Research Statement
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My current research interests lie in the area of Dynamical Systems and Ergodic Theory. In my work, I extensively use methods from Functional and Complex Analysis and Hyperbolic Geometry. I then apply my results to solve problems in other areas of mathematics, such as Fractal Geometry, Hyperbolic Geometry, Number Theory, and Probability. Some arguments are computer-assisted. Numerical values I give in theorems are accurate to all decimal places specified, and all calculations involved have rigorously verified bounds on numerical errors.

The main objective of the modern theory of dynamical systems was set by Kolmogorov [22]. It can be summarised as "To describe the behaviour of almost all trajectories of almost all systems". In the simplest possible setting this means the following. Suppose we ask someone to draw a graph of a function on an interval; we call this function f. Then somebody else picks a point x from the interval. Without knowing neither the function nor the point in advance, can we predict properties of the sequence $\{x, f(x), f(f(x)), \dots, f^n(x) \dots\}$?

In order to study questions of this nature, many characteristics describing properties of trajectories of dynamical systems have been introduced during the past century. They allow us to classify a given dynamical system and to answer the basic question: what is most likely to happen with a randomly chosen point in the phase space?

The main theme of my latest research is characteristics of proven significance, that quantify important statistical and geometric properties of dynamical systems:

— Hausdorff dimension of the limit set in Dynamical Systems;
— Dimension of stationary measures in Probability Theory;
— Resonances for geodesic flows in Spectral geometry; and
— Lyapunov exponents in Ergodic Theory.

I have developed and implemented rigorous computer algorithms for estimating these values.

1. Hausdorff dimension of the limit set

The Hausdorff dimension of certain fractal subsets of the real line turned out to be the key ingredient for studying important problems in number theory, in particular the density-one Zaremba conjecture and the Markov and Lagrange spectra. A dynamical approach to estimating the dimension is to construct a system which has the set in question as a limit set, i.e., the set of all limit points of all trajectories.

In a joint work with Pollicott [30], we provide a fast and effective algorithm which allows us to obtain estimates, with rigorously validated error bounds, on the Hausdorff dimension of subsets of $\mathbb{R}$ described in terms of continued fractions of their elements. This makes it possible to study sets which are far beyond the limits of applicability of previously existing methods [16], [26].

The first application is an attack on the Zaremba conjecture. In [2] Bourgain (1994 Fields medalist), in a joint work with Kontorovich, showed the following result (popularised under the name of density-one Zaremba conjecture).

Theorem 1 (Bourgain and Kontorovich)**.** *Consider the set of all rational numbers that can be represented as continued fractions with partial denominators $1 \le a_j \le m$. Let $D_m(N)$ be the* *number of values $1 \le q \le N$ which occur as denominators of these rational numbers. Then*

$$\lim_{N\to+\infty} \frac{D_{50}(N)}{N} = 1.$$

Let us introduce $E_m := \{x = [0; a_1, a_2, \dots] \mid a_n \in \{1, 2, \dots, m\} \text{ for all } n \in \mathbb{N}\}$. The original proof of Theorem 1, based on the circle method, uses the fact that the Hausdorff dimension $\dim E_{50}$ is very close to 1. In subsequent work Huang [15] and Kan [18], assuming heuristic estimates $\dim E_5 > \frac{5}{6}$ and $\dim E_4 > \frac{\sqrt{19}-2}{3}$, proved results analogous to Theorem 1 for $m = 5$ and $m = 4$, respectively. We show [30] that both inequalities hold true, thus removing the extra assumptions:

Theorem 2 (Pollicott and **P.V.).**
$\dim E_5 = 0.836829443680 \pm 10^{-12} > \frac{5}{6}$ *and* $\dim E_4 = 0.788945557483 \pm 10^{-12} > \frac{\sqrt{19}-2}{3}$.

Hence, we have $\frac{D_4(N)}{N} \to 1$ as $N \to \infty$.

