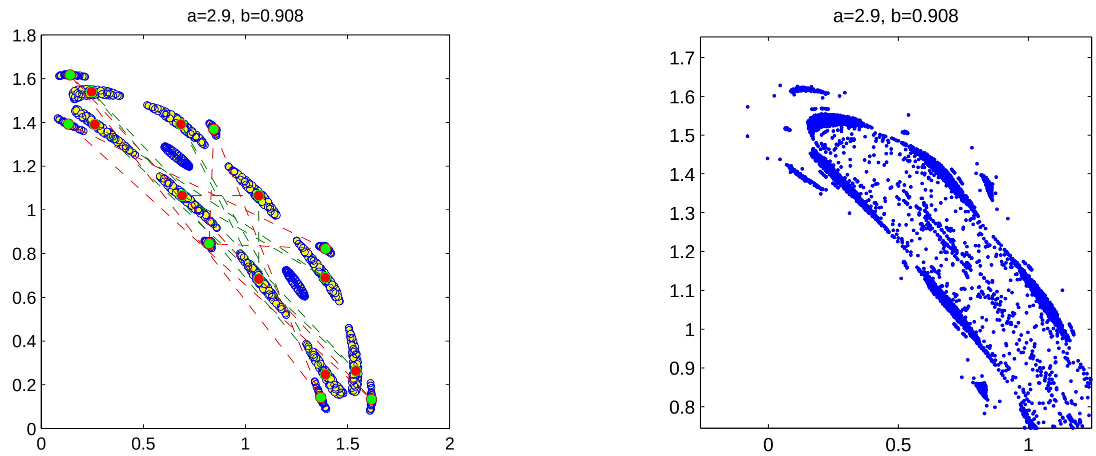
$(b, a) \in D_6$. One half of unstable 2-cycle, fixed point is unstable.



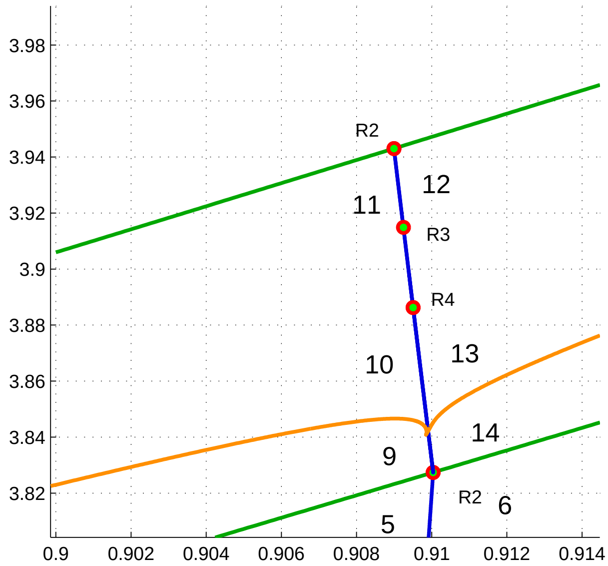
$(b, a) \in D_7$. Unstable 2-cycle, unstable fixed point. An orbit around 2-cycle belong to the stable manifold of the fixed point.



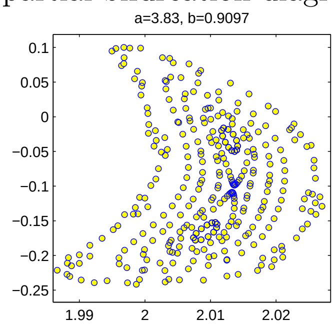
Stable 7-cycle, stable invariant curve and unstable 2-cycle. Fixed point is unstable and not shown. On the right figure we see a "hole" around a point of 2-cycle, indicating that it's unstable. Stable 9-cycle (see next figure) exists very close to invariant curve and not shown.

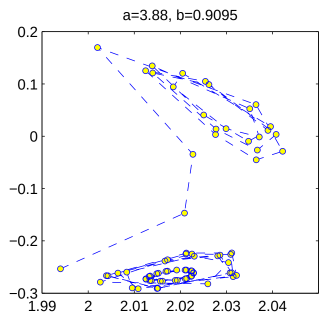


Stable 7-cycle doesn't belong to invariant curve. Stable 9-cycle belong to invariant curve, unstable 9-cycle (not-shown) also belong to invariant curve, and lives in holes. 2-cycle is unstable, unstable fixed point not shown.

