Hausdorff dimension estimates applied to Lagrange and Markov spectra, Zaremba theory, and limit sets of Fuchsian groups

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Abstract

In this note we will describe a simple and practical approach to get rigorous bounds on the Hausdorff dimension of limits sets for some one dimensional Markov iterated function schemes. The general problem has attracted considerable attention, but we are particularly concerned with the role of the value of the Hausdorff dimension in solving conjectures and problems in other areas of mathematics. As our first application we confirm, and often strengthen, conjectures on the difference of the Lagrange and Markov[1] spectra in Diophantine analysis, which appear in the work of Matheus and Moreira [36]. As a second application we (re-)validate and improve estimates connected with the Zaremba conjecture in number theory, used in the work of Bourgain–Kontorovich [2], Huang [21] and Kan [31]. As a third more geometric application, we rigorously bound the bottom of the spectrum of the Laplacian for infinite area surfaces, as illustrated by an example studied by McMullen [41].

In all approaches to estimating the dimension of limit sets there are questions about the efficiency of the algorithm, the computational effort required and the rigour of the bounds. The approach we use has the virtues of being simple and efficient and we present it in Section 3 in a way that is straightforward to implement.

These estimates apparently cannot be obtained by other known methods.

1 Introduction

We want to consider some interesting problems where a knowledge of the exact value of the Hausdorff dimension of some appropriate set plays an important role in an apparently unrelated area. For instance we consider the applications to Diophantine approximation and the difference between the Markov and Lagrange spectra, denominators of finite continued fractions and the Zaremba conjecture, and the spectrum of the Laplacian on certain Riemann surfaces. A common feature is that the progress on

*The first author is partly supported by ERC-Advanced Grant 833802-Resonances and EPSRC grant EP/T001674/1 the second author is partly supported by EPSRC grant EP/T001674/1.

[1]Markov's name will appear in this article in two contexts, namely those of Markov spectra in number theory and the Markov condition from probability theory. Since the both notions are associated with the same person (A. A. Markov, 1856–1922), we have chosen to use the same spelling, despite the conventions often used in these different areas.

these topics depends on accurately computing the dimension of certain limit sets for iterated function schemes.

The sets in question are dynamically defined sets given by Markov iterated function schemes. The traditional approach to estimating the dimension of such sets is to use a variant of what is sometimes called a finite section method. This typically involves approximating the associated transfer operator by a finite rank operator and deriving approximations to the dimension from its maximal eigenvalue. This method originated with traditional Ulam method and there are various applications and refinements due to Falk—Nussbaum [9], Hensley [20], McMullen [41] and others. A second approach, which we will call the periodic point approach, uses fixed points for combinations of contractions in the iterated function schemes [25]. This approach works best for a small number of analytic branches whereas the finite section method often works more generally. However, in both of these approaches additional work is needed to address the important issue of validating numerical results. In the case of periodic point method there has been recent progress in getting rigorous estimates for Bernoulli systems [26], but it can still be particularly difficult to get rigorous bounds in the case of Markov maps. In the case of Ulam's method the size of matrices involved in approximation can make it hard to obtain reasonable bounds.

In this note we want to use a different approach which has the twin merits of giving both effective estimates on the dimension and ensuring the rigour of these values. This is based on combining elements of the methods of Babenko–Yur'ev [1] and Wirsing [55] originally developed for the Gauss map. We will describe this in more detail in § 3.

To complete the introduction we will discuss our main applications.

Application I: Markov and Lagrange spectra

As our first application, we can consider the work of Matheus and Moreira [36] on estimating the size of the difference of two subsets of the real line called the Markov spectrum $\mathcal{M} \subset \mathbb{R}^+$ and the Lagrange spectrum $\mathcal{L} \subset \mathbb{R}^+$. The two sets play an important role in Diophantine approximation theory and an excellent introduction to topics in this subsection is [8].

By a classical result of Dirichlet from 1840, for any irrational number α there are infinitely many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$. For each irrational α we can choose the largest value $\ell(\alpha) > 1$ such that the inequality $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\ell(\alpha) q^2}$ still has infinitely many solutions with $\frac{p}{q} \in \mathbb{Q}$. An equivalent definition would be

$$\ell(\alpha) \colon = \left(\inf_{p,q \in \mathbb{Z}, q \neq 0} |q(q\alpha - p)| \right)^{-1}.$$

For example, we know that $\ell\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$, $\ell\left(1 - \sqrt{2}\right) = \sqrt{8}$, etc. The Hurwitz irrational number theorem states that for any irrational α there are infinitely many rationals $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{5}}$. This implies, in particular, that $\ell(\alpha) \geq \sqrt{5}$ for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 1.1. *The set* $\mathcal{L} = \{\ell(\alpha) \colon \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$ *is called the* Lagrange spectrum*.*

There is another characterisation of elements of Lagrange spectrum in terms of continued fractions [8]. We denote the infinite continued fraction of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ by

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. Assume that for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have

$$\alpha = [a_0; a_1, a_2, \ldots] = [a_0; a_1, a_2, \ldots, a_n, \alpha_{\mathbf{n+1}}],$$

in other words for the n'th rational approximation we may write

$$\left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{\left(\alpha_{n+1} + \frac{q_{n-1}}{q_n} \right) q_n^2}.$$

Then

$$\ell(\alpha) = \limsup_{n \to \infty} \left(\alpha_{n+1} + \frac{q_{n-1}}{q_n} \right).$$

Replacing lim sup in the latter formula by supremum, we get the definition of the Markov spectrum.

Definition 1.2. *In the notation introduced above, let*

$$\mu(\alpha) = \sup_n \left(\alpha_{n+1} + \frac{q_{n-1}}{q_n} \right).$$

The set $\mathcal{M} = \{\mu(\alpha) \colon \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$ *is called the* Markov spectrum.

There is an equivalent definition of the Markov spectrum in terms of quadratic forms. Both notions were suggested by Markov in 1879–80 [38], [39].

Naturally, the sets $\mathcal{L}$ and $\mathcal{M}$ have many similarities. The smallest value for each is $\sqrt{5}$ and in $[\sqrt{5}, 3]$ both sets are countable and agree, i.e.,

$$\mathcal{L} \cap [\sqrt{5}, 3] = \mathcal{M} \cap [\sqrt{5}, 3] = \{\sqrt{5}, \sqrt{8}, \sqrt{221}/5 \cdots\}.$$

Furthermore, Freiman [14], following earlier work of Hall [18], computed an explicit constant, called Freiman constant $\sqrt{20} < c_F < \sqrt{21}$, such that

$$\mathcal{L} \cap [c_f, +\infty) = \mathcal{M} \cap [c_F \ldots, +\infty) = [c_F, +\infty).$$

The half-line $[c_F, +\infty)$ is known as Hall's ray. Nevertheless, these two sets are actually different. In particular, Tornheim [52] showed $\mathcal{L} \subseteq \mathcal{M}$ and Freiman [13] showed $\mathcal{L} \neq \mathcal{M}$.

In a recent work Matheus and Moreira [36], §B.2 give upper bounds on the Hausdorff dimension $\dim_H(\mathcal{M} \setminus \mathcal{L})$ in terms of the Hausdorff dimension of limits sets of specific Markov Iterated Function Schemes. Using the approach presented in this article we compute the Hausdorff dimensions of the sets concerned, and combining our numerical estimates in §4.1 with the intricate analysis of [36] we obtain the following result (the proof is computer-assisted).

Theorem 1.3. *We have the following bounds on the dimension of parts of* $\mathcal{M} \setminus \mathcal{L}$

1. $\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{5}, \sqrt{13})) < 0.7281096$;

2. $\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{13}, 3.84)) < 0.8552277$;
3. $\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (3.84, 3.92)) < 0.8710525$;
4. $\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (3.92, 4.01)) < 0.8110098$; *and*
5. $\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{20}, \sqrt{21})) < 0.8822195$.

In particular, taking into account the known bound $\dim_H(\mathcal{M} \setminus \mathcal{L} \cap (4.01, \sqrt{20})) < 0.873316$ ([36], (B.6)) on the remaining interval we obtain an upper bound of

$$\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.8822195.$$

Note that this confirms the conjectured upper bound $\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.888$ ([36], (B.1)) and improves on the earlier rigorous bound of $\dim_H(\mathcal{M} \setminus \mathcal{L}) < 0.986927$ ([36], Corollary 7.5 and [42], Theorem 3.6).

For the purposes of comparison, we present the bounds in Theorem 1.3 with the previous rigorous bounds given on different portions of $\mathcal{M} \setminus \mathcal{L}$ in Figure 1.

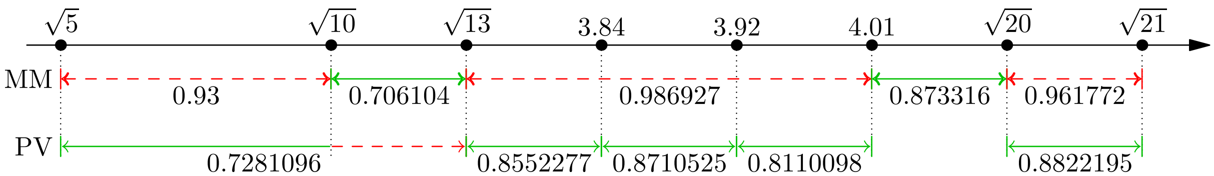

Figure 1: Comparison of old and new upper bounds.

In the present work we will be looking for both lower and upper bounds. In order to give a clearer presentation of the results we will use the following notation

Notation 1.4. *We abbreviate* $a \in [b-c, b+c]$ *as* $a = b \pm c$.

In order to get a lower bound on the difference of the Lagrange and Markov spectra Matheus and Moreira ([36], Theorem 5.3) showed that there is a lower bound $\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim_H(E_2)$ where $E_2 \subset [0,1]$ denotes the Cantor set of irrational numbers with infinite continued fraction expansions whose digits are either 1 or 2.[2] The study of the dimension of this set was initiated by Good in 1941 [17]. There are various estimates on $\dim_H(E_2)$ including [26] where the dimension was computed to 100 decimal places using periodic points. In §7 we will recover and improve on this estimate giving an estimate accurate to 200 decimal places and thus we deduce the following result.

Theorem 1.5. [3]

$$\begin{aligned}\dim_H(E_2) = 0.&5312805062\,7720514162\,4468647368\,4717854930\,5910901839\,8779888397\\ &8039275295\,3564383134\,5918109570\,1811852398\,8042805724\,3075187633\\ &4223893394\,8082230901\,7869596532\,8712235464\,2997948966\,3784033728\\ &7630454110\,1508045191\,3969768071\,3 \pm 10^{-201}.\end{aligned}$$

Details on the proof of this bound appear in §4.1.2. Whereas it may not be clear why a knowledge of $\dim_H(E_2)$ to 200 decimal places is beneficial, it at least serves to illustrate the effectiveness of the method we are using compared with earlier approaches.

[2] A slight improvement on this lower bound is described in the book "Classical and Dynamical Markov and Lagrange Spectra: Dynamical, Fractal and Arithmetic Aspects" by D. Lima, C. Matheus, C. G Moreira, S. Romana.

[3] The calculation was done using Mathematica. The validity of the estimate depends on the internal error estimates of the software.

Application II: Zaremba theory

A second application is to problems on finite continued fractions related to the Zaremba conjecture. The Zaremba conjecture [57] was formulated in 1972, motivated by problems in numerical analysis. It deals with the denominators that can occur in finite continued fraction expansions using a uniform bound on the digits. A nice account appears in the very informative survey of Kontorovich [33].

Zaremba conjecture. For any natural number $q \in \mathbb{N}$ there exists p (coprime to q) and $a_1, \cdots, a_n \in \{1,2,3,4,5\}$ such that

$$\frac{p}{q} = [0; a_1, \cdots, a_n] : = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$

Let us denote for each for $N \geq 1$ and $m \geq 2$,

$$D_m(N) :=$$
$$\mathrm{Card}\left\{1 \leq q \leq N \mid \exists p \in \mathbb{N}, (p,q)=1,\ a_1, \cdots, a_n \in \{1,2,\cdots,m\} \text{ with } \frac{p}{q} = [0; a_1, \cdots, a_n]\right\},$$

i.e., the number of $1 \leq q \leq N$ which occur as denominators of finite continued fractions using digits $|a_i| \leq m$. The Zaremba conjecture would correspond to $D_5(N) = N$ for all $N \in \mathbb{N}$. The conjecture remains open, but Huang [21], building on work of Bourgain and Kontorovich [2], proved the following version of Zaremba conjecture.

Theorem 1.6 (Bourgain—Kontorovich, Huang)**.** *There is a density one version of the Zaremba conjecture, i.e.,*

$$\lim_{N \to +\infty} \frac{D_5(N)}{N} = 1.$$

There have been other important refinements on this result by Frolenkov–Kan [15], [16], Kan [29], [30], Huang [21] and Magee–Oh–Winter [35].

Let us introduce for each $m \geq 2$,

$$E_m : = \{[0; a_1, a_2, \cdots] \mid a_n \in \{1,2,\cdots,m\} \text{ for all } n \in \mathbb{N}\}$$

which is a Cantor set in the unit interval. Originally, Bourgain—Kontorovich [2] proved an analogue to Theorem 1.6 for $D_{50}(N)$. Amongst other things, their argument, related to the circle method, used the fact that the Hausdorff dimension $\dim_H(E_{50})$ is sufficiently close to 1 (more precisely, $\dim_H(E_{50}) > \frac{307}{312}$). In Huang's refinement of their approach, he reduced m to 5, i.e. replaced the alphabet $\{1,2,\cdots,50\}$ with $\{1,2,3,4,5\}$, as in the statement of Theorem 1.6. In Huang's approach, it was sufficient to show that $\dim_H(E_5) > \frac{5}{6}$. In [26] there is an explicit rigorous bound on the Hausdorff dimension of this set which confirms this inequality. The approach used there is the periodic point method, whereas in this article we use a different method to confirm and improve on these bounds.

As another example, we recall the following result for $m = 4$ and the smaller alphabet $\{1,2,3,4\}$.

Theorem 1.7 (Kan [30])**.** *For the alphabet* $\{1, 2, 3, 4\}$ *there is a positive density version of the Zaremba conjecture, i.e.,*

$$\liminf_{N\to+\infty} \frac{D_4(N)}{N} > 0.$$

The proof of the result is conditional on the lower bound $\dim_H(E_4) > \frac{\sqrt{19}-2}{3}$. In [30] this inequality is attributed to Jenkinson [24], where this value was, in fact, only heuristically estimated. In [26] there is an explicit rigorous bound on the Hausdorff dimension of this set which confirms this inequality. The approach used there is the periodic point method, whereas in this article we give a different method to confirm and improve on these bounds, as well as give new examples. These results are presented in §4.2.