The second application of the new approach is the study of geometry of the Lagrange (L) and Markov (M) spectra. These sets were defined by Markov in 1879–80 and have been studied with considerable interest ever since. For instance, in [27] Moreira (a plenary ICM-2018 speaker) established that

$$\dim((-\infty, t) \cap M) = \dim((-\infty, t) \cap L) := \mathcal{D}(t). \tag{1}$$

Using a refinement of the new method, we were able to give an accurate estimate on the first transition point where the Markov and Lagrange spectra acquire the full dimension [25]:

Theorem 3 (Matheus, Moreira, Pollicott, and **P.V.).** *Markov and Lagrange spectra acquire full dimension at* $t_1 := \inf\{t \mid \mathcal{D}(t) = 1\} = 3.334384\dots$, *where the given value is accurate to all decimal places presented.*

The technique used to compute the first transition point can be easily adapted to study the continuous, but not Hölder continuous, function $\mathcal{D}$ defined by (1). The same technique allowed us to study fractal properties of the set $M \setminus L$ and to improve estimates on the dimension [25].

Theorem 4 (Matheus, Moreira, Pollicott, and **P.V.).** *The dimension of the set difference of Markov and Lagrange spectra satisfies* $0.537152 < \dim(M \setminus L) < 0.796445$, *where the given values are accurate to all decimal places presented.*

Future research direction #1

Since the algorithm presented in [30] for computing the Hausdorff dimension and Lyapunov exponents proved very effective, it is a pressing problem to generalise it to two-dimensional settings. In this direction, the first step is to extend the method to Julia sets and to Lyapunov exponents for random walks on Kleinian groups. A more ambitious project I have in mind is possible generalisations to parabolic settings, where no existing method guarantees convergence. In particular, the computation of Hausdorff dimension of the Apollonian gasket has attracted attention of many mathematicians, including C. T. McMullen [26] (1998 Fields medalist), but remains open.

2. Dimension of stationary measures

The study of stationary measures of affine iterated function schemes of similarities, the best known example of which is Bernoulli convolutions, dates back to the work of Erdős [7] from the 1930s. The dimension of the stationary measure provides information on the distribution of the limit points of the underlying system.

Bernoulli convolution measures μ_λ correspond to iterated function schemes of two contractions $f_1(x) = \lambda x$ and $f_2(x) = \lambda x + 1$ taken with equal probabilities $p_1 = p_2 = \frac{1}{2}$. For $0 < \lambda < \frac{1}{2}$ the stationary measure is supported on a Cantor set; for $\lambda = \frac{1}{2}$ the stationary measure is a rescaled Lebesgue measure; and the case $\frac{1}{2} < \lambda < 1$ has been the subject of intensive research for the past 80 years [28]. It is well known that the main characteristics of these measures, such as absolute continuity and dimension, are very sensitive to algebraic properties of λ. In particular, Erdős [7] showed that if λ is the reciprocal of a Pisot number, then the measure μ_λ is not absolutely continuous. Garsia [10] improved on Erdős result and proved that if λ is the reciprocal of a Pisot number, than $\dim \mu_\lambda < 1$. The question whether or not Pisot numbers are the only numbers with this property remains open. Varjú [35] recently demonstrated that, for any transcendental values of λ, the Bernoulli convolution measure μ_λ has full Hausdorff dimension.

In a joint work with Kleptsyn and Pollicott [20] we give an algorithm for computing *uniform* lower bounds for the dimension of the stationary measure of affine iterated function schemes. We demonstrate the effectivity of our method by computing lower bounds for the dimension of Bernoulli convolutions and of the stationary measure corresponding to a scheme of three contractions $f_0(x) = \lambda x$, $f_1(x) = \lambda x + 1$, $f_3(x) = \lambda x + 3$ taken with equal probabilities. (The latter is often referred to as the $\{0, 1, 3\}$–problem see, e.g., an ICM survey by Hochman [14, §2.2].) For each system, our lower bounds have the form of a piecewise constant function with 10000 intervals. A plot of this function for Bernoulli convolutions measures, for the most interesting interval of parameter values, which contains infinitely many accumulation points of Pisot numbers, is shown in Figure 1.

On a less detailed level we give the following uniform bounds.