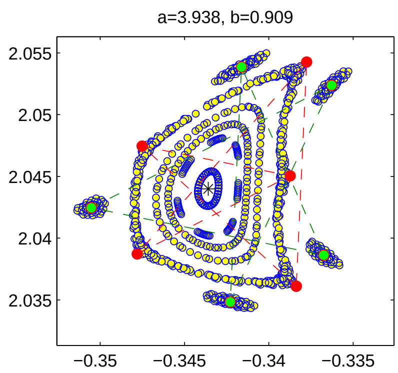


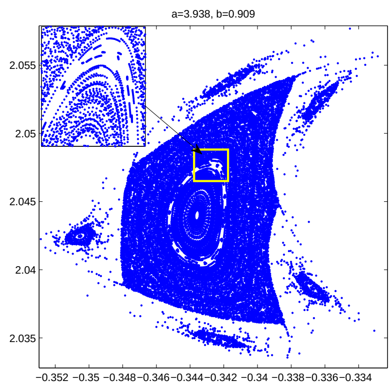
Zoomed partial bifurcation diagram, a part corresponding to the forth iteration.



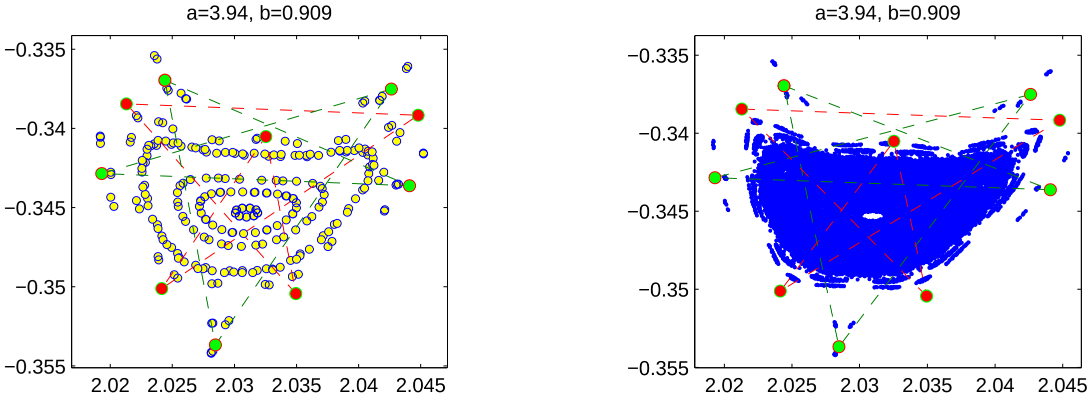


One half of the stable 4-cycle. A stable invariant curve around (left $(b, a) \in D_9$). Invariant curve destroyed via a homoclinic tangency (right $(b, a) \in D_{10}$).