Application III: Schottky group limit sets

A third application belongs to the area of hyperbolic geometry. The two dimensional hyperbolic space can represented as the Poincaré disc $\mathbb{D}^2 = \{z \in \mathbb{C} \colon |z| < 1\}$ with the Poincaré metric $ds^2 = 4(1 - |z|^2)^{-2}$. A Fuchsian group Γ is a discrete group of isometries of the two dimensional hyperbolic space. In particular, the factor space $\mathbb{D}^2/\Gamma$ is a surface of constant curvature $\kappa = -1$.

We can associate to the Fuchsian group Γ the limit set $X_\Gamma \subseteq \partial D = \{z \in \mathbb{C} \colon |z| = 1\}$ defined to be the Euclidean limit points of the orbit $\{g0 \colon g \in \Gamma\}$. In the event that Γ is cocompact, the quotient $\mathbb{D}^2/\Gamma$ is a compact surface, and thus the limit set will be equal to the entire unit circle. On the other hand, if Γ is a Schottky group then the limit set will be a Cantor subset of the unit circle (of Hausdorff dimension strictly smaller than 1).

In the particular case that Γ is a Schottky group the space $\mathbb{D}^2/\Gamma$ is a surface of infinite area. It is known [6] that the classical Laplace—Beltrami operator has positive real spectra and in particular, its smallest eigenvalue $\lambda_\Gamma > 0$ is strictly positive. There is a close connection between the spectral value λ_Γ and the Hausdorff dimension $\dim_H(X_\Gamma)$. More precisely, we have a classical result (see [51])

$$\lambda_\Gamma = \min\left\{\frac{1}{4}, \dim_H(X_\Gamma)(1 - \dim_H(X_\Gamma))\right\}.$$

Next we want to consider a concrete example of a Schottky group.

Example 1.8. *McMullen [41] considered the Schottky group* $\Gamma = \langle R_1, R_2, R_3\rangle$ *generated by reflections* $R_1, R_2, R_3 : \mathbb{D}^2 \to \mathbb{D}^2$ *in three symmetrically placed geodesics with end points* $e^{\pi i(2j+1)/6}$ *with* $j = 1, \cdots, 6$ *on the unit circle (Figure 2).*

The limit set X_Γ can be written as a limit set of a suitable Markov iterated function scheme, or, more precisely, a directed Markov graph system [37]. Applying the method described in this article we can estimate the dimension of the limit set and thus the lowest eigenvalue of the Laplacian.

Theorem 1.9. *In notation introduced above, the dimension of the limit set of the Schottky group* Γ *satisfies*

$$\dim_H(X_\Gamma) = 0.295546475 \pm 5 \cdot 10^{-9}$$

and the smallest value of the Laplacian satisfies

$$\lambda_\Gamma = 0.2081987565 \pm 2.5 \cdot 10^{-9}$$

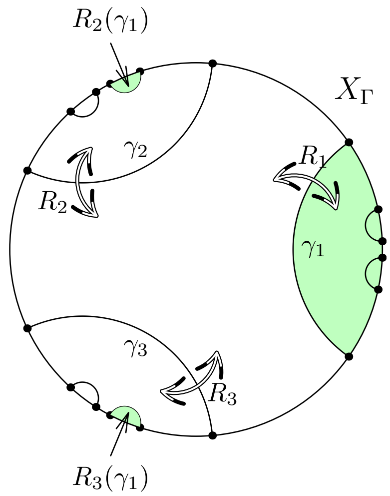

Figure 2: Group Γ generated by reflections R_1, R_2, and R_3 in the geodesics γ_1, γ_2, and γ_3.

Finally, the same approach we have used in these applications can also be used to estimate the Hausdorff dimension of the various dynamically defined limit sets of iterated function schemes which have been considered by other authors cf. [20]. We will return to this in §4.9, where we verify and improve estimates of Hensley [20] and Jenkinson [24] for some iterated function schemes and estimates of Chousionis at al. [7] for some countable iterated function systems.

In the next section we will describe the general setting of one dimensional Markov iterated function schemes which is the main focus of our study and the key to proving these theorems.

2 Definitions and Preliminary results

In this section we collect together some of the background material we require.

2.1 Hausdorff Dimension

The following classical definition of the Hausdorff dimension is well known, and an excellent general reference is the textbook of Falconer [11]. Given $X \subset \mathbb{R}^+$ for each $0 < s < 1$ and $\delta > 0$ we define the s-dimensional *Hausdorff content* of X by

$$H^s_\delta(X) = \inf_{\substack{\mathcal{U}=\{U_i\},\\ |U_i|\leq\delta}} \sum_i |U_i|^s,$$

where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ of X with each open set U_i having diameter $|U_i| \leq \delta$. The s-dimensional *Hausdorff outer measure* of X is given by

$$H^s(X) = \lim_{\delta\to 0} H^s_\delta(X) \in [0, +\infty].$$

Finally, the *Hausdorff dimension* of X is defined as infimum of values s for which the outer measure vanishes:

$$\dim_H(X) = \inf\{s \mid H^s(X) = 0\}.$$

2.2 Markov iterated function schemes

We say that the contractions $T_j\colon [0,1] \to [0,1]$ $(j = 1, \cdots, d)$ satisfy the *Open Set Condition* if there exists a non-empty open set $U \subset [0,1]$ such that the images $\{T_j U\}_{j=1}^d$ are pairwise disjoint. This will be the case in all the examples we consider.

We begin with the definition of a one dimensional Markov iterated function scheme. Recall that a matrix M is called aperiodic if there exists $n \geq 1$ such that $M^n > 0$, i.e., all of the entries are strictly positive.

Definition 2.1. *Let $d \geq 2$. A Markov iterated function scheme consists of:*

1. *a family $T_j\colon [0,1] \to [0,1]$ $(j = 1, \cdots, d)$ of $C^{1+\alpha}$ contractions satisfying the Open Set Condition; and*
2. *an aperiodic $d \times d$ matrix M with entries 0 and 1, which gives the* Markov condition.

We can define the limit set *of $\{T_j\}_{j=1}^d$ with respect to the matrix M by*

$$X_M = \left\{ \lim_{n\to+\infty} T_{j_1} \circ \cdots \circ T_{j_n}(0) \mid j_k \in \{1, \cdots, d\}, M(j_k, j_{k+1}) = 1, 1 \leq k \leq n-1 \right\}.$$

Remark 2.2. In one of the examples we consider, the matrix M has an entire column of zeros. In this case, we can remove the contraction corresponding to this column from the iterated function scheme without changing the limit set. This corresponds to removing the row and the column corresponding to this contraction from the matrix M.

Remark 2.3. More generally, let M be a $k \times k$ matrix with entries 0 or 1, and assume its rows and columns are indexed by $\{1, \cdots, k\}$. Given $r, s \in \{1, \cdots, k\}$ we say s is accessible from r if there exists $n \geq 1$ with $M^n(r,s) \geq 1$. After reordering the index set, if necessary, we can assume that if $n \geq 1$ with $M^n(r,s) \geq 1$ then $s \geq r$. We can then define an equivalence relation on $\{1, \cdots, k\}$ by $r \sim s$ if both s is accessible from r and also r is accessible from s and assume that there are q distinct equivalence classes $[j_1], \cdots, [j_q]$. The matrix M then takes the form of sub-matrices $M_1, \cdots, M_q$ on the diagonal indexed by the equivalence classes $[j_1], \cdots, [j_q]$, say, with any other non-zero entries appearing only above the main diagonal [**?**, Ch. 1].

In particular, each matrix M_j is irreducible, i.e. for pair of indices (r,s) there exists $n = n(j,r,s)$ such that $M_j^n(r,s) \geq 1$. The period d_j of M_j is the greatest common divisor of $n(j,r,s)$ for all pairs of indices (r,s). Finally, after further reordering of the index set, if necessary, the dth power M_j^d takes the form of aperiodic sub-matrices $M_{j1}, M_{j2}, \cdots, M_{jp}$ on the diagonal (i.e., there exists $n = n(j,k) \geq 1$ such that $\forall r, s$ in the index set for M_{jk} we have $M_{jk}^n(r,s) \geq 1$).

We can also consider the iterated function schemes $X_{M_{jk}}$ associated to the matrices M_{jk}, where the corresponding contractions being d-fold compositions of the original contractions. Then the iterated function scheme with contractions T_i where $i \in \{1, \cdots, k\}$ with the limit set X_M has dimension $\dim(X_M) = \max_{j,k} \dim(X_{M_{jk}})$.

Definition 2.1 is a special case of a more general graph directed Markov system [37], where the contractions T_j may have different domains and ranges, to which our analysis also applies. However, the above definition suffices for the majority of our applications, although in the case of Fuchsian—Schottky groups a more general setting is implicitly used.

Remark 2.4. For Markov iterated function schemes, the Hausdorff dimension coincides with the box counting dimension [45, Ch. 5], see also [46], which has a slightly easier definition. More precisely, for $\varepsilon > 0$ we denote by $N(\varepsilon)$ the smallest number of ε-intervals required to cover X. We define the Box dimension to be

$$\dim_B(X_M) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

provided the limit exists. Then $\dim_H(X_M) = \dim_B(X_M)$. However, we needed to introduce the definition of Hausdorff dimension, for the benefit of the statements of results on Markov and Lagrange spectra.

2.3 Pressure function

We would like to use the Bowen—Ruelle formula [49] to compute the value of the Hausdorff dimension of the limit set of a Markov iterated function scheme. We will use the following notation.

Notation 2.5. *In what follows, $A(t) \lessapprox B(t)$ denotes that there exists $C > 0$ such that $A(t) \leq C \cdot B(t)$. We write $A(t) \asymp B(t)$ if there exist constants $C_1, C_2 > 0$ such that $C_1 A(t) \leq B(t) \leq C_2 A(t)$ for all sufficiently large t.*

Before giving the statement we first need to introduce the following defintion.

Definition 2.6. *A pressure function $P\colon [0,1] \to \mathbb{R}$ associated to a system of contractions $\{T_j\}_{j=1}^k$ with Markov condition defined by a matrix M is given by*

$$P(t) = \lim_{n \to +\infty} \frac{1}{n} \log \left(\sum_{\substack{M(j_1,j_2)=\cdots= \\ M(j_{n-1},j_n)=1}} \left| (T_{j_1} \circ \cdots \circ T_{j_n})'(0) \right|^t \right),$$

where the summation is taken over all compositions $T_{j_1} \circ \cdots \circ T_{j_n} \colon [0,1] \to [0,1]$ which are allowed by the Markov condition and the summands are the absolute values of the derivatives of these contractions at a fixed reference point (which for convenience we take to be 0) raised to the power t.

The pressure function depends on the matrix M but we omit this in the notation. In the present context the function is well defined as the limit in this definition of $P(t)$ always exists. There are various other definitions of the pressure, and we refer the reader to the books [44] and [54] for more details. A sketch of the graph of the pressure function is given in Figure 3.

The following well known connection between $\dim_H(X_M)$ and the pressure function is useful for practical applications.

Lemma 2.7 (Bowen [3], Ruelle [49, Proposition 4])**.** *In the setting introduced above, the pressure function of a Markov Iterated Function Scheme has the following properties:*

1. *$P(t)$ is a smooth monotone strictly decreasing analytic function; and*
2. *The Hausdorff dimension of the limit set is the unique zero of the pressure function i.e., $P(\dim_H(X_M)) = 0$.*

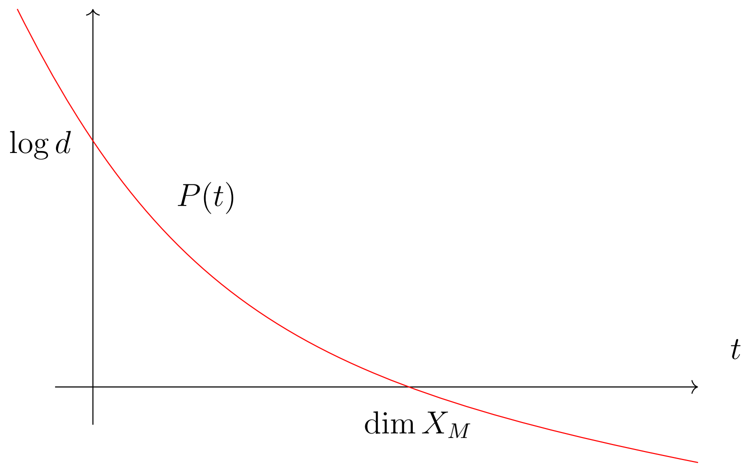

Figure 3: A typical plot of the pressure function P.

The general result of Ruelle extended a more specific posthumous result of Bowen on the Hausdorff dimension for quasi-circles. Thus the problem of estimating $\dim_H(X_M)$ is reduced to the problem of locating the zero of the pressure function.

Remark 2.8. To understand the connection between the pressure and the Hausdorff dimension, described in Lemma 2.7, we can consider covers of X_M of the form $\mathcal{U} = \{T_{i_1} \circ \cdots \circ T_{i_n} U\}$, where $T_{i_1} \circ \cdots \circ T_{i_n}$ are allowed compositions (i.e. $M(i_1, i_2) = \ldots = M(i_{n-1}, i_n) = 1$) and $U \supset [0,1]$ is an open set. The diameters of the elements of this cover for large n are comparable to the absolute values of the derivatives $(T_{i_1} \circ \cdots \circ T_{i_n})'(0)$. In particular, by the mean value theorem, $\mathrm{diam}(T_{i_1} \circ \ldots \circ T_{i_n}(U)) \le \sup_{y \in U} |(T_{i_1} \circ \ldots \circ T_{i_n})'(y)|$ and taking into account

$$\frac{|(T_{i_1} \circ \ldots \circ T_{i_n})'(y)|}{|(T_{i_1} \circ \ldots \circ T_{i_n})'(0)|} \le \sup_{z \in U} \exp\left(\left(\log |(T_{i_1} \circ \ldots \circ T_{i_n})'(z)|\right)'\right) < +\infty.$$

If $P(t) = 0$ then for any $t_0 > t$ we have that $P(t_0) < 0$ and therefore the Hausdorff content

$$H_\delta^{t_0}(X_M) \lessapprox \sum_{\substack{M(i_1,i_2)=\cdots \\ =M(i_{n-1},i_n)=1}} \left|(T_{i_1} \circ \cdots \circ T_{i_n})'(0)\right|^{t_0}$$

for n sufficiently large. In particular, letting $n \to +\infty$ we can deduce from the definition of pressure that the right hand side of the inequality tends to zero and therefore $H^{t_0}(X) = 0$. We then conclude that outer measure vanishes and thus $\dim_H(X_M) \le t_0$ and, since $t_0 > t$ was arbitrary, we have $\dim_H(X_M) \le t$ (see §2.1). For the reverse inequality see Remark 2.14.