Theorem 5 (Kleptsyn, Pollicott, and **P.V.**). *The Hausdorff dimension of Bernoulli convolutions μ_λ for any $\frac{1}{2} < \lambda < 1$ satisfies $\dim \mu_\lambda > 0.96399$.*

This improves on the previously known estimate of 0.82 by Hare and Sidorov [12]. The lower bounds we obtain in [20] can be potentially improved with extra computational time and with further refinement of the parameter interval $(\frac{1}{2}, 1)$.

Future research direction #2

In a joint work with Kleptsyn [21], we continue the study of dimension-type characteristics of stationary measures (such as the correlation dimension and the Frostman dimension) of affine iterated function schemes, with a goal to understand how they are interconnected.

3. Resonances for geodesic flows

To a geodesic flow on a hyperbolic infinite-area surface we can associate a zeta function, called the dynamical zeta function, which is defined in terms of closed geodesics by analogy with the classical Riemann zeta function. Let Γ be the set of the primitive closed geodesics on the

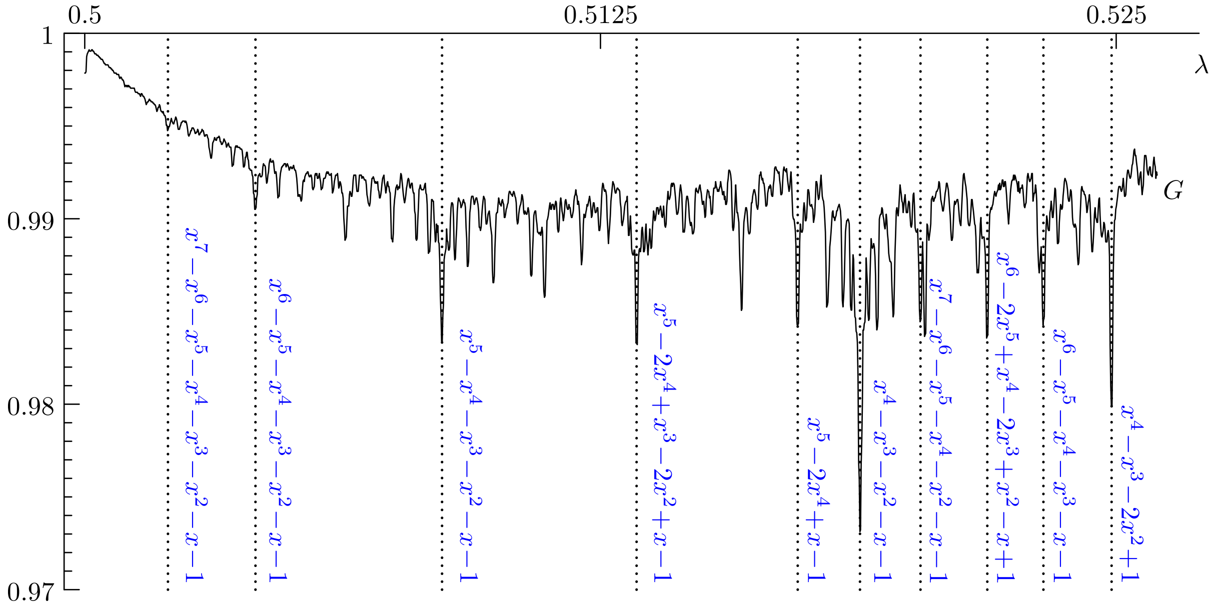

Figure 1: A plot of a piecewise constant function that gives a lower bound on the dimension of the Bernoulli convolution measure $\dim \mu_\lambda$. Some minimal polynomials of algebraic numbers which seem to correspond to the largest drops are shown. These algebraic numbers are reciprocals of Pisot numbers.

surface and let $\ell(\gamma)$ be the length of $\gamma \in \Gamma$. The dynamical zeta function for the geodesic flow in the case of infinite area surfaces is defined [34] by

$$\zeta(s) = \prod_{\gamma\in\Gamma} \left(1 - e^{-s\ell(\gamma)}\right). \tag{2}$$

In the special case of infinite-area surfaces of constant negative curvature, the zeros are the poles of the resolvent of the Laplacian operator and are referred to as *resonances* [3]. From the dynamical viewpoint, they provide information on the long-term behaviour of the flow.