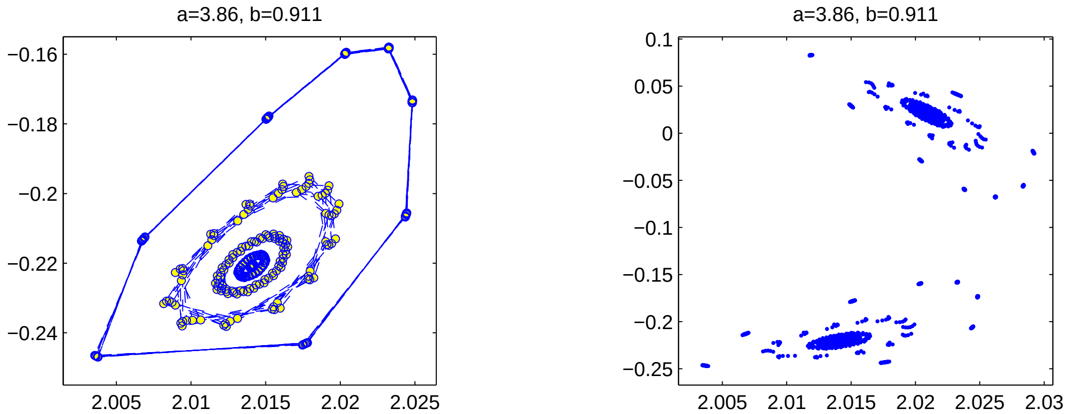




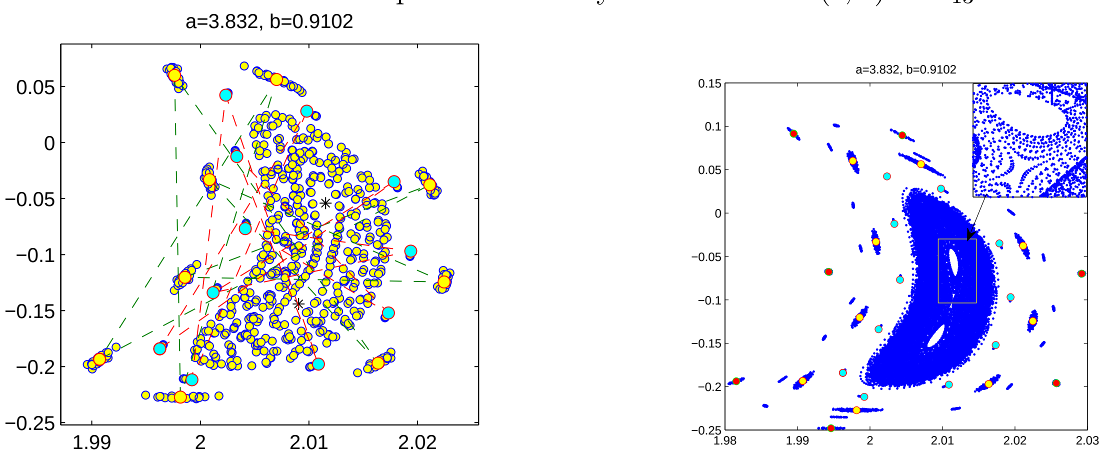
One quater of the stable 4-cycle (black star), on the right we see it's atrraction domain. Unstable 20-cycle (red) reached unstable invarinat curve, that is the boundary of attraction domain of 4-cycle. Stable 5-cycle (green) is a focus. $(b, a) \in D_{11}$.



One quater of the unstable 4-cycle (not shown, lives in the center "hole"). Unstable 5-cycle (red) is a saddle and stable 5-cycle (green) is a focus. $(b, a) \in D_{12}$.



One quater (left) and one half (right) of the unstable 4-cycle. There exists a 7-cycle, that is a stable focus. Multiplier of the 4-cycle is about 4. $(b, a) \in D_{13}$.



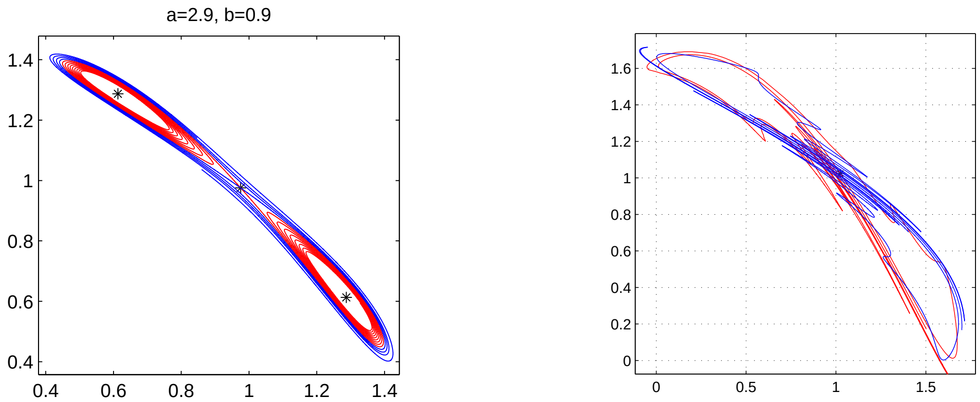
Big invariant curve exists around unstable 4-cycle (black dots mark one half of it). There are a lot of periodic points among them 11-cycle (blue), 9-cycle (yellow) and 7-cycle (green) are shown. $(b, a) \in D_{14}$.

4.2 Evolution of invariant manifolds

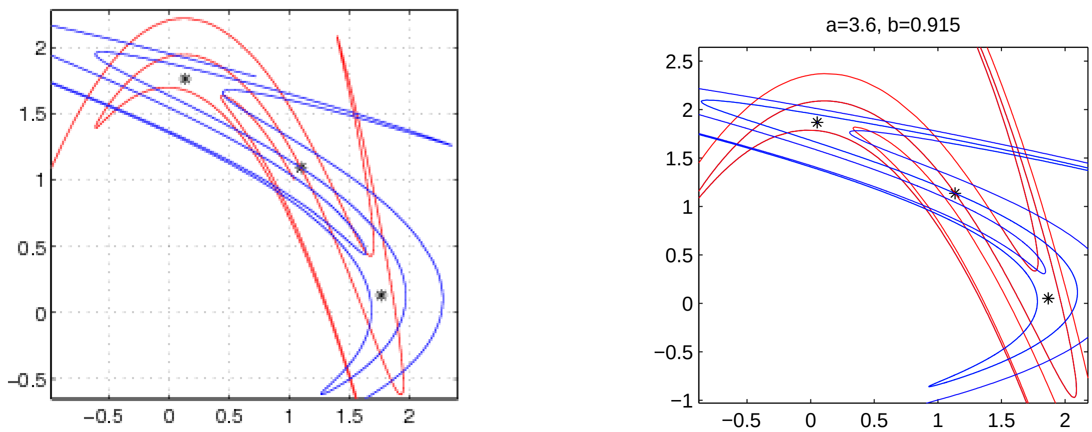
We want to show that “geometry” of stable and unstable manifolds of the saddle fixed points for first, second, 4th, 8th, etc. iterations are very similar. To do that we plot stable and unstable manifolds for maps with $b = 0.93 \pm \varepsilon$, $0 < \varepsilon < 0.03$ and varying a: $3.1 < a < 4.0$. Actually, if our conjecture on convergence of the sequence of strong 1:2 resonances is correct, then the limit map g has an unstable periodic point of period 2^n for any $n \in \mathbb{N}$. Since we claim that the limit map g is a fixed point of the doubling operator of the form

$$f \mapsto A^{-1} \circ f \circ f \circ A, \quad A \text{ is a linear map } \mathbb{R}^2 \to \mathbb{R}^2$$