2.4 Transfer operators for Markov iterated function schemes

We explained in the previous subsection that the dimension $\dim_H(X_M)$ corresponds to the zero of the pressure function P. This function can also be expressed in terms of a linear operator on a suitable space of Hölder functions.

In order to accommodate the Markov condition it is convenient to consider the space consisting of d disjoint copies of $[0,1]$, which we denote by $S := \oplus_{j=1}^d [0,1] \times \{j\}$

and to introduce the maps

$$T_j\colon S \to S, \qquad T_j\colon (x,k) \mapsto (T_j(x), j).$$

We omit the dependence on d where it is clear.

The transfer operator associated to $\{T_j\}$ is a linear operator acting on the space of Hölder functions $C^\alpha(S)\colon = \oplus_{j=1}^d C^\alpha([0,1]\times\{j\})$, where $0<\alpha\le 1$. This is a Banach space with the norm $\|f_1,\cdots,f_d\| = \max_{1\le j\le d}\{\|f_j\|_\alpha + \|f_j\|_\infty\}$ where $\|f_j\|_\alpha = \sup_{x\neq y}\frac{|f_j(x)-f_j(y)|}{|x-y|^\alpha}$. For many applications we can take $\alpha = 1$.

Definition 2.9. *For $0\le t\le 1$ the transfer operator $\mathcal{L}_t\colon C^\alpha(S)\to C^\alpha(S)$ for the scheme $T_j\colon S\to[0,1]\times\{j\}$ is defined by the formula $\mathcal{L}_t\colon (f_1,\cdots,f_d)\mapsto(F_1^t,\cdots,F_d^t)$ where*

$$F_k^t(x,k) = \sum_{j=1}^d M(j,k)\cdot f_j(T_j(x,k))|T_j'(x,k)|^t.$$

We omit the dependence on M where it is clear.

Remark 2.10. For some applications where the contractions form a Bernoulli system (i. e. there are no restrictions and all the entries of M are 1) it is sufficient to take one copy of the interval $[0,1]$, i. e. $d=1$. This applies, for example, in all of the Zaremba examples. In this case the transfer operator takes simpler form

$$\mathcal{L}_t : C^\alpha([0,1]) \to C^\alpha([0,1]) \qquad \mathcal{L}_t : f \mapsto \sum_{j=1}^d f(T_j)|T_j'|^t. \tag{1}$$

Definition 2.11. *We say that a function $\underline{f}\in C^\alpha(S)$ is positive if $f=(f_1,\ldots,f_d)$ and each $f_j\in C^\alpha([0,1]\times\{j\})$ takes only positive values.*

The connection between the linear operator $\mathcal{L}_t$ and the value of the pressure function $P(t)$ comes from the first part of the following version of a standard result, cf. [47].

Lemma 2.12 (after Ruelle)**.** *Assume that the matrix M is aperiodic. In terms of the notation introduced above, the spectral radius of $\mathcal{L}_t$ is $e^{P(t)}$. Furthermore,*

1. *$\mathcal{L}_t$ has an isolated maximal eigenvalue $e^{P(t)}$ associated to a positive eigenfunction $\underline{h}\in C^\alpha(S)$ and a positive eigenprojection[4] $\eta\colon C^\alpha(S)\to\langle\underline{h}\rangle$; and*
2. *for any $\underline{f}\in C^\alpha(S)$ we have*

$$\|e^{-nP(t)}\mathcal{L}_t^n\underline{f} - \eta(\underline{f})\|_\infty \to 0 \text{ as } n\to+\infty.$$

Proof. For the reader's convenience we sketch a proof adapted to the present context (see [48] for a different proof).

[4]For a general introduction see [32], Ch.3, §6; we say that the eigenprojection is positive if elements of the positive cone are mapped into $\mathbb{R}^+\underline{h}$.

Part 1. Let $c = \max_{1\le i\le d} \|T_i'\|_\infty < 1$ and $A = \max_{1\le i\le d} \|\log |T_i'|\|_\alpha < 1$ and choose B sufficiently large that $A + Bc < B$. We can define a convex space

$$\mathcal{C} = \left\{ \underline{f} = (f_1, \cdots, f_d) = C^\alpha(S) \colon \begin{array}{l} \forall 1 \le j \le d, \forall 0 \le x \le 1, f_j(x) \le 1 \\ \forall 1 \le j \le d, \forall 0 \le x, y \le 1, f_j(x) \le f_j(y) e^{B|x-y|^\alpha} \end{array} \right\}$$

which is uniformly compact by the Arzela–Ascoli theorem. Given $\underline{f} \in \mathcal{C}$ and $n \ge 1$ we can define $\Phi_n(\underline{f}) = \frac{\mathcal{L}_t(\underline{f}+\frac{1}{n})}{\|\mathcal{L}_t(\underline{f}+\frac{1}{n})\|_\infty} \in C^\alpha(S)$. For all $0 \le x, y \le 1$ and $\underline{f} = (f_1, \cdots, f_d) \in \mathcal{C}$ and each $1 \le j \le d$, using $0 \le t \le 1$, we get

$$F_k^t(x) \le e^{(A+Bc)|x-y|^\alpha} \sum_{j=1}^{d} M(j,k) \cdot f_j(T_j(y)) |T_j'(y)|^t \le e^{B|x-y|^\alpha} F_k^t(y).$$

Hence for each $n \ge 1$ and $0 \le x, y \le 1$ we have $\Phi_n(\underline{f})(x) \le e^{B|x-y|^\alpha} \Phi_n(\underline{f})(y)$ and thus $\|\Phi_n(\underline{f})(x)\|_\infty = 1$. Therefore, for each $n \ge 1$ the map $\Phi_n : \mathcal{C} \to \mathcal{C}$ is well defined and has a nontrivial fixed point $\underline{h}_n = \Phi_n(\underline{h}_n) \ne 0$ by the Schauder fixed point theorem. We can take an accumulation point $\underline{h}$ of $\{\underline{h}_n\} \subset \mathcal{C}$. This is an eigenfunction of $\mathcal{L}_t$ for the eigenvalue $\lambda = \|\mathcal{L}_t \underline{h}\|_\infty$ and we can choose x_0 and j, say, with $|h_j(x_0)| = \|h\|_\infty$.

Since M is aperiodic we can choose N sufficiently large such that for arbitrary k and j_1 we have a contribution $h_k(T_{j_1} \circ \cdots \circ T_{j_N} x_0) |(T_{j_1} \circ \cdots \circ T_{j_N})'(x_0)|^t > 0$ to $\mathcal{L}_t^N$. It follows from the second condition of the definition of $\mathcal{C}$ that $h_k(x_1) > 0$ where we let $x_1 = T_{j_1} \circ \cdots \circ T_{j_N} x_0$. Since $\underline{h} \in \mathcal{C}$ we conclude that for any x we have $h_k(x) \ge h_k(x_1) e^{-B} > 0$. By Definition 2.6 of the pressure function,

$$P(t) = \lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{L}_t^n \underline{1}\| = \lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{L}_t^n \underline{h}\| = \log \lambda.$$

Part 2. Let $\Delta(\underline{h}) : C^\alpha(S) \to C^\alpha(S)$ denote multiplication by $\underline{h}$ and then introduce the linear operator $\mathcal{K} = \frac{1}{\lambda} \Delta(\underline{h})^{-1} \mathcal{L}_t \Delta(\underline{h})$, which now satisfies $\mathcal{K}\underline{1} = \underline{1}$. This implies that for $\underline{f} \in C^\alpha(S)$ we have that for each $\underline{x} \in S$:

(a) the sequence $\inf_{0\le x\le 1} \mathcal{K}^n \underline{f}(\underline{x})$ $(n \ge 1)$ is monotone increasing and bounded, and

(b) the sequence $\sup_{0\le x\le 1} \mathcal{K}^n \underline{f}(\underline{x})$ $(n \ge 1)$ is monotone decreasing and bounded.

The limits are fixed points for $\mathcal{K}$, and thus without too much effort we can deduce that $\mathcal{K}^n \underline{f}$ converges uniformly to a constant function. Reformulating this for the original operator $\mathcal{L}_t$ gives the claimed result. □

Remark 2.13. The properties of the transfer operator in Lemma 2.12 now clarify the reasoning behind the remaining parts of Lemma 2.7. With a little more work (and the Fortet—Doeblin inequality) one can show that $e^{P(t)}$ is an isolated eigenvalue of $\mathcal{L}_t$ and thus has an analytic dependence on t (compare with [44]).

Remark 2.14. To explain the idea behind Lemma 2.7, it remains to recall why if $P(t) = 0$ then $\dim_H(X) \ge t$.

Let $\mathcal{M}$ be the space of probability measures supported on S with the weak star topology. By Alaoglu's theorem this space is compact. The map $\Psi : \mathcal{M} \to \mathcal{M}$ defined by $[\Psi\mu](\underline{g}) = e^{-P(t)} \mu(\mathcal{L}_t \underline{g})$ for $\underline{g} \in C^\alpha(S)$ has the fixed point η.

We would like to apply the mass distribution principle (cf. [11, p. 67]) to the measure η. In other words, to show that t corresponding to $P(t) = 0$ is a lower bound on $\dim_H(X)$, it is sufficient to show that there exists $C > 0$ for which on any small interval V we have $\eta(V) \le C|V|^t$. Moreover, it suffices to consider intervals of the form $V = T_{i_1} \circ \ldots \circ T_{i_n} U$ for some $i_1, \ldots, i_n$. In particular, providing $P(t) = 0$ for an allowed sequence $i_1, \ldots, i_n$ we have

$$\eta(T_{i_1} \circ \cdots \circ T_{i_n} U) \asymp \int \mathcal{L}_t^n \chi_{T_{i_1} \circ \cdots \circ T_{i_n} U} d\eta \asymp |(T_{i_1} \circ \cdots \circ T_{i_n})'(0)|^t.$$

By compactness of the closure $cl(T_{i_1} \circ \ldots \circ T_{i_n} U)$ for $x \in U$ we have

$$\frac{|(T_{i_1} \circ \ldots \circ T_{i_n})'(0)|}{|(T_{i_1} \circ \ldots \circ T_{i_n})'(x)|} \le \sup_{y \in U} \exp\left(\left(\log |(T_{i_1} \circ \ldots \circ T_{i_n})'(y)|\right)'\right) < +\infty$$

together with the mean value theorem we get $|(T_{i_1} \circ \ldots \circ T_{i_n})'(0)| \asymp \operatorname{diam}(T_{i_0} \circ \ldots \circ T_{i_n} U)$. Hence we conclude $\eta(T_{i_1} \circ \cdots \circ T_{i_n} U) \asymp \operatorname{diam}(T_{i_1} \circ \cdots \circ T_{i_n} U)^t$. The mass distribution principle gives

$$\dim_H(X) \ge \lim_{n \to +\infty} \frac{\log \eta(T_{i_1} \circ \cdots \circ T_{i_n} U)}{\log \operatorname{diam}(T_{i_1} \circ \cdots \circ T_{i_n} U)^t} = t.$$

This completes the preparatory material. In the next section we will explain our basic methodology.

3 Hausdorff dimension estimates

Effective estimates on the Hausdorff dimension come from two ingredients.

1. Min-max type inequalities presented in §3.1, which give a way to rigorously bound the largest eigenvalue $e^{P(t)}$ of the transfer operator $\mathcal{L}_t$ using a *suitable test function* $\underline{f}$.
2. The Lagrange—Chebyshev interpolation scheme described in §3.2, which gives a polynomial that can serve as the test function $\underline{f}$.

The first is inspired by the corresponding result for the second eigenvalue of the transfer operator for the Gauss map in the work of Wirsing [55]. The second part is inspired by the work of Babenko–Yur'ev [1] on the problem of Gauss.

The accuracy of the estimates which come out from the min-max inequalities depend on the test function. Lagrange—Chebyshev interpolation is a very classical method of approximating holomorphic functions and perhaps first has been used in this setting by Babenko–Yur'ev [1]. Whereas various interpolation schemes have been used by many authors to estimate $e^{P(t)}$, it is the combination of these two ingredients that leads to particularly effective and accurate estimates.

3.1 The min-max inequalities

Our analysis is based on the maximal eigenvalue $e^{P(t)}$ for $\mathcal{L}_t$ being bounded using the following simple result.

Lemma 3.1. *1. Assume there exists $a > 0$ and a positive function $\underline{f} \in C^\alpha(S)$ such that for all $x \in S$*

$$a\underline{f}(x) \le \mathcal{L}_t \underline{f}(x)$$

then $a \le e^{P(t)}$.

2. *Assume there exists $b > 0$ and a positive function $\underline{g} \in C^\alpha(S)$ such that for all $x \in S$*

$$\mathcal{L}_t \underline{g}(x) \le b \underline{g}(x)$$

then $e^{P(t)} \le b$.

Proof. By iteratively applying $\mathcal{L}_t$ to both sides of the inequality in part 1 we have that for all $x \in S$ and $n \ge 1$

$$a^n \underline{f}(x) \le \mathcal{L}_t^n \underline{f}(x)$$

and thus taking n'th roots and passing to the limit as $n \to +\infty$ we have for all $x \in S$

$$a \le \limsup_{n \to +\infty} |\mathcal{L}_t^n \underline{f}(x)|^{1/n} = e^{P(t)}$$

since $\underline{f}(x)$ is strictly positive and $e^{-nP(t)} \mathcal{L}_t^n \underline{f}$ converges uniformly to $\eta(\underline{f}) > 0$ by the second part of Lemma 2.12. This completes the proof of part 1.

The proof of part 2 is similar. □

Below we will use the following shorthand notation when working with the Banach space $C^\alpha(S)$.