In a beautiful experimental work, Borthwick [4] showed that, in some cases, resonances seem to lie on well-defined curves. In a joint work with Pollicott [29] we explain this phenomenon in the highly symmetric case of surfaces with three funnels with boundary geodesics of the same length, by combining validated numerics and rigorous mathematical estimates. Figure 2 shows a plot of zeros of the dynamical zeta function for this surface.

Future research direction #3

In general, in the case of geodesic flows on infinite-area surfaces, the location of zeros of the dynamical zeta function is poorly understood. Experimental work of my own [36] seems to indicate that the location and distribution of resonances is very sensitive to the Diophantine properties of the ratios of the shortest closed geodesics on the surface. In addition, numerical experiments by Weich [38] and others suggest that resonance-free regions are typical. Similar phenomena have also been observed in the setting of Laplacians on metric graphs. There is an imperative to develop analytic tools to explain these heuristic results.

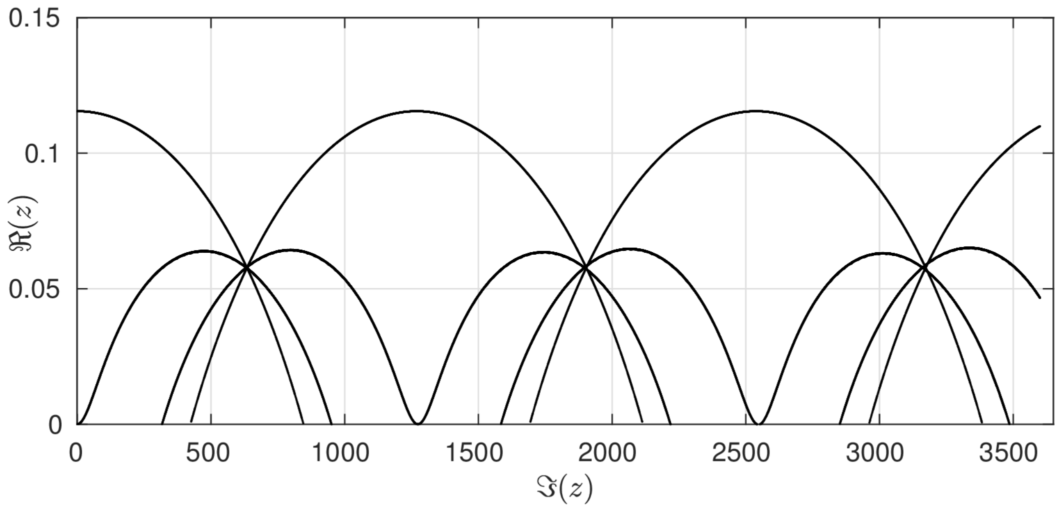

Figure 2: The zeros of the dynamical zeta function (2) associated to a symmetric pair of pants in the critical strip. The individual zeros are so close to one another in the plot that this creates an illusion that they lie on well-defined smooth curves.

4. Lyapunov exponents

Lyapunov exponents are used to characterise stability of typical orbits. We are currently working on an adaptation of the method we developed in [30] for estimating the Hausdorff dimension of limit sets to estimating Lyapunov exponents in various hyperbolic settings. More precisely, we are considering expanding piecewise analytic Markov maps of the interval [31], random matrix products [32], and random walks on Fuchsian groups with respect to probability measures with finite support [33].

The problem of estimating Lyapunov exponents for random matrix products is an important problem in subadditive ergodic theory [19]. Furthermore, these products have recently been connected to the problem of computing the Hausdorff dimension of Bernoulli convolutions for Pisot parameter values [11], [13]. In a joint work with Pollicott we present a high-accuracy method for the computation of Lyapunov exponents for random matrix products [32]. We do not require the matrices to be positive, instead, we impose an extra condition of irreducibility; in other words, we require that the matrices have no a common invariant subspace.

The problem of estimating Lyapunov exponents for random walks on a Fuchsian group arises in the study of harmonic measures on the boundary of the hyperbolic plane. More precisely, let Γ_0 be a finite set of generators of a Fuchsian group and let ν be the probability measure on Γ_0 which gives equal weights to the generators. Let $h(\nu)$ be the Avez asymptotic random walk entropy associated to ν and let $\ell(\nu)$ be the Lyapunov exponent of the random walk.