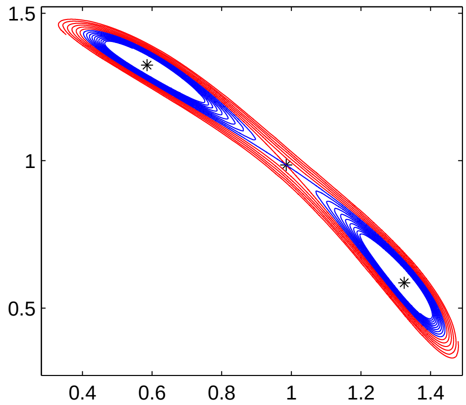
then the linear operator A should map the stable and unstable manifolds of the unstable fixed point of the map g onto stable and unstable manifolds of the unstable fixed point the map $g^{\circ 2}$. The later two manifolds should be mapped by A onto stable and unstable manifolds of a saddle fixed point of the map $g^{\circ 4}$, etc. (At least in some neighborhood of the fixed points). Thus pairs of stable and unstable manifolds should be related by a linear transformation.



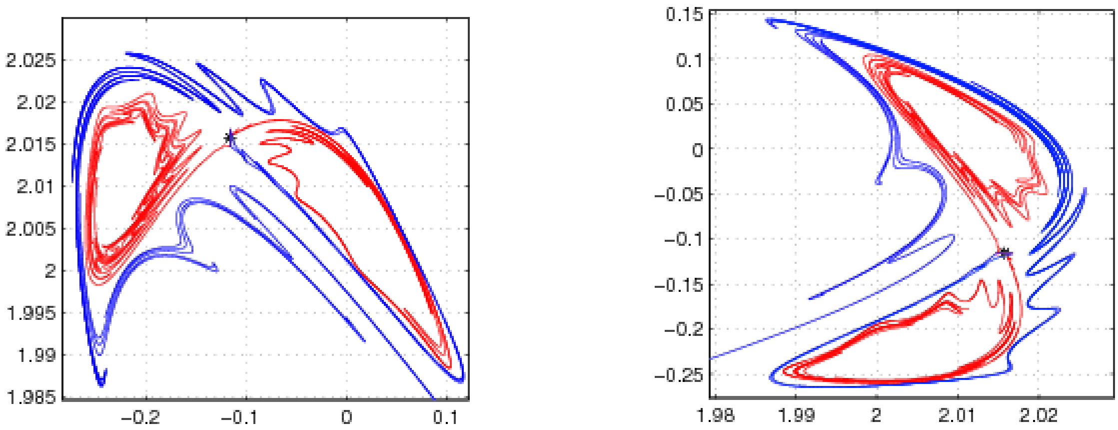
Left: Stable and unstable manifolds for GHM map with $(b; a) = (0.9; 2.9) \in D_3$. Right: Stable and unstable manifolds for GHM map with $(b; a) = (0.9; 3.1) \in D_4$. Period two cycle is stable.



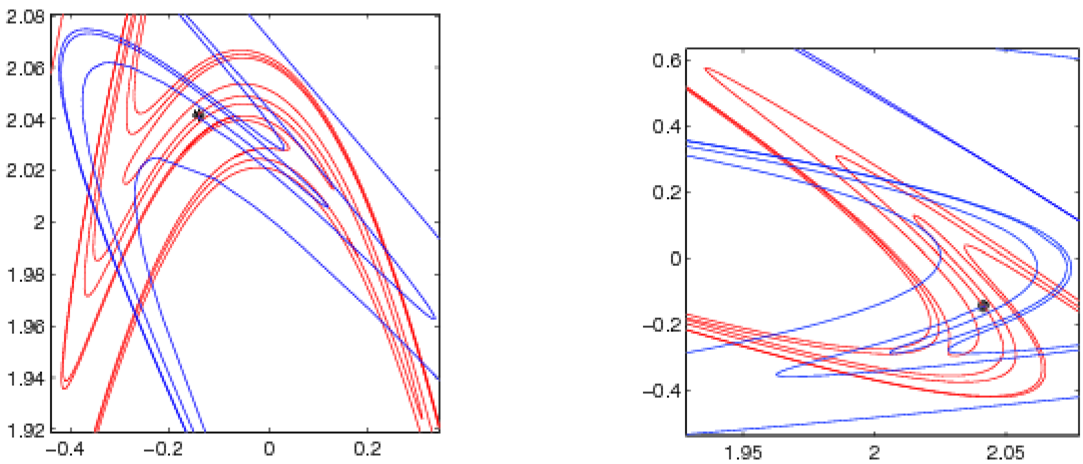
Left: Stable and unstable manifolds for GHM map with $(b; a) = (0.9; 3.6) \in D_5$. Period two cycle is stable. Right: Stable and unstable manifolds for GHM map with $(b; a) = (0.915; 3.6) \in D_6$. Period two cycle is unstable.