Notation 3.2. *Given $f, g \in C^\alpha(S)$ we abbreviate*

$$\sup_S \frac{f}{g} := \sup_{1 \le j \le d} \sup_{x \in S} \frac{f_j(x)}{g_j(x)} \qquad \text{and} \qquad \inf_S \frac{f}{g} := \inf_{1 \le j \le d} \inf_{x \in S} \frac{f_j(x)}{g_j(x)}.$$

In particular, we can use Lemma 3.1 to deduce a technical fact, which is a basis for validation of all numerical results in the present work.

Lemma 3.3. *The Hausdorff dimension $\dim_H X \in (t_0, t_1)$ if and only if there exists two positive functions $\underline{f}, \underline{g} \in C^\alpha(S)$ such that the following inequalities hold*

$$\inf_S \frac{\mathcal{L}_{t_0} \underline{f}}{\underline{f}} > 1 \quad \text{and} \quad \sup_S \frac{\mathcal{L}_{t_1} \underline{g}}{\underline{g}} < 1, \tag{2}$$

Proof. Assume first that $\dim_H X \in (t_0, t_1)$ then $P(t_0) > 0 > P(t_1)$ which is equivalent to $e^{P(t_0)} > 1 > e^{P(t_1)}$. Then by part 1 of Lemma 2.12 there exist positive eigenfunctions $\underline{h}_0$ and $\underline{h}_1$ of $\mathcal{L}_{t_0}$ and $\mathcal{L}_{t_1}$, respectively, such that

$$\frac{\mathcal{L}_{t_0} \underline{h}_0}{\underline{h}_0} = e^{P(t_0)} > 1 \quad \text{and} \quad \frac{\mathcal{L}_{t_1} \underline{h}_1}{\underline{h}_1} = e^{P(t_1)} < 1.$$

Now assume that (2) hold true for some $\underline{f}$ and $\underline{g}$. Then by Lemma 2.7 the first inequality in (2) implies that the hypothesis of part 1 of Lemma 3.1 holds with $a = \inf_S \frac{\mathcal{L}_{t_0} \underline{f}}{\underline{f}} > 1$. We deduce that $e^{P(t_0)} \ge a > 1$ and thus $P(t_0) > 0$. The second inequality in (2) implies that the hypothesis of part 2 of Lemma 3.1 holds with $b = \sup_S \frac{\mathcal{L}_{t_1} \underline{g}}{\underline{g}} < 1$. We deduce that $e^{P(t_1)} \le b < 1$, thus $P(t_1) < 0$. By the intermediate value theorem applied to the strictly monotone decreasing continuous function P we see that the unique zero $t = \dim_H(X)$ for P satisfies $t_0 < \dim_H X < t_1$, as required. □

Therefore our aim in applications is to make choices of $\underline{f} = (f_i)_{i=1}^d > 0$ and $\underline{g} = (g_i)_{i=1}^d > 0$ in Lemma 3.3 so that t_0 and t_1 are close, in order to get good estimates on $\dim_H(X)$.

Remark 3.4 (Domains of test functions)*.* It is apparent from the proof that we need only to consider the minima and maxima of the ratios $\mathcal{L}\underline{f}(x)/\underline{f}(x)$ and $\mathcal{L}\underline{g}(x)/\underline{g}(x)$ for those $x \in S$ lying in the limit set. However, a compromise which simplifies the use of calculus would be to consider the minima and maxima over the smallest interval containing the limit set.

Remark 3.5 (Min-max theorem)*.* A more refined version which we won't require is the min-max result:

$$e^{P(t)} = \sup_{\underline{f}>0} \inf_S \frac{\mathcal{L}_t\underline{f}}{\underline{f}} = \inf_{\underline{f}>0} \sup_S \frac{\mathcal{L}_t\underline{f}}{\underline{f}}.$$

To see this, observe that by Lemma 3.1 for any $\underline{f}, \underline{g} > 0$ we have

$$\inf_{x\in S} \frac{\mathcal{L}_t\underline{f}(x)}{\underline{f}(x)} \le e^{P(t)} \le \sup_{x\in S} \frac{\mathcal{L}_t\underline{g}(x)}{\underline{g}(x)},$$

and therefore

$$\sup_{\underline{f}>0} \inf_S \frac{\mathcal{L}_t\underline{f}}{\underline{f}} \le e^{P(t)} \le \inf_{\underline{g}>0} \sup_S \frac{\mathcal{L}_t\underline{g}}{\underline{g}}.$$

In particular, by the Ruelle operator theorem, the equality is realized when $\underline{f} = \underline{g} = \underline{h} > 0$ is the eigenfunction associated to $e^{P(t)}$. We refer the reader to ([40], p.88) and ([23], §2.4.2) for more details.

3.1.1 Applying Lemma 3.3 in practice

In order to obtain good estimates on the Hausdorff dimension based on Lemma 3.3 it is necessary to construct a pair of functions $\underline{f}$ and $\underline{g}$ and to rigorously verify the inequalities (2). We shall now explain how the verification has been realised in practice, i.e. in our computer program. To simplify the exposition, we demonstrate the method in the case of Bernoulli scheme. It will be clear how to generalise it to treat a more general Markov case.

Evidently for any interval $[t_0, t_1] \subset [0, 1]$ and for any $x \in [t_0, t_1]$ we have[5]

$$\left| \frac{[\mathcal{L}_t f](t_0)}{f(t_0)} - \frac{[\mathcal{L}_t f](x)}{f(x)} \right| \le \left\| \Bigl(\frac{\mathcal{L}_t f}{f}\Bigr)' \right\|_\infty (t_1 - t_0), \tag{3}$$

therefore if we can get an upper bound on $\bigl\|\bigl(\frac{\mathcal{L}_t f}{f}\bigr)'\bigr\|_\infty$, then we can rigorously estimate the ratio by taking a partition of $[0, 1]$ and applying (3) on each interval of the partition. Furthermore, it is clear that providing $\bigl\|\bigl(\frac{\mathcal{L}_t f}{f}\bigr)'\bigr\|_\infty$ is small, we can allow a relatively coarse partition.

We have the following simple useful fact.

Lemma 3.6. *Let h be a positive eigenfunction of $\mathcal{L}_t$ corresponding to the eigenvalue λ. Then there exists a constant $r_1 > 0$ such that for any approximation f such that $\|f - h\|_{C^1} < \varepsilon$ we have $\bigl\|\bigl(\frac{\mathcal{L}_t f}{f}\bigr)'\bigr\|_\infty < r_1\varepsilon$.*

Proof. By the hypothesis of the Lemma, we may write $f = h + f_\varepsilon$, where $\|f_\varepsilon\|_{C^1} = \max |f_\varepsilon| + \max |f'_\varepsilon| < \varepsilon$, and $f > r_0 > 0$. Then the condition $\|f_\varepsilon\|_{C^1} < \varepsilon$ implies

[5] Here by $\|f\|_\infty$ we understand $\sup_{[t_0,t_1]} |f|$.

$\|f\|_{C^1} \leq \|h\|_{C_1} + \varepsilon$, furthermore, we calculate

$$\begin{aligned} \big|([\mathcal{L}_t f_\varepsilon](x))'\big| &\leq \sum_{j=1}^{d} \Big|\big(|T_j'(x)|^s f_\varepsilon(T_j(x))\big)'\Big| \\ &\leq \sum_{j=1}^{d} \big(\big|(|T_j'(x)|^s)' f_\varepsilon(T_j(x))\big| + \big||T_j'(x)|^{s+1} f_\varepsilon'(T_j(x))\big|\big) \leq r_2\varepsilon, \end{aligned}$$

where $r_2 = 2d\max(\|(|T_j'|^s)'\|_\infty, \||T_j'(x)|^{s+1}\|_\infty)$. By linearity of the transfer operator we have that $(\mathcal{L}_t f)' = (\mathcal{L}_t h)' + (\mathcal{L}_t f_\varepsilon)' = \lambda h' + (\mathcal{L}_t f_\varepsilon)'$. Now we have for the derivative of the ratio

$$\begin{aligned} \Big|\Big(\frac{\mathcal{L}_t f}{f}\Big)'\Big| &= \Big|\frac{(\mathcal{L}_t f)' f - f'(\mathcal{L}_t f)}{f^2}\Big| \\ &\leq \Big|\frac{(\lambda h' + (\mathcal{L}_t f_\varepsilon)') \cdot (h + f_\varepsilon) - (h' + f_\varepsilon') \cdot (\lambda h + \mathcal{L}_t f_\varepsilon)}{r_0^2}\Big| \\ &\leq \frac{\varepsilon}{r_0^2}(r_2\|f\|_\infty + \lambda\|h'\|_\infty + \|\mathcal{L}_t f\|_\infty + r_2\|h'\|_\infty). \end{aligned}$$

Taking into account that $\|f\|_{C^1} \leq \|h\|_{C^1} + \varepsilon$ we may now choose r_1, which depends on the norm of the eigenfunction $\|h\|_{C^1}$, but is independent of the choice of approximation f, such that $\big\|\big(\frac{\mathcal{L}_t f}{f}\big)'\big\|_\infty \leq r_1\varepsilon$. □

In practice, the derivative $\big(\frac{\mathcal{L}_t f}{f}\big)'$ can be effectively estimated using the following computer-assisted approach to arbitrary precision and without excessive effort. We construct our test function f as a polynomial of degree m and our IFS consists of linear-fractional transformations $T_j(x) = \frac{a_j x + b_j}{c_j x + d_j}$, the image functions $F^t = \mathcal{L}_t f$ can be written as

$$F^t(x) = \sum_{j=1}^{d} \frac{|a_j d_j - b_j c_j|^t}{|c_j x + d_j|^{2t}} f\left(\frac{a_j x + b_j}{c_j x + d_j}\right) = \sum_{j=1}^{d} \frac{|a_j d_j - b_j c_j|^t}{(c_j x + d_j)^{2t}(c_j x + d_j)^m} p_j(x),$$

where p_j are polynomials of degree m, whose coefficients can be computed with arbitrary precision. Then for the derivative of the ratio we obtain

$$\left(\frac{F^t(x)}{f(x)}\right)' = \sum_{j=1}^{d} \frac{|a_j d_j - b_j c_j|^t}{(c_j x + d_j)^{2t}(c_j x + d_j)^{m+1} f^2(x)} \widehat{p}_j(x), \tag{4}$$

where

$$\widehat{p}_j(x) = -(m + 2t)p_j(x)f(x) - (c_j x + d_j)(f(x)p_j'(x) - f'(x)p_j(x))$$

is a polynomial of degree $2m$ whose coefficients can be computed explicitly with arbitrary precision chosen. The computation of these coefficients, together with the coefficients of f, allowed us to obtain accurate estimates on the derivative $\big(\frac{F^t}{f}\big)'$ on the entire interval $[0, 1]$ using ball arithmetic [28] in all the examples we considered.

Remark 3.7. Since the ratios are analytic functions, one would expect that they can be approximated by polynomials. A heuristic observation suggests that for an IFS of analytic contractions, $\varepsilon \sim 10^{-3m/4}$ in Lemma 3.6, where m is the degree of the approximating polynomial.

Remark 3.8. The formulae for F^t, $\frac{F^t}{f}$ and $\widehat{p}_j$ given above have been used in the actual computer program written in C to study the Examples we have in the paper. For the iterated function schemes which are not linear fractional transformations, the formulae, indeed, will be different, and in particular, $\widehat{p}_j$ may not be a polynomial. The same method applies, but the computation might require more computer time. The implicit constant r_2 also affects the accuracy of the estimate.

3.2 Lagrange—Chebyshev Interpolation

We might fancifully note that if we had an *a priori* knowledge of the true eigenfunction $\underline{h}$ for $\mathcal{L}_t$ corresponding to the maximal eigenvalue and took this choice for $\underline{f}$ in Lemma 3.1 then we would immediately have $a = b = e^{P(t)}$. However, in the absence of a knowledge of the eigenfunction our strategy is to find an approximation.

(a) Choose $t_0 < t_1$ as potential lower and upper bounds, respectively, in Lemma 3.3 based on an heuristic estimate on the dimension (for example using the periodic point method, or bounds that we would like to justify from any other source); and

(b) Find candidates for $\underline{f}$ and $\underline{g}$ which are close to the eigenfunctions $\underline{h}_{t_1}$ and $\underline{h}_{t_0}$ for the operators $\mathcal{L}_{t_1}$ and $\mathcal{L}_{t_0}$, respectively. We do this by approximating the two operators by finite dimensional versions and using their eigenfunctions for $\underline{f}$ and $\underline{g}$ (in the next section).

There are different possible ways to find the functions we require in (b) in the previous section. We will use classical Lagrange interpolation [53].

Step 1 (Points). Fix $m \geq 2$. We can then consider the zeros of the Chebyshev polynomials:

$$x_k = \cos\left(\frac{\pi(2k-1)}{2m}\right) \in [-1,1], \text{ for } 1 \leq k \leq m.$$

In the present context it is then convenient to translate them to the unit interval by setting $y_k = (x_k + 1)/2 \in [0,1]$.

Step 2 (Functions). We can use the values $\{y_k\}$ to define the associated Lagrange interpolation polynomials:

$$lp_k(x) = \frac{\prod_{i\neq k}(x - y_i)}{\prod_{i\neq k}(y_k - y_i)}, \text{ for } 1 \leq k \leq m \tag{5}$$

which are the polynomials of the minimal degree with the property that $lp_k(y_k) = 1$ and $lp_k(y_j) = 0$ for $j \neq k$ for all $1 \leq j, k \leq m$. These polynomials span an m-dimensional subspace $\langle lp_1, \cdots, lp_m\rangle \subset C^\alpha([0,1])$.

To allow for the Markov condition we need to consider the $(d \times m)$-dimensional subspace of $C^\alpha(S)$. To define it we simply consider d copies $lp_{k,i} : [0,1] \times \{i\} \to \mathbb{R}$ $(1 \leq i \leq d)$ of the Lagrange polynomials given by $lp_{k,i}(x,i) \equiv lp_k(x)$.