Theorem 6 (after Ledrappier [23])**.** *The dimension of the harmonic measure μ associated to the random walk (Γ, ν) satisfies*

$$\dim(\mu) = \min\left\{1, \frac{h(\nu)}{\ell(\nu)}\right\}.$$

Future research direction #4

An open conjecture by Kaimanovich—Le Prince [17] states that, for a discrete group of Moebius transformations and any probability measure ν the harmonic measure μ, is always singular. The method for computing Lyapunov exponents of the random walks on Fuchsian groups that we are currently working on will allow us to study this conjecture systematically.

5. The kinematic fast dynamo program

It is well known that stars possess rapidly changing magnetic fields. However, it is not clear how the stars acquired this magnetic field and how they maintain it. The fast dynamo theory seeks an answer to the following simple question: assume that at the time of creation the star has got a seed of magnetic field. Is it possible that the processes taking place in the star amplify the magnetic field so much that the feedback action by the Lorentz force has an influence on the matter of the star? A discrete version of the system of partial differential equations which describe evolution of the magnetic field takes the form [1]

$$\mathcal{D}_\varepsilon \colon B \mapsto \exp(\varepsilon\Delta)(g_* B),$$

where $g\colon M \to M$ is a volume-preserving diffeomorphism of a 3-dimensional domain $M \subset \mathbb{R}^3$, g_* is the induced action on vector fields, Δ is the Laplacian operator in $\mathbb{R}^3$. The question asks whether there exists a g and a divergence-free vector field B_0 such that

$$\liminf_{\varepsilon\to+0} \liminf_{n\to+\infty} \frac{1}{n} \log \|\mathcal{D}_\varepsilon^n B_0\|^2_{L_2(M)} > 0. \tag{3}$$

It is known that for small ε the growth rate is bounded from above by the topological entropy of g. The parameter ε is called the diffusion coefficient, and its positivity prevents the energy $\|\mathcal{D}_\varepsilon^n B_0\|^2_{L_2(M)}$ from growing exponentially, since it mixes vectors of the field B pointing in opposite directions.

In my PhD thesis [37], some tools to study Poincaré maps of flows of this type have been developed and the following result has been established.

Theorem 7 (P.V.). *There exists an area-preserving piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$*

$$\lim_{\varepsilon\to0} \lim_{n\to\infty} \frac{1}{n} \ln \|(\exp(\varepsilon\Delta)F_*)^n B_0\|_{\mathcal{L}_1} > 0.$$

The map F can be realised as a Poincaré map of an incompressible fluid flow filling a compact domain in $\mathbb{R}^3$ (an immersed 3-dimensional manifold with a boundary).

The construction relied on the particulars of the setting and the proof was based on a careful analysis of the action of g_* on vector fields, which was grounded in the Markov partitions approach. This and other limitations prevent the argument from extending to three dimensions. Nevertheless, an important feature of the two-dimensional construction is that it can be realised as a Poincaré map of a flow in $\mathbb{R}^3$, which is a realistic example of the fast dynamo.

Future research direction #5

It is an open problem to extend the results obtained for Poincaré maps to partially hyperbolic flows in the case of unbounded return time in the presence of small random perturbations.

Very recent developments in the theory of anisotropic spaces, in particular the works of Butterley at al. [5], Dyatlov and Zworski [6], Faure and Roy [8], and Galatolo [9] provide us with powerful and flexible tools to analyse dynamical properties of diffeomorphisms and flows by studying spectral properties of the associated transfer operator acting on a suitable anisotropic space and establishing a Lasota–Yorke type inequality. In this rôle, anisotropic spaces essentially replaced Markov partitions, which were used in 1980-90s.

Therefore I am planning to carefully study different spaces of anisotropic distributions to extract the maximum advantage from each method, with a goal to develop a suitable theory of anisotropic spaces which would enable a study of induced actions on vector fields. An application of this theory of a particular interest would be the kinematic fast dynamo problem described above.

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