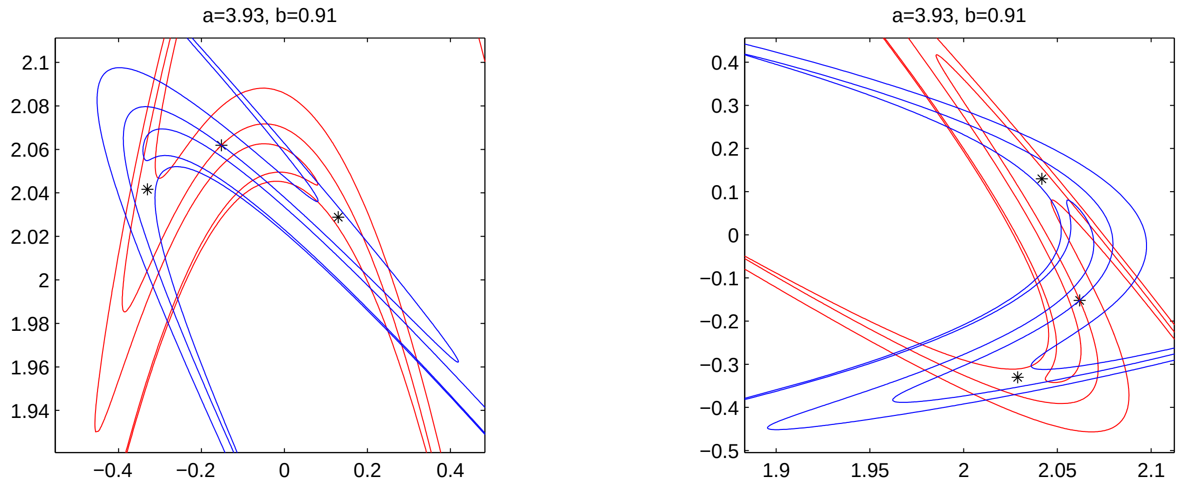
Stable and unstable manifolds for GHM map with $(b; a) = (0.91; 2.9) \in D_8$.



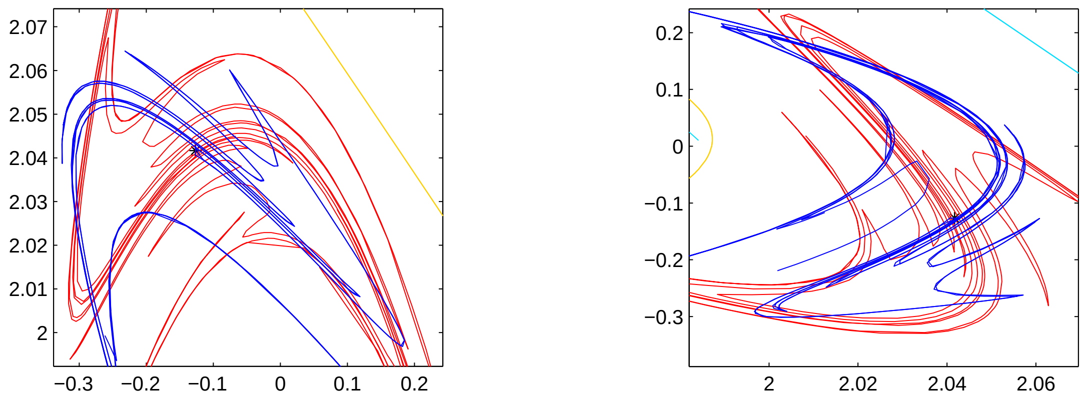
Stable and unstable manifolds for unstable 2-cycle. $(b; a) = (0.9; 3.82) \in D_9$.