Step 3 (Matrix). The polynomials $lp_{k,i}$ for $i = 1, \ldots, d$ and $k = 1, \ldots, m$ constitute a basis of an $(d \times m)$-dimensional subspace $\mathcal{E} := \langle lp_{k,i}\rangle \subset C^\alpha(S)$. Using definition 2.9

of the transfer operator we can write for any $0 < t < 1$ and $(f_1, \ldots, f_d) \in \mathcal{E}$:

$$(\mathcal{L}_t(f_1, \ldots, f_d))_j = \sum_{i=1}^{d} M(i,j) f_i(T_i)|T_i'|^t, \quad 1 \le i \le d, \tag{6}$$
$$(\mathcal{L}_t(f_1, \ldots, f_d))_j \colon [0,1] \times \{j\} \to \mathbb{R}.$$

For each $1 \le i, j \le d$ and $1 \le k, l \le m$ we can introduce the $m \times m$ matrix $B^t_{ij}(l,k)$

$$B^t_{ij}(l,k) \colon = lp_{k,i}(T_i(y_{l,j})) \cdot |T_i'(y_{l,j})|^t. \tag{7}$$

Then we can apply the operator $\mathcal{L}_t$ to a basis function $(0, \cdots, 0, lp_{k,i}, 0, \cdots, 0) \in \mathcal{E}$ and evaluate the resulting function at the nodes $y_{l,j} = (y_l, j) \in [0,1] \times \{j\}$, $1 \le l \le m$, $1 \le j \le d$

$$\begin{aligned}(\mathcal{L}_t(0, \cdots, 0, lp_{k,i}, 0, \cdots, 0))_j(y_{l,j}) &= M(i,j) lp_{k,i}(T_i(y_{l,j}))|T_i'(y_{l,j})|^t \\ &= M(i,j) \cdot B^t_{i,j}(l,k).\end{aligned}$$

Taking into account that the polynomials $lp_{k,i}$ constitute a basis of the subspace $\mathcal{E}$ we can approximate the restriction $\mathcal{L}_t|_{\mathcal{E}}$ by a $md \times md$ matrix:

$$B^t = \begin{pmatrix} M(1,1)B^t_{11} & \cdots & M(1,d)B^t_{1d} \\ \vdots & \ddots & \vdots \\ M(d,1)B^t_{d1} & \cdots & M(d,d)B^t_{dd} \end{pmatrix}. \tag{8}$$

For large values of m the maximal eigenvalue of the matrix B^t will be arbitrarily close to the maximal eigenvalue $e^{P(t)}$ of the transfer operator $\mathcal{L}_t$, see §3.4.

Step 4 (Eigenvector). We can consider the left eigenvector of B^t

$$v^t = (v^t_{1,1}, \cdots, v^t_{1m}, v^t_{21}, \cdots, v^t_{2,m} \cdots, v^t_{d1}, \cdots v^t_{dm}) \tag{9}$$

corresponding to the maximal eigenvalue and use it to define a function $(f^t_1, \cdots, f^t_d) \in \mathcal{E}$ as a linear combination of Lagrange polynomials

$$f^t_i = \sum_{j=1}^{m} v^t_{ij} lp_{i,j}, \quad 1 \le i \le d. \tag{10}$$

Remark 3.9. In many cases the calculation can be simplified. More precisely, in the construction above, the points $y_{l,j} \equiv y_l$ do not depend on j. Therefore the matrices $B_{ij}(l,k)$ do not depend on j either, and instead of computing d^2 matrices $B_{ij}(l,k)$ it is sufficient to compute d matrices $B^t_i(l,k)$.

It is not immediately clear that the polynomials given by (10) are positive. In Proposition 3.10 in §3.4 below we show that for an iterated function scheme of analytic contractions the algorithm presented above gives positive functions provided m is sufficiently large. However, we don't have a priori bounds on m. Therefore, for every example we consider, we rigorously verify that the function constructed is positive using the following simple method (and if the function turns out not to be positive, we increase m).

Since our f^t_j's are polynomials, their derivatives are easy to compute symbolically. We then take a uniform partition of the interval into 2^{10} intervals. For each interval (a,b) of the partition, we compute the following:

1. The middle point $c = \frac{1}{2}(a+b)$ and half-length $r = \frac{1}{2}(b-a)$.
2. The first $m-1$ derivatives at c: $f_j^{(k)}(c)$ for $k = 1, \ldots, m-1$.
3. The image of the interval under the m'th derivative: $(a_1, b_1) = |f_j^{(m)}(a,b)|$, this is done using ball arithmetic. The inequality $\max_{(a,b)} |f_j^{(m)}| \le b_1$ is guaranteed by the Arb library [28].

Then we can calculate a lower bound on f_j on (a,b):

$$f_j(x) \ge f_j(c) - (r|f_j^{(1)}(c)| + r^2|f_j^{(2)}(c)| + \ldots + r^{m-1}|f_j^{(m-1)}(c)| + r^m b_1) \text{ for all } x \in (a,b).$$

3.3 Bisection method

The approach described in the previous two sections can be used not only to verify given estimates $t_0 < \dim_H X < t_1$ but also to compute the Hausdorff dimension of a limit set of a Markov iterated function scheme with any desired accuracy using a basic bisection method. Assume that given $\varepsilon > 0$ we would like to find an interval $\dim_H X \in (d_0, d_1)$ of length $d_1 - d_0 = \varepsilon$.

We begin by fixing a value of m, say $m = 6$. Then we pick $t_0 < t_1$ for which we know $P(t_0) > 1 > P(t_1)$ (for a one-dimensional Iterated Function Scheme a safe choice is $t_0 = 0$, $t_1 = 1$) and compute $q = \frac{1}{2}(t_0 + t_1)$. Using Lagrange–Chebyshev interpolation, the method described in §3.2, we compute the matrix B^q defined by (8). We then calculate its left eigenvector v^q using the classical power method and construct the corresponding function $\underline{f}^q \in \mathcal{E}$ according to (10) as a sum of Lagrange polynomials. We verify that $\underline{f}^q$ is positive, using the approach explained above. Having the function $\underline{f}^q$ we compute the minima and the maxima of the ratio

$$a' := \inf_S \frac{\mathcal{L}_q \underline{f}^q}{\underline{f}^q}, \qquad b' := \sup_S \frac{\mathcal{L}_q \underline{f}^q}{\underline{f}^q}$$

Then there are three possibilities

1. if $a' > 1$ we deduce by Lemma 3.3 that $\dim_H X \ge q$ and move the left bound of the interval to $t_0 = q$,
2. if $b' < 1$ we deduce from Lemma 3.3 that $\dim_H X \le q$ and move the right bound of the interval to $t_1 = q$,
3. if $a' \le 1 \le b'$ we increase m,

and repeat the process described above, see Figure 4 for a flowchart.

3.4 The convergence of the algorithm

In this section we show that our method gives estimates on the dimension which are arbitrarily close to the true value, thus opening up the possibility of arbitrarily close estimates with further computation.

All of the examples we consider in the present work have an iterated scheme consisting of one-dimensional real analytic contractions. For simplicity we state the next Proposition for a single interval I, which corresponds to the Bernoulli case with $d = 1$ in §3.2, but it will be clear how to extend this case to the more general Markov setting.

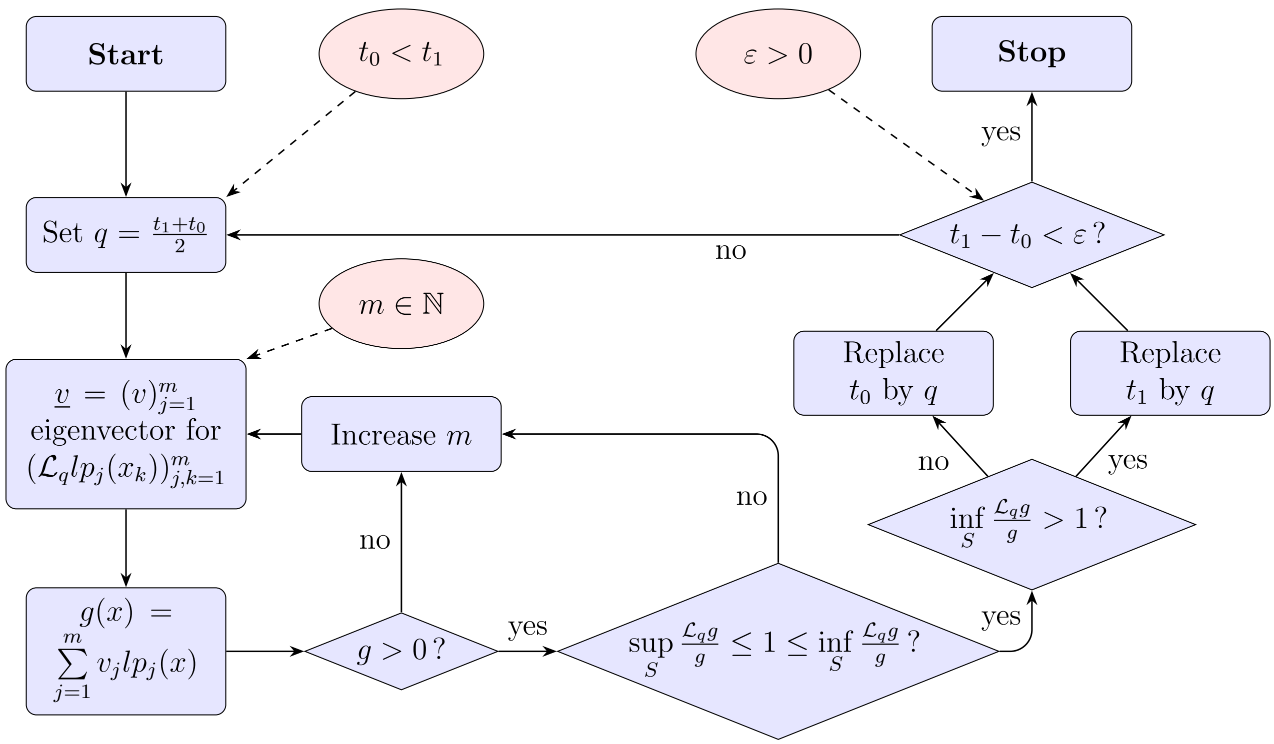

Figure 4: The flow diagram summarizes how a computer programme implements the bisection procedure to find a small interval $[t_0, t_1]$ of small size $\varepsilon > 0$ containing $\dim_H(X)$.

Proposition 3.10. *Let $T_1, \cdots, T_d : I \to I$ be an iterated function scheme with real analytic contractions with $0 < \inf_{x\in I} |T_j'(x)| \leq \sup_{x\in I} |T_j'(x)| < 1$ for $j = 1, \cdots, d$. Assume that $P(t) \neq 0$ (i.e. $\dim_H(X) \neq t$). Then for any m sufficiently large the polynomial f^t defined by* (10) *satisfies one of the inequalities of Lemma 3.3, in other words we have either $\inf_I \frac{\mathcal{L}_t f^t}{f^t} > 1$ or $\sup_I \frac{\mathcal{L}_t f^t}{f^t} < 1$.*

The proof of the Proposition which we will give here consists of two steps: The first step is to construct a subspace of analytic functions in $C^\alpha(I)$ such that the restriction of the transfer operator onto it has the right spectral properties. The second step is to construct an approximation of the operator acting on the subspace of analytic functions by a finite rank operator acting on the subspace of polynomials of degree m. We begin our preparations for the proof of the Proposition by introducing the subspace of analytic functions and defining the operator there. First, we need to introduce a suitable domain of analyticity.

Definition 3.11. *Given $\rho > 1$ we define an ellipse with the foci* 0 *and* 1 *by*

$$\partial U_\rho = \left\{ z = \frac{1}{2} + \frac{1}{4}\left(\rho e^{i\theta} + \frac{e^{-i\theta}}{\rho}\right) : 0 \leq \theta < 2\pi \right\}. \tag{11}$$

It is often referred to as a Bernstein ellipse *[53].*

In the setting and under the hypothesis of Proposition 3.10 without loss of generality we may assume the following:

(a) There exists $\rho > 1$ such that each contraction T_j extends to a complex domain $U_\rho \supset [0,1]$ bounded by the ellipse ∂U_ρ, such that $T_j'(z) \neq 0$ for any $z \in U_\rho$; and

(b) The closures $\mathrm{cl}(T_j U_\rho)$ of the images $T_j U_\rho$ satisfy $\mathrm{cl}(T_i U_\rho) \subset U_\rho$ for all $j = 1, \cdots, k$.

It is clear that an ellipse satisfying (a) and (b) exists. More precisely, since we assume real analyticity of the T_j we can choose an elliptical domain sufficiently close to $[0, 1]$ and by the hypotheses of the Proposition we can deduce (a). Since the T_j contract we can choose the ellipse in (a) sufficiently close to I (by making ρ close to 1) that (b) holds. In what follows, we shall simplify notation and omit the index ρ.

After introducing the domain of analyticity, we now define a Banach space of analytic functions.

Definition 3.12. *Let H^∞ denote the space of bounded analytic functions on U_ρ with the norm $\|f\| = \sup_{z \in U_\rho} |f(z)|$.*

The space H^∞ is special case of Hardy spaces (hence the choice of notation), and it is known to be a Banach space [34]. For any function $f \in H^\infty$ the restriction $f|_I$ is a continuously differentiable function on I and, in particular, it is contained in $C^\alpha(I)$ for any $0 < \alpha \le 1$. More precisely, we can choose a simple closed curve $\Gamma \subset U_\rho$ close to ∂U_ρ and use Cauchy's theorem to write the derivative for the restriction $f|_I$ by

$$f'(x) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - x)^2} dz,$$

which is continuous as a function of $x \in I$.

Note that T_j' is real analytic and non-vanishing on I and the same holds true for $|T_j'|^t$. Thus by slight abuse of notation we may also denote by $|T_j'(z)|^t$ its analytic extension to the domain bounded by the sufficiently small Bernstein ellipse U_ρ. Therefore we may now introduce a restriction of the transfer operator (1) onto H^∞.

Definition 3.13. *For $t > 0$, the transfer operator $\mathcal{L}_t : H^\infty \to H^\infty$ will again be given by the formula*

$$[\mathcal{L}_t f](z) = \sum_{j=1}^{k} |T_j'(z)|^t f(T_j z), \quad z \in U_\rho. \tag{12}$$

The operator $\mathcal{L}_t \colon H^\infty \to H^\infty$ given by (12) is actually compact and even nuclear [40],[47] although this will not be needed. We will use the following simpler fact instead.