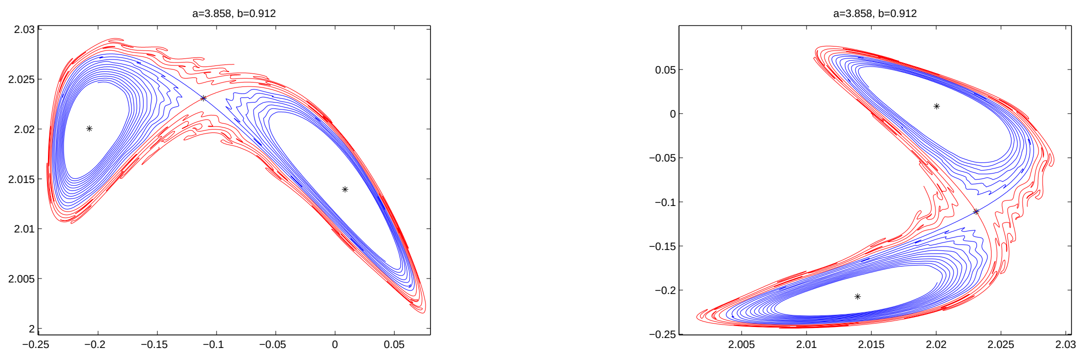
Stable and unstable manifolds for unstable 2-cycle. $(b;a) = (0.9; 3.87) \in D_{11}$.



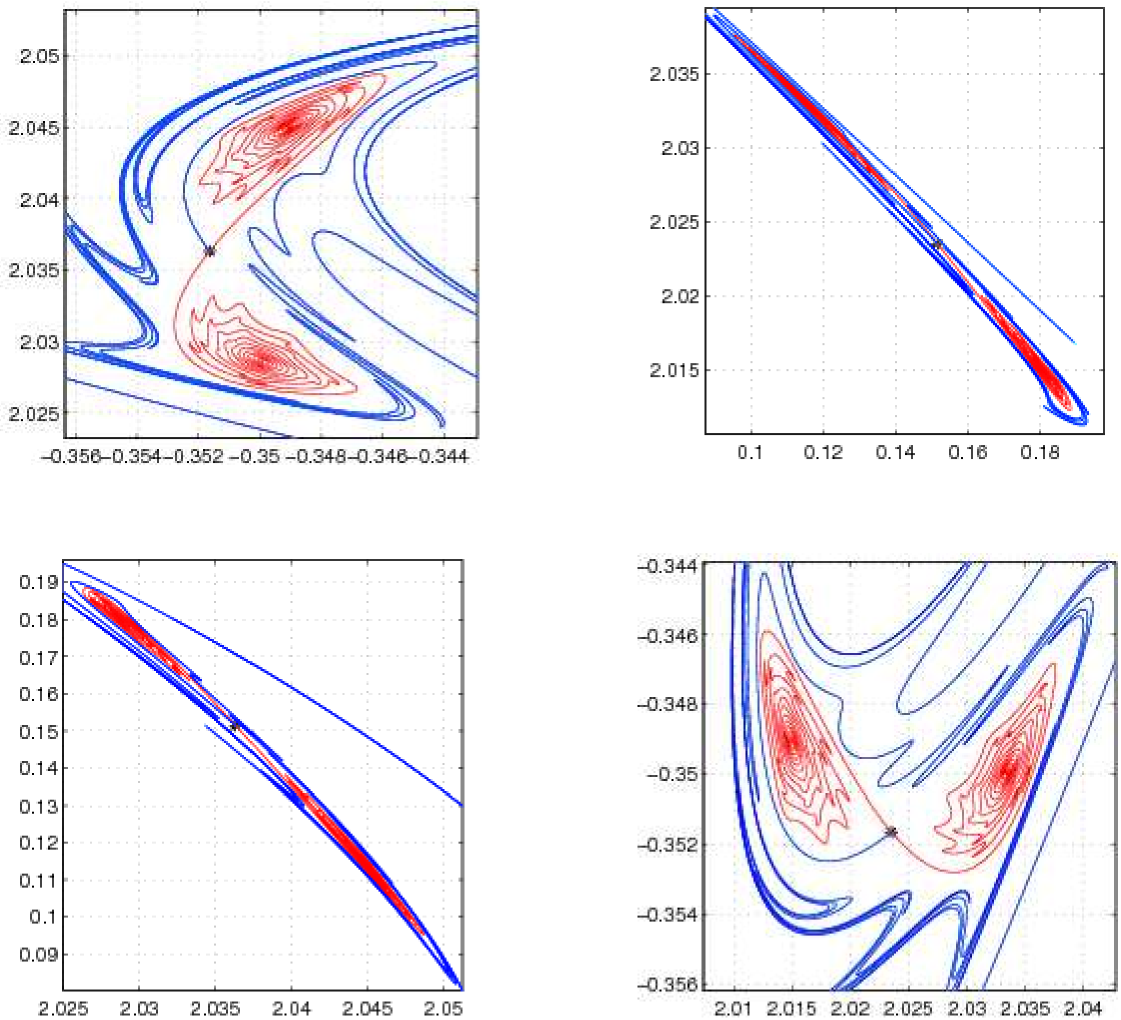
Stable and unstable manifolds for unstable 2-cycle. $(b;a) = (0.91; 3.93) \in D_{12}$.