Lemma 3.14. *The operator $\mathcal{L}_t \colon H^\infty \to H^\infty$ shares the same maximal eigenvalue $\lambda = e^{P(t)}$ as $\mathcal{L}_t \colon C^\alpha(I) \to C^\alpha(I)$ and the rest of the spectrum is contained in a disk of strictly smaller radius.*

Proof. Indeed, if we let $\mathbb{1}$ denote the constant function on $[0, 1]$ then by Lemma 2.12 we see that $e^{-nP(t)} \mathcal{L}_n^t \mathbb{1}$ converges uniformly to $\eta(\mathbb{1}) \in \langle \underline{h} \rangle$. However, since $\mathbb{1} \in H^\infty$ and $\mathcal{L}_t^n$ preserves H^∞ we can conclude that $\eta(\mathbb{1}) \in H^\infty$ and thus $\underline{h}$ has an extension in H^∞ and $e^{P(t)}$ is a simple eigenvalue for $\mathcal{L}_t \colon H^\infty \to H^\infty$. Similarly, any eigenvalue for $\mathcal{L}_t \colon H^\infty \to H^\infty$ must be an eigenvalue for $\mathcal{L}_t \colon C^\alpha(I) \to C^\alpha(I)$ since $H^\infty \subseteq C^\alpha(I)$. Therefore, since $\mathcal{L}_t \colon C^\alpha(I) \to C^\alpha(I)$ has the rest of the spectrum in a disk of the radius strictly small than $e^{P(t)}$ this property persists for the operator on H^∞. □

This completes the first step of the proof of Proposition 3.10 outlined above.

Recall that the Lagrange polynomials $lp_1, \ldots, lp_m$ given by (5) form a basis of the subspace of polynomials of degree $m-1$ in H^∞; in §3.2 we named this subspace $\mathcal{E}$. Let $\mathcal{P}_m : H^\infty \to \mathcal{E}$ be the natural projection given by the collocation formula

$$[\mathcal{P}_m f](x) = \sum_{j=1}^{m} f(x_j) lp_j(x), \quad x \in I. \tag{13}$$

We see immediately that the restriction $\mathcal{P}_m|_\mathcal{E}$ of $\mathcal{P}_m$ to $\mathcal{E}$ is the identity.

The second step is to approximate the transfer operator on H^∞ by a finite rank operator. We now recall an estimate on the norm of the difference $\|\mathcal{L}_t - \mathcal{L}_t\mathcal{P}_m\|_{H^\infty}$.

Lemma 3.15 (see [5], Theorem 3.3)**.** *For the transfer operator on H^∞ given by* (12) *there exist $C > 0$ and $0 < \theta < 1$ such that $\|\mathcal{L}_t - \mathcal{L}_t\mathcal{P}_m\|_{H^\infty} \le C\|\mathcal{L}_t\|_{H^\infty}\theta^m$ for $m \ge 1$.*

This is also implicit in [56, §2.2].

Remark 3.16. The proof of Lemma 3.15 relies on the fact that for any function f analytic on a domain bounded by a Bernstein ellipse the image $\mathcal{L}_t f$ is analytic on a larger domain bounded by another Bernstein ellipse. This form of analyticity improving property is also essential in showing that $\mathcal{L}_t$ is nuclear [47].

It follows from Lemmas 3.14, 3.15 and classical analytic perturbation theory (see the book of Kato [32, Chapter VII]) that we have the following:

Corollary 3.17. *For any $\varepsilon > 0$ there exists $\delta > 0$ sufficiently small such that for all m sufficiently large we have*

1. *$\mathcal{L}_t\mathcal{P}_m : H^\infty \to H^\infty$ has a simple maximal eigenvalue λ_m with $|\lambda_m - \lambda| < \varepsilon$;*
2. *The rest of the spectrum of $\mathcal{L}_t\mathcal{P}_m$ is contained in $\{z \in \mathbb{C} \colon |z| < \lambda - 2\delta\}$; and*
3. *The eigenfunctions h_m of $\mathcal{L}_t\mathcal{P}_m$ converge to the eigenfunction h of $\mathcal{L}_t$, more precisely $\|h_m - h\|_{H^\infty} \to 0$ as $m \to \infty$.*
4. *The exists a constant $c > 0$, independent of m, such that the eigenfunction h_m for $\mathcal{L}_t\mathcal{P}_m$ corresponding to λ_m satisfies $|h_m(z)| > c$ for all $z \in U_\rho$.*

We are now ready to prove Proposition 3.10.

Proof (of Proposition 3.10). It follows from (13) that the restriction $\mathcal{P}_m\mathcal{L}_t|_\mathcal{E}$ to $\mathcal{E}$ is a finite rank operator $\mathcal{P}_m\mathcal{L}_t \colon \mathcal{E} \to \mathcal{E}$ given by

$$\mathcal{P}_m\mathcal{L}_t \colon f \mapsto \sum_{j=1}^{n} [\mathcal{L}_t f](x_j) \cdot lp_j.$$

In the basis of Lagrange polynomials $\{lp_j\}_{j=1}^m$ the operator $\mathcal{P}_m\mathcal{L}_t$ is given by the $m \times m$-matrix $B^t = B^t(j,k)$, $j,k = 1, \ldots, m$, where

$$B^t(j,k) = \sum_{i=1}^{m} [\mathcal{L}_t lp_j](x_i) \cdot lp_i(x_k) = [\mathcal{L}_t lp_j](x_k) \qquad \text{for } 1 \le j,k \le m;$$

which agrees with the matrix given by (7) in a special case $d = 1$. A straightforward computation gives that the eigenvalue λ_m for $\mathcal{L}_t\mathcal{P}_m$ is also an eigenvalue for the matrix B^t corresponding to the eigenvector $\mathcal{P}_m h_m \in \mathcal{E}$. Since we have chosen the basis of Lagrange polynomials to define the matrix B^t, we conclude that

$$[\mathcal{P}_m h_m](x) = \sum_{j=1}^{m} v_j lp_j(x),$$

where $(v_1, \ldots, v_m)$ is the eigenvector of B^t.

To see that $\mathcal{P}_m h_m$ is a positive function, we apply a classical result by Chebyshev [53], which gives

$$\sup_I |h_m - \mathcal{P}_m h_m| \leq \frac{1}{2^{m-1} m!} \sup_I |h_m^{(m)}|. \tag{14}$$

Since $h_m \to h$ as $m \to \infty$ in the space of analytic functions H^∞, the derivatives $h_m^{(m)}$ are uniformly bounded on I. Therefore the positivity of h_m on I guarantees the positivity of the projection $\mathcal{P}_m h_m \in \mathcal{E}$ on I for sufficiently large m.

It remains to show that one of the inequalities in Lemma (3.3) holds true for $\mathcal{P}_m h_m$. Without loss of generality we may assume that $P(t) > 0$, which implies $\frac{\mathcal{L}_t h}{h} = e^{P(t)} > 1$. Therefore the part 4 of Corollary 3.17 together with (14) gives $\|\mathcal{P}_m h_m - h\|_{H^\infty} \to 0$ as $m \to \infty$. Hence for m sufficiently large we shall have $\frac{\mathcal{L}_t \mathcal{P}_m h_m}{\mathcal{P}_m h_m} > 1$ everywhere on I. The case $P(t) < 0$ is similar.

This completes the proof of Proposition 3.10. □

Remark 3.18. To avoid confusion we should stress that positivity of the polynomial $P_m h_m \in \mathcal{E}$ and positivity of the entries of B^t are not related; furthermore, in the examples we consider the matrices B^t typically have entries of both signs.

The convergence of the algorithm we presented in §3.2-§3.3 follows immediately from Proposition 3.10.

Corollary 3.19. *After applying the bisection method sufficiently many times we obtain an approximation to* $\dim_H X$ *which is arbitrarily close to the true value.*

Remark 3.20. We would like to note that although Proposition 3.10 guarantees the convergence of our algorithm, it doesn't give any explicit estimates on how large m one will need to take in practical realisation. Our heuristic observation shows that it takes of order $N = -\frac{\log \varepsilon}{\log 2}$ iterations of the bisection method to find an interval of length ε containing the value of $\dim_H X$ for a scheme of analytic contractions.

For instance, for many of our examples it is sufficient to take $m = 6$ to obtain an estimate accurate to 4 decimal places which makes the matrices small and the computation very fast. However, when we require greater accuracy we need to choose m larger to provide test functions for Lemma 3.3 which will be a more close approximation of the eigenfunction. For example, to verify $\dim_H(E_2)$ to over 200 decimal places we choose $m = 275$.

Example 3.21 (Contractions with less regularity)**.** *When the contractions have less regularity the present interpolation method is not very efficient. For instance Falk and Nussbaum ([9], §3.3) considered the limit set* X *corresponding to the contractions*

$$T_1(x) = \frac{x + ax^{7/2}}{3 + 2a} \quad \text{and} \quad T_2(x) = \frac{x + ax^{7/2}}{3 + 2a} + \frac{2 + a}{3 + 2a}$$

for $0 < a < 1$*. For example, when* $a = \frac{1}{2}$ *they show that*

$$\mathbf{0.7334774}5730\,00780 \leq \dim_H(X) \leq \mathbf{0.7334774}6222\,22678$$

giving an estimate accurate to 6 *decimal places. By increasing the number of Chebyshev nodes to* $m = 100$ *we obtain test functions that, using Lemma 3.3 give*

$$\dim_H(X) = 0.7334746151\,5 \pm 1.5 \cdot 10^{-10}.$$

This is a modest improvement in accuracy to 9 *decimal places.*

4 Applications

We will now apply the method in §3 to the problems described in the introduction.

4.1 Markov—Lagrange theorems

The bounds on the Hausdorff dimension on various parts of the difference of the Markov and Lagrange spectra as stated in Theorem 1.3 are built on the comprehensive analysis of Matheus and Moreira [36], where the Hausdorff dimension bounds are given in terms of the Hausdorff dimension of certain linear-fractional Markov Iterated Function Schemes.

Therefore we set ourselves the task of establishing with an accuracy of $\varepsilon = 10^{-5}$, say, the Hausdorff dimension of the limit sets involved. We need to consider five different Iterated Function Schemes corresponding to five cases of Theorem 1.3.

We begin by formulating a general framework which embraces many of our numerical results. In the proof of Theorem 1.3 the set X_M is given in terms of sequences from an alphabet $\{1, 2, \dots, d\}$ with certain forbidden words $\underline{w}$ (of possibly varying lengths). For example, in Parts 2 and 3 the basic forbidden words are of length 2 and Markov condition is easy to write. However, in Parts 1, 4 and 5 the basic forbidden words are of length 3 and 4. If the maximum length of a word is N, then we can consider the alphabet of "letters" $\underline{w} = w_1 w_2 \cdots w_{N-1} \in \{1, 2, \cdots, d\}^{N-1}$, which are sequences of length $N-1$ and define a $dN \times dN$ matrix M', where

$$M_{jk} = \begin{cases} 1 & \text{if } w_2^j \cdots w_{N-1}^j = w_1^k \cdots w_{N-2}^k \text{ and } w_1^j \cdots w_{N-1}^j w_{N-1}^k \text{ is allowed;} \\ 0 & \text{otherwise.} \end{cases}$$

where we say that the word is allowed, if it doesn't contain any forbidden subwords. We can then define a matrix M by changing the entry in the row indexed by a word $i_1 i_2 \cdots i_N$ and column indexed by $j_1 j_2 \cdots j_N$ to 0 whenever the concatenation $i_1 i_2 \cdots i_{N-1} j_{N-1}$ contains any of the forbidden words as a substring[6]. The transformation associated to the word $\underline{w}$ is defined by the first term, i.e., $T_{\underline{w}} = T_{w_1}$.

4.1.1 Proof of Theorem 1.3

We can divide the proof of Theorem 1.3 into its five constituent parts.

Part 1: $(\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{5}, \sqrt{13})$. It is proved in §B.1 of [36] that

$$\dim_H(\mathcal{M} \setminus \mathcal{L} \cap (\sqrt{5}, \sqrt{13})) \le 2 \dim_H(X_M). \tag{15}$$

where $X_M \subset [0, 1]$ is a set of numbers whose continued fractions expansions contain only digits 1 and 2, except that subsequences 121 and 212 are not allowed. We can relabel the contractions T_{ij}, $i, j \in \{1, 2\}$ for the Markov iterated function scheme on $[0, 1]$ as

$$T_1(x) = T_2(x) = \frac{1}{1+x}, \; T_3(x) = T_4(x) = \frac{1}{2+x} \; \text{ and } M = \begin{pmatrix} 1&1&0&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&0&1&1 \end{pmatrix}.$$

[6] In the special case $N = 2$ this corresponds to the usual 1-step Markov condition.

The associated transfer operator is acting on the Hölder space of functions $C^\alpha(S)$ where $S = \oplus_{j=1}^4 [0,1] \times \{j\}$. We choose $m = 8$ and apply the bisection method starting with the bounds $t_0' = 0.35$, $t_1' = 0.38$. It gives the estimate

$$t_0 \colon = 0.3640546 < \dim_H X_M < 0.3640548 \tag{16}$$

To justify the bounds, the functions $\underline{f} = (f_1, \dots, f_4)$, $\underline{g} = (g_1, \dots, g_4) \in C^\alpha(S)$ are explicitly computed using eigenvectors of the matrices B^{t_0} and B^{t_1}, respectively. These are polynomials of degree 7 given by

$$f_j(x) = \sum_{k=0}^{7} a_k^j x^k, \qquad g_j = \sum_{k=0}^{7} b_k^j x^k, \quad j = 1, \dots, 4.$$

whose coefficients a_k^j, b_k^j are tabled in §A.1.1. We can obtain the following inequalities

$$\sup_S \frac{\mathcal{L}_{t_1} \underline{f}}{\underline{f}} < 1 - 10^{-9}; \qquad \inf_S \frac{\mathcal{L}_{t_0} \underline{g}}{\underline{g}} > 1 + 10^{-7}, \tag{17}$$

and the bound (16) follows from Lemma 3.3. Substituting the bounds from (16) into the inequality (15) we obtain

$$\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{5}, \sqrt{13})) < 0.7281096.$$

□

The estimates from (16) confirm the conjectured upper bound of $\dim_H(X_M) < 0.365$ obtained in [36], §B.1 using the periodic point method.