Stable and unstable manifolds for unstable 2-cycle. $(b;a) = (0.915; 3.9) \in D_{13}$.

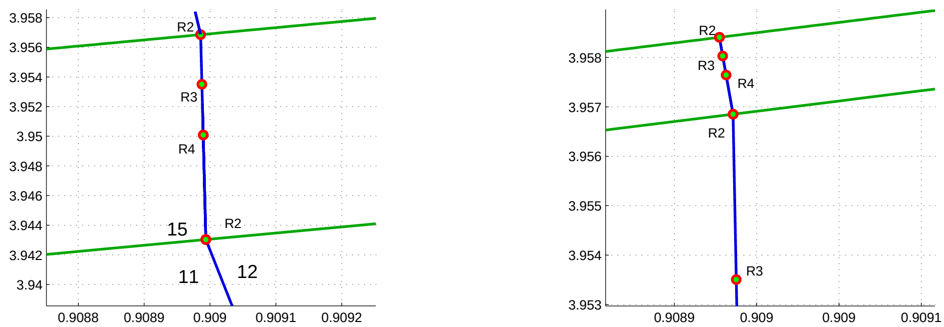


Stable and unstable manifolds for unstable 2-cycle. $(b;a) = (0.912; 3.858) \in D_{14}$.



Stable and unstable manifolds for unstable 4-cycle. $(b; a) = (0.9; 3.91)$.

Without unfolding, we provide parts of bifurcations diagrams for the 8th and 16th interate near Neimar-Sacker curve. We see that they support the convergence conjecture.

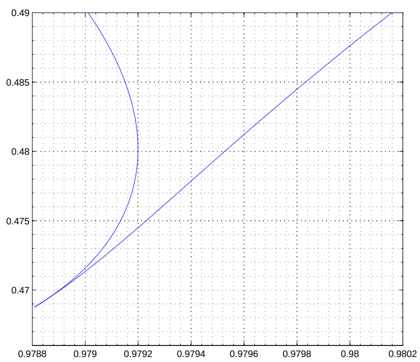


Neimark-Sacker and period-doubling curve for $F^{\circ 8}$ (left) and $F^{\circ 16}$ (right).

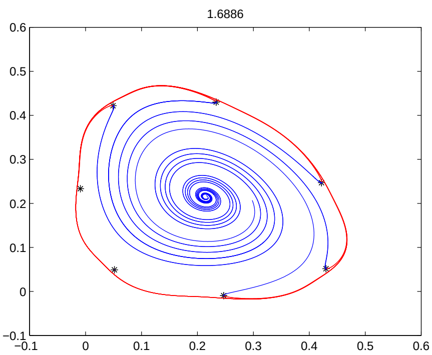
4.3 Periodic points of high iterations

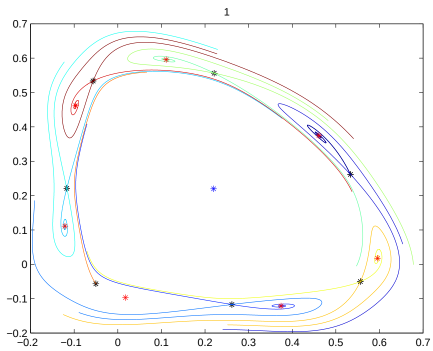
There are a lot of resonances on Neimark-Sacker curves. It's well known that Arnold tongue grows up from each resonance. This tongue is formed by two curves of limit point of a periodic point of the corresponding period.

To check that our numerical algorithms are correct, we compute several Arnold tongues and stable and unstable manifolds of unstable cycles. Our goal is to get classical phase portraits with invariant curve formed by unstable manifolds of periodic points and understand destroying scenario of the invariant curve.

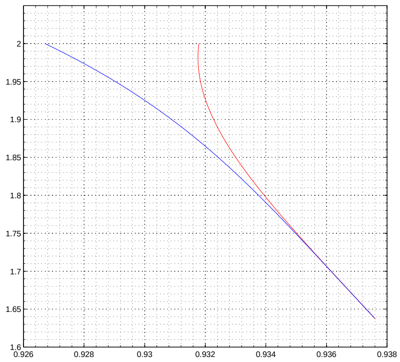


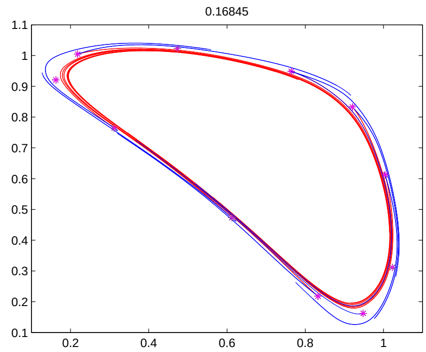
It is known that unstable manifolds of the saddle cycle form an invariant curve only on for parameter values closer to the Neimark-Sacker curve than any codimension two bifurcation of the periodic point. So we don't consider cycles of period 5, since it happens that $F_r^{\circ 5}$ has a strong $1{:}1$ resonance very close to the Neimark-Sacker curve of F_r. Thus we proceed to a cycle of period 7.