Part 2: $(\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{13}, 3.84)$. In [36], §B.2 it is shown that

$$\dim_H \Big((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{13}, 3.84) \Big) < 0.281266 + \dim_H X_M, \tag{18}$$

where $X_M \subset [0,1]$ is a set of numbers whose continued fractions expansions contain only digits $1, 2, 3$, except subsequences 13 and 31 are not allowed. This is also the limit set of the IFS

$$T_1(x) = \frac{1}{1+x}, \ T_2(x) = \frac{1}{2+x}, \ T_3(x) = \frac{1}{3+x}, \quad M = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix} \right).$$

Following the same strategy as above, with $S = \oplus_{j=1}^3 [0,1] \times \{j\}$ and $m = 8$, and using $t_0' = 0.56$ and $t_1' = 0.58$ as initial guesses for bisection method[7] we get the following upper and lower bounds

$$t_0 \colon = 0.5739612 \le \dim_H(X_M) \le 0.5739617 =\colon t_1. \tag{19}$$

The leading eigenvectors of the corresponding matrices B^{t_0} and B^{t_1} give polynomial functions

$$f_j(x) = \sum_{k=0}^{7} a_k^j x^k, \qquad g_j = \sum_{k=0}^{7} b_k^j x^k, \quad j = 1, 2, 3.$$

[7]Based on [36], §B.2.

whose coefficients a_k^j, b_k^j are tabled in §A.1.2. We can obtain the following inequalities

$$\sup_S \frac{\mathcal{L}_{t_1}\underline{f}}{\underline{f}} < 1 - 10^{-7}; \qquad \inf_S \frac{\mathcal{L}_{t_0}\underline{g}}{\underline{g}} > 1 + 10^{-7}, \tag{20}$$

and the bound (19) follows from Lemma 3.3. Substituting the bounds from (19) into the inequality (18) we obtain

$$\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{13}, 3.84)) < 0.855228.$$

□

The upper bound in estimate (19) confirms the conjectural bound $\dim_H(X_M) < 0.574$ ([36], §B.2) which has been obtained using the periodic point approach.

Part 3: $(\mathcal{M} \setminus \mathcal{L}) \cap (3.84, 3.92)$. The following inequality was established in [36] §B.3:

$$\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (3.84, 3.92)) < \dim_H(X_M) + 0.25966, \tag{21}$$

where $X_M \subset [0,1]$ is the set of all numbers such that its continued fraction expansions contain only digits 1, 2, and 3 with an extra condition that subsequences 131, 132, 231, and 313 are not allowed. This corresponds to a Markov IFS with 9 contractions

$$T_{ij} = \frac{j+x}{1+i(j+x)}, \quad i, j \in \{1, 2, 3\},$$

with Markov condition given by the 9×9 matrix

$$M_{ij,kl} = \begin{cases} 0 & \text{if } \{ijk, jkl\} \cap \{131, 132, 231, 313\} \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Ordering the 2-element sequences ij in lexicographical order, we see that some columns of the resulting 9×9 matrix agree, more precisely, $M(j,1) \equiv M(j,2)$ and $M(j,4) \equiv M(j,5) \equiv M(j,6)$ for all $1 \le j \le 9$. Therefore, the matrix B^t defined by (8) has the same property. Let $v^t = (\overline{v_1}, \dots, \overline{v_d})$, where $\overline{v}_j \in \mathbb{R}^m$ be its left eigenvector corresponding to the leading eigenvalue λ. Then

$$\sum_{k=1}^{d} \overline{v_k} M(k,j) \cdot B_k^t = \lambda \overline{v_j}, \quad 1 \le j \le 9.$$

Therefore the equality between matrix columns implies $\overline{v_1} = \overline{v_2}$ and $\overline{v_4} = \overline{v_5} = \overline{v_6}$.

The bisection method with $m = 8$ gives the following lower and upper bounds on dimension

$$t_0 \colon = 0.6113922 < \dim_H X_M < 0.6113925 =: t_1. \tag{22}$$

The coefficients of the test functions $\underline{f} = (f_1, \dots, f_9)$ for $\mathcal{L}_{t_1}$ and $\underline{g} = (g_1, \dots, g_9)$ for $\mathcal{L}_{t_0}$, which are polynomials of degree 7 are tabled in §A.1.3. The equalities between elements of the eigenvectors imply $f_1 = f_2$, $f_4 = f_5 = f_6$ and $g_1 = g_2$, $g_4 = g_5 = g_6$. Using ball arithmetic we get bounds for the ratios

$$\sup_S \frac{\mathcal{L}_{t_1}\underline{f}}{\underline{f}} < 1 - 10^{-7}; \qquad \inf_S \frac{\mathcal{L}_{t_0}\underline{g}}{\underline{g}} > 1 + 10^{-7}. \tag{23}$$

and the estimates (22) follow from Lemma 3.3. Substituting the upper bound from (22) into the inequality (21) we obtain

$$\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (3.92, 4.01)) < 0.8710525.$$

The estimates (22) confirm the heuristic bound of $\dim_H(X_M) < 0.612$ given in [36], §B.3.

In the remaining two cases, corresponding to intervals $(3.92, 4.01)$ and $(\sqrt{20}, \sqrt{21})$ the Markov condition is a little more complicated and instead of numbering the contractions defining the Iterated Function Scheme by single numbers it is more convenient to index them by pairs or triples.

Part 4: $\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(3.92, 4.01))$. In [36], §B.4 the following inequality is proved:

$$\dim_H((\mathcal{M} \setminus \mathcal{L}) \cap (3.92, 4.01)) < \dim_H(X_M) + 0.167655. \tag{24}$$

where $X_A \subset [0, 1]$ is the set of all numbers such that its continued fraction expansions contain only digits 1, 2, and 3 with an extra condition that subsequences 131, 313, 2312, and 2132 are not allowed. This corresponds to a Markov iterated function scheme

$$T_{ijk} = \frac{1 + j(k + x)}{i + (ij + 1)(k + x)}, \quad i, j, k \in \{1, 2, 3\},$$

with Markov condition given by the 27×27 matrix

$$M(ijk, qrs) = \begin{cases} 0 & \text{if } \{ijk, jkq, kqr, qrs\} \cap \{131, 313\} \neq \varnothing, \text{ or} \\ 0 & \text{if } \{ijkq, jkqr, kqrs\} \cap \{2312, 2132\} \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Again we see that there are equalities between columns of the matrix M. In particular, for any triple ijk

$$\begin{aligned} M(ijk, 111) &= M(ijk, 112) = M(ijk, 113) \\ M(ijk, 121) &= M(ijk, 122) = M(ijk, 123) \\ M(ijk, 131) &= M(ijk, 313) = 0 \\ M(ijk, 321) &= M(ijk, 322) = M(ijk, 323) \\ M(ijk, 331) &= M(ijk, 332) = M(ijk, 333) \\ M(ijk, 211) &= M(ijk, 2rs) \quad 1 \le r, s \le 3. \end{aligned} \tag{25}$$

As in the previous case, these equalities imply that the corresponding components of the eigenfunctions are identical. The bisection method with $m = 8$ and $\varepsilon = 6 \cdot 10^{-8}$ and gives the following lower and upper bounds on dimension

$$t_0 \colon = 0.6433544 < \dim_H X_A < 0.6433548 =: t_1. \tag{26}$$

The coefficients of the test functions $\underline{f}$ for $\mathcal{L}_{t_1}$ and $\underline{g}$ for $\mathcal{L}_{t_0}$, which are polynomials of degree 7 are tabled in §A.1.4. Using ball arithmetic we get bounds for the ratios

$$\sup_S \frac{\mathcal{L}_{t_1}\underline{f}}{\underline{f}} < 1 - 10^{-7}; \qquad \inf_S \frac{\mathcal{L}_{t_0}\underline{g}}{\underline{g}} > 1 + 10^{-7}, \tag{27}$$

and the estimates (26) follow from Lemma 3.3. Substituting the upper bound from (26) into the inequality (24) we obtain

$$\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(\sqrt{20},\sqrt{21}))<0.8110098.$$

The upper bound from (26) makes rigorous the heuristic bound $\dim_H(X_M)<0.65$ in [36], §B.5 which was based on the non-validated periodic point method.

Remark 4.1. In [36] there are also bounds on $\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(4.01,\sqrt{20}))$ which use estimates on $\dim_H(X_{1,2,3})$ due to Hensley. We reconfirm and improve these in Table 3.

Part 5: $\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(\sqrt{20},\sqrt{21}))$. In [36], §B.5 Matheus and Moreira established the following inequality:

$$\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(\sqrt{20},\sqrt{21})<\dim_H(X_M)+0.172825, \tag{28}$$

where $X_M\subset[0,1]$ is a set of numbers whose continued fractions expansions contain only digits 1, 2, 3, and 4, except that subsequences 14, 24, 41 and 42 are not allowed. This is the limit set of the Markov Iterated Function Scheme

$$T_1(x)=\frac{1}{1+x},\ T_2(x)=\frac{1}{2+x},\ T_3(x)=\frac{1}{3+x},\ T_4(x)=\frac{1}{4+x};\ M=\begin{pmatrix}1&1&1&0\\1&1&1&0\\1&1&1&1\\0&0&1&1\end{pmatrix}.$$

The bisection method with $m=10$ and $\varepsilon=6\cdot10^{-8}$ as before and the initial guess $t_0'=0.7$, $t_1'=0.71$ gives upper and lower bounds on the dimension:

$$t_0:=0.7093943<\dim_H X_M<0.7093945=:t_1. \tag{29}$$

The coefficients of the test functions $\underline{f}$ for $\mathcal{L}_{t_1}$ and $\underline{g}$ for $\mathcal{L}_{t_0}$, which are polynomials of degree 9 are tabled in §A.1.5. Using ball arithmetic we get bounds for the ratios

$$\sup_S\frac{\mathcal{L}_{t_1}\underline{f}}{\underline{f}}<1-10^{-7};\qquad\inf_S\frac{\mathcal{L}_{t_0}\underline{g}}{\underline{g}}>1+10^{-8}, \tag{30}$$

and the estimates (29) follow from Lemma 3.3. Substituting the upper bound from (29) into the inequality (28) we obtain

$$\dim_H((\mathcal{M}\setminus\mathcal{L})\cap(3.84,3.92))<0.8822195.$$

The upper bound from (29) confirms a heuristic estimate $\dim_H(X_M)<0.715$ (in [36], §B.6) obtained using the periodic points method.

Remark 4.2. We can very easily increase the accuracy significantly, at the expense of greater computation, but the present estimates seem sufficient for our needs.

Remark 4.3. The initial calculations were carried out on a MacBook Pro with a 2.8 GHz Quad-Core Intel i7 with 16GB DDR3 2133MHz RAM running MacOS Catalina using Mathematica. The bounds have been confirmed rigorously using a C program written by the second author based on Arb library for arbitrary precision ball arithmetic.

Remark 4.4. The periodic point method does not allow such accurate estimates on the computational error. In particular, it is possible to estimate the error in the case of Bernoulli iterated function schemes, but the estimate is ineffective for Markov systems. Similarly, for the McMullen algorithm [41] the errors are more difficult to estimate.

4.1.2 Lower bounds and the proof of Theorem 1.5

A further estimate on the Hausdorff dimension of the difference of the Lagrange and Markov spectra uses the dimension of the Cantor set of numbers in $(0,1)$ whose continued fraction expansion contains only digits 1 and 2:

$$E_2 \colon = \{[0; a_1, a_2, a_3, \ldots] \mid a_n \in \{1,2\} \text{ for all } n \in \mathbb{N}\}. \tag{31}$$

In particular, Matheus and Moreira [36] showed that $\mathcal{M} \setminus \mathcal{L}$ contains the image under a Lipschitz bijection of the set E_2 ([36], Theorem 5.3) from which they immediately deduce the following.

Theorem 4.5 (Matheus—Moreira)**.** $\dim_H(\mathcal{M} \setminus \mathcal{L}) \geq \dim_H(E_2)$.

Remark 4.6. Theorem 4.5 is a corollary of the slightly stronger local result that $\dim_H(\mathcal{M} \setminus \mathcal{L} \cap (3.7096, 3.7097)) \geq \dim_H(E_2)$ cf. ([36], Corollary 5.4).

It is easy to see that the set E_2 is the limit set for the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+x}.$$

In [25] there is a validated value for the Hausdorff dimension of $\dim_H(E_2)$ to 100 decimal places using periodic points method and a careful analysis of the error bounds. In this case the error estimates are easier because the system is Bernoulli. In [41] Theorem 4.5 is combined with the numerical value from [25] to give a lower bound on the dimension of the difference of the Markov and Lagrange spectrum.

We can use the approach in §3 this note to rigorously (re-)verify this bound. We begin with the lower bound. We choose $S = [0,1]$ since the iterated function scheme is Bernoulli. Let $m = 120$ and

$$\begin{aligned} t_0' &= 0.5312805062\,7720514162\,4468647368\,4717854930\,5910901839 \\ &\quad 8779888397\,8039275295\,3564383134\,5918109570\,1811852398 \end{aligned} \tag{32}$$

$$t_1' = t_0 + 10^{-100} \tag{33}$$

(taken from [25]). We can then use the Chebyshev—Lagrange interpolation to find a test function $g \colon [0,1] \to \mathbb{R}$ which is a polynomial of degree 119. We can then check that

$$\inf_S \frac{\mathcal{L}_{t_0'} g}{g} > 1 + 10^{-100}$$

and then applying Lemma 3.3 we can deduce that $\dim_H(E_2) > t_0$.

We proceed similarly to verify the upper bound t_1'. Namely, the Chebyshev—Lagrange interpolation gives another test function $f \colon [0,1] \to \mathbb{R}$ which is also a polynomial of degree 119 with the property that

$$\sup_S \frac{\mathcal{L}_{t_1'} f}{f} < 1 - 10^{-101}.$$

Thus by Lemma 3.3 we conclude that $\dim_H(E_2) < t_1'$.

Remark 4.7. This example demonstrates that, despite the fact that at first sight estimating the ratio of the image of the test function and the function itself could be potentially very time consuming and challenging, for many systems of particular interest, the derivative $\left(\frac{\mathcal{L}_q f}{f}\right)'$ turns out to decrease sufficiently fast as $m \to \infty$ to make realisation possible in practice.

In [25] the estimates involved computing $2^{25} = 33,554,432$ periodic points up to period 25 and the exponentially increasing amount of data needed makes it impractical to improve the rigorous estimate on $\dim_H(E_2)$ significantly. On the other hand, using the approach via Chebyshev—Lagrange interpolation and Lemma 3.3 we were able to confirm this result same accuracy using only two 120×120 matrices, and it would require about 600 matrices (of increasing size from 6×6 up to 120×120) in total to recompute this estimates starting with the initial guess $t_0 = 0$ and $t_1 = 1$. This represents a significant saving in memory usage at expense of computing 400 coefficients for the derivative estimates.