Left: Stable and unstable manifolds of 7-cycle of GHM map with $(b;a) = (0.97945; 0.48) \in D_1$. Right: Stable and unstable manifolds of 7-cycle of GHM map with $(b;a) = (0.97845; 0.49) \in D_1$.





Left: Arnold tongue at 11-resonance. Right: Stable and unstable manifolds of 11-cycle of GHM map with $(b; a) = (0.9325; 1.85) \in D_1$.

5 Conjecture on renormalization operator

A doubling operator T has been introduced above. Our goal is to find an appropriate linear operator A.

Since the linear operator should map invariant manifolds of the fixed point of the first iteration onto the invariant manifolds of the fixed point of the second iteration, I suggest to construct it as following. First of all, the operator A should map eigenvectors of the fixed point in to eigenvectors. Since we require only preserving of directions, this choice fixes two coefficients of the matrix of A. To choose another two coefficients, we should fix image of some point. It seems to be reasonable to fix the image of a point on a primary intersection of two manifolds. Intersection of stable and unstable manifolds is a two-sides infinite sequence:

$$(M^s \cap M^u)_p = \{\dots x_{-n}, x_{-n+1}, \dots, x_{-1}, x_0, x_1, \dots x_{n-1}, x_n, \dots\}.$$

Two subsequences of negative and positives indices converge to a fixed point: $x_{-n} \to x^*$, $x_n \to x^*$ as $n \to \infty$. Adding a point x^* to the intersection, we get a closed set: $(M^s \cap M^u) \cup \{x^*\}$ is closed. Thus a maximum of the distance between fixed point and this set is well-defined. Let fix image of the point that serve this maximum for kth iteration:

$$x^{(k)}_{max} = \max_{x \in (M^s \cap M^u)_p \cup \{x^*\}} \|x - x^*\|$$

$$Ax^{(1)}_{max} = x^{(2)}_{max}$$

I guess that this point plays some role in dynamics. However, I don't know now how to formulate it properly.

For numerical goals this point could be found as following. Pick up a point in the unstable direction. Consider, say, $N > 200$ iterations. If the point belong to the stable manifold as well, then $\|f^{\circ N} - x^*\| \to 0$ as $N \to \infty$. Thus in this way we can find points at the intersection effectively. From the first 100 points one could pick up the point at the maximum distance and find operator A solving linear equation.

Of course, this operator is well-defined only for maps with unstable fixed point and 2-cycle.

In the article [26] it is suggested to construct renormalization operator based on the eigenvalues. It is claimed that eigenvalues of all periodic points of period $2^{\circ n}$ for the limit map is the same. Our computations show that for a map with parameter values close to 1:2 of the 32 iteration the eigenvalues are almost the same for all periodic points of periods 1, 2, 4, and 8. However, for a 16th and 32th periods they are different. We conjecture that the reason is that behaviour near periodic points up to 8th order have stabilized and doesn't change anymore.

per	x_0	λ_1	λ_2	$\lambda_1 \cdot \lambda_2$
1	(1.21834 ; 1.21834)	-0.50099	-2.05751	1.0308
2	(-0.16817; 2.07714)	-0.48441	-2.05662	0.99625
4	(-0.36499; 2.04977)	-0.49024	-2.04078	1.00046
8	(-0.35767; 2.03547)	-0.48001	-2.08303	0.99989
16	(-0.35428; 2.03501)	-0.58875	-1.69832	0.99989
32	(-0.35463; 2.03486)	-0.51867	-1.92757	0.99976

6 Conclusion

Although our studies are far from eing complete, we summarize our results.

First of all, from the normal coeeficients we see that types of strong 1:2 alternate, since the second normal form coefficient changes sign, when we procced from $F^{\circ 2^n}$ to $F^{\circ 2^{n+1}}$.

Next, a thin tongue of homoclinic tangencies emerges from each strong 1:2 resonance. The branches of the tongue intersect next period doubling curve far from the Neimark-Sacker curve.

On the bifurcations diagrams we see that the angle between primary and secondary Neimak-Sacker curve turns to zero with respect to number of iterations. This also indicates convergence of 1:2 resonances.

Analysis of invariant manifolds shows that there exists a homoclinic structure in the neighborhood of every point of period 2^n. Altogether this structures gives us an idea of construction of renormalization operator. If our conjecture is correct, the next step is to proof numerically the existence of the fixed point of renormalization operator and to compute its eigenvalues. Another works in this direction gives us a hope, that fixed point a saddle, with two-dimensional unstable manifold and stable manifold of codimension two.

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