Moreover, we can now easily improve on the estimate using the bisection method combined with interpolation and Lemma 3.3 where the amount of data required by our analysis *grows linearly* with the accuracy required. Indeed, letting $m = 270$ and $\varepsilon = 10^{-200}$ we apply the bisection method choosing t_0' given by (32) and t_1' given by (33) as initial guess. It gives

$$\begin{aligned} t_0 = 0.&5312805062\,7720514162\,4468647368\,4717854930\,5910901839\,8779888397 \\ &8039275295\,3564383134\,5918109570\,1811852398\,8042805724\,3075187633 \\ &4223893394\,8082230901\,7869596532\,8712235464\,2997948966\,3784033728 \\ &7630454110\,1508045191\,3969768071\,2. \end{aligned} \tag{34}$$

and

$$t_1 = t_0 + 2 \cdot 10^{-201} \tag{35}$$

Then we can use the interpolation method to construct test functions f and g which are polynomials of degree 269 defined on the unit interval[8].

We can then explicitly compute

$$\inf_S \frac{\mathcal{L}_{t_0} f}{f} > 1 + 10^{-213}, \qquad \sup_S \frac{\mathcal{L}_{t_1} g}{g} < 1 - 10^{-211}.$$

which implies that $t_0 \le \dim_H(E_2) \le t_1$.

4.2 Zaremba Theory

In the introduction we described interesting results of Bourgain—Kontorovich [2], Huang [21], and Kan [29], [30], [31], which made progress towards the Zaremba Conjecture. These results have a slightly more general formulation, which we will now recall. For a finite alphabet set $A \subset \mathbb{N}$ consider the iterated function scheme

$$T_n \colon [0,1] \to [0,1], \qquad T_n(x) = \frac{1}{x+n} \text{ for } n \in A.$$

and denote its limit set by X_A.

[8] We omit a detailed listing of all the 540 coefficients of $\underline{f}$ and $\underline{g}$. However, they are easily recovered Mathematica.

For any $N \in \mathbb{N}$ we can in addition consider a set

$$D_A(N) \colon = \left\{ q \in \mathbb{N} \mid 1 \leq q \leq N, \exists p \in \mathbb{N}, (p,q) = 1; a_1, \cdots, a_n \in A \text{ with } \frac{p}{q} = [0; a_1, \cdots, a_n] \right\}.$$

In particular, when $A = \{1, 2, \cdots, m\}$ then D_A reduces to D_m as defined in the introduction.

We begin with the density one result [21], [22].

Theorem 4.8 (Bourgain—Kontorovich, Huang)**.** *Let $A \subset \mathbb{N}$ be a finite subset for which that associated set X_A satisfies $\dim_H(X_A) > \frac{5}{6} = 0.83\dot{3}$. Then*

$$\lim_{N \to +\infty} \frac{\# D_A(N)}{N} = 1.$$

The statement of Theorem 1.6 corresponds to the particular choice of alphabet $A = \{1, 2, 3, 4, 5\}$ in Theorem 4.8. Similarly, Theorem 1.7 has a slightly more general formulation (from [29]) as a positive density result.

We begin by recalling the following useful notation. Given two real-valued functions f and g we say that $f \gg g$ if there exist a constant c such that $f(x) > cg(x)$ for all x sufficiently large.

The statement of Theorem 1.7 corresponds to the particular choice of alphabet of $A = \{1, 2, 3, 4\}$ in Theorem 4.9.

Theorem 4.9 (Kan [29], Theorem 1.4)**.** *Let $A \subset \mathbb{N}$ be a finite set for which the associated limit set X_A has dimension $\dim_H(X_A) > \frac{\sqrt{19}-2}{3} = 0.7862\ldots$ Then*

$$\# \left\{ q \in \mathbb{N} \mid 1 \leq q \leq N \colon \exists p \in \mathbb{N}; a_1, \cdots, a_n \in A \text{ with } \frac{p}{q} = [0; a_1, \cdots, a_n] \right\} \gg N.$$

The derivation of Theorem 4.9 is conditional on the inequality $\dim_H(E_4) > \frac{\sqrt{19}-2}{3}$ which was based on the empirical computations by Jenkinson [24], but which were rigorously justified in [26]. We will rigorously (re)confirm this inequality in the next section using the approach in §3.

In the case that the Hausdorff dimension of the limit set X_A is smaller, in particular, $\dim_H(X_A) < \frac{5}{6}$ there are still some interesting lower bounds on $\# D_A(N)$. For convenience we denote (omitting dependence on A)

$$\delta \colon = \dim_H(X_A)$$

then a classical result of Hensley showed that $\# D_A(N) \gg N^{2\delta}$ [20]. Subsequently, this was refined in different ranges of δ as follows:

i) If $\frac{1}{2} < \delta < \frac{5}{6}$ then $\# D_A(N) \gg N^{\delta + (2\delta - 1)(1-\delta)/(5-\delta) - \varepsilon}$, for any $\varepsilon > 0$ [2].

ii) If $\frac{\sqrt{17}-1}{4} < \delta < \frac{5}{6}$ then $\# D_A(N) \geq N^{1-\varepsilon}$ for all $\varepsilon > 0$ [29].

iii) If $3 - \sqrt{5} < \delta < \frac{\sqrt{17}-1}{4}$, then $\# D_A(N) \geq N^{1 + \frac{2\delta^2 + 5\delta - 5}{2\delta - 1} - \varepsilon}$ [31].

As a concrete application, Kan considered the finite set $A = \{1, 2, 3, 5\}$ and by applying the inequality in iii) obtained the following lower bound.

Theorem 4.10 (Kan [31], Theorem 1.5, Remark 1.3). *For the alphabet $A = \{1, 2, 3, 5\}$ one has $\#D_A(N) \gg N^{0.85}$.*

This gave an improvement on the bound of $\#D_A(N) \gg N^{0.80}$ arising from i). However, this required that $\dim_H(X_A) > 3-\sqrt{5}$, an estimate conjectured by Jenkinson in [24], but which was not validated. We will present a rigorous bound in the next subsection.

To further illustrate this theme we will use the present method to confirm the following local version of the Zaremba conjecture proposed by Huang.

Theorem 4.11 (after Huang). *Let $A = \{1, 2, 3, 4, 5\}$ and consider $D_A = \cup_{n\in\mathbb{N}} D_A(n)$, in other words*

$$D_A\colon = \left\{ q \in \mathbb{N} \mid \exists p \in \mathbb{N}, (p, q) = 1 \text{ and } a_1, \cdots, a_n \in A \text{ with } \frac{p}{q} = [0; a_1, \cdots, a_n] \right\}.$$

Then for every $m > 1$ we have that $D_A = \mathbb{N}(\bmod m)$. In other words, for every $m > 1$ and every $q \in \mathbb{N}$ we have $q \,(\bmod m) \in D_A$.

4.2.1 Dimension estimates for E_5.

A crucial ingredient in the analysis in the proof of density one Theorem 1.6 used in [21] is that the limit set E_5 for the iterated function scheme

$$T_j : [0, 1] \to [0, 1], \quad T_j(x) = \frac{1}{j + x}, \quad 1 \le j \le 5$$

satisfies $\dim_H(E_5) > \frac{5}{6} = 0.83\dot{3}$. In [26], this was confirmed with rigorous bounds

$$\dim_H(E_5) = 0.836829445 \pm 5 \cdot 10^{-9}$$

using the periodic points method. For this particular example, the error estimates are more tractable because the iterated function scheme is Bernoulli, rather than just Markov.

However, we can use the method in this note to reconfirm this bound, and improve it, with very little effort, to the following:

Theorem 4.12.

$$\dim_H(E_5) = 0.836829443680 \pm 10^{-12}.$$

Proof. We can choose $S = [0, 1]$ since the iterated function scheme is Bernoulli. Applying the bisection method with $m = 15$ and $\varepsilon = 10^{-11}$, we get lower and upper bounds

$$t_0 = 0.83682944368\,02, \text{ and}$$
$$t_1 = t_0 + 2 \cdot 10^{-12} = 0.83682944368\,20.$$

The Chebyshev—Lagrange interpolation method then gives two polynomials of degree 14 that can serve as test functions. Their coefficients are listed in §A.2.1.

We can then explicitly compute

$$\sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-13}; \qquad \inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-13} \tag{36}$$

and the result follows from Lemma 3.3. □

Remark 4.13. In terms of the practical application to Theorem 1.6, there is no need to have accurate estimates of $\dim_H(E_5)$, it is sufficient to show $\dim_H(X) > \frac{5}{6}$. This can be achieved using a simple calculation "by hand". We can take

$$f(x) = \frac{2}{3} - \frac{11}{20}x + \frac{1}{3}x^2 - \frac{1}{10}x^3.$$

We can then compute that $\frac{\mathcal{L}_{5/6}f(x)}{f(x)} > 1.0029$. It then follows from Lemma 3.3 that $\dim_H(E_5) > \frac{5}{6}$.

4.2.2 Dimension estimates for E_4.

A crucial ingredient in the analysis in the proof of Theorem 1.7 used in [30] is that the limit set E_4 for the iterated function scheme

$$T_j : [0,1] \to [0,1], \quad T_j(x) = \frac{1}{j+x}, \quad 1 \le j \le 4$$

satisfies $\dim_H(E_4) > \frac{\sqrt{19}-2}{3} \approx 0.7862\ldots$

We can validate this result by showing the following bounds on the dimension

Theorem 4.14. $\dim_H(E_4) = 0.7889455574\,83 \pm 10^{-12}$

Proof. We can choose $S = [0,1]$ since the iterated function scheme is Bernoulli. Applying the bisection method with $m = 15$ and $\varepsilon = 10^{-11}$, we obtain the lower and upper bounds

$$t_0 = 0.7889455574\,81 \text{ and}$$
$$t_1 = t_0 + 2 \cdot 10^{-12} = 0.7889455574\,84.$$

The Chebyshev—Lagrange interpolation method then gives two polynomials of degree 14 that can serve as test functions. Their coefficients are listed in §A.2.2.

We can then explicitly compute

$$\sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-13}; \qquad \inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-12} \tag{37}$$

and the result follows from Lemma 3.3.

□

Remark 4.15. In terms of the practical application to Theorem 1.7, there is no need to have accurate estimates of $\dim_H(E_4)$, it is sufficient to show $\dim_H(X) > \frac{\sqrt{19}-2}{3}$. This can be achieved using a simple calculation "by hand". We can take

$$f(x) = \frac{27}{50} - \frac{11}{25}x + \frac{33}{100}x^2 - \frac{11}{50}x^3 + \frac{21}{200}x^4 - \frac{1}{40}x^5.$$

We can then compute that $[\mathcal{L}_{\frac{\sqrt{19}-2}{3}} f](x)/f(x) > 1.00205$. It then follows from Lemma 3.3 that $\dim_H(E_4) > \frac{\sqrt{19}-2}{3}$.

4.2.3 Dimension estimates for alphabet $A = \{1, 2, 3, 5\}$.

Theorem 4.16. *Let* $A = \{1, 2, 3, 5\}$*. Then* $\dim_H(X_A) = 0.7709149399\,375 \pm 1.5 \cdot 10^{-12}$.

Proof. The estimate can be recovered following the same approach as in the proof of theorem 4.14 and coefficients of the corresponding test functions listed in §A.2.3. We choose $S = [0, 1]$. Applying the bisection method with $m = 16$ and $\varepsilon = 3 \cdot 10^{-12}$, we obtain the lower and upper bounds

$$t_0 = 0.7709149399\,36 \text{ and}$$
$$t_1 = t_0 + 3 \cdot 10^{-12} = 0.7709149399\,39.$$

The Chebyshev—Lagrange interpolation method then gives two polynomials of degree 15 that can serve as test functions. Their coefficients are listed in §A.2.3.

We can then explicitly compute

$$\sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-12}, \qquad \inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-12};$$

and the result follows from Lemma 3.3.

□

Remark 4.17. In terms of the practical application to Theorem 4.10, there is no need to have accurate estimates of $\dim_H(E_{\{1,2,3,5\}})$, it is sufficient to show $\dim_H(X) > 3 - \sqrt{5}$. This can be achieved using a simple calculation "by hand". We can take

$$f(x) = \frac{9}{10} - \frac{2}{5}x.$$

We can then compute that $[\mathcal{L}_{3-\sqrt{5}} f](x)/f(x) > 1.00042$. It then follows from Lemma 3.3 that $\dim_H(X) > 3 - \sqrt{5}$.

Remark 4.18. The periodic point method and McMullen's approach [41] cannot give such accurate estimates because of the prohibitive computer resources required. In a recent paper [10] Falk and Nussbaum computed Hausdorff dimension of the sets E_5, E_4, E_{1235} and some of the Hensley examples we give below. Their method is also rooted in the interpolation, but uses different machinery.

4.3 Counter-example to a conjecture of Hensley

In [2] Bourgain and Kontorovich gave a counter-example to a conjecture of Hensley ([20], Conjecture 3, p.16). The conjecture stated that for any finite alphabet $A \subset \mathbb{N}$ for which the Hausdorff dimension of the associated limit set (corresponding to the iterated function scheme with contractions $T_j(z) = \frac{1}{j+x}$ for $j \in A$) satisfies $\dim_H(X_A) > \frac{1}{2}$ the analogue of the Zaremba conjecture holds true for q sufficiently large, i.e, there exists $q_0 > 0$ such that for any natural number $q \geq q_0$ there exists $p < q$ and $a_1, \cdots, a_n \in A$ such that $\frac{p}{q} = [a_1, \cdots, a_n]$.

The construction of the counter-example hinged on the observation that the denominators q corresponding to such restricted continued fraction expansions cannot ever satisfy $q = 3(\bmod 4)$ and on showing that for the iterated function scheme $\{T_j = \frac{1}{x+j} \mid a = 2, 4, 6, 8, 10\}$ the limit set X has dimension $\dim_H(X) > \frac{1}{2}$ (cf. [2,

p. 139]). This was rigorously confirmed in [26, Theorem 7] by a fairly elementary argument, where it was also suggested by a non-rigorous computation that

$$\dim_H X = 0.5173570309\,3701730466\,6628474836\,4397337\ldots \tag{38}$$

To rigorously justify this estimate, we apply the bisection method with $S = [0,1]$, $m = 40$, $\varepsilon = 10^{-36}$ and the initial guess

$$t_0 = 0.5173570309\,3701730466\,6628474836\,4397337$$
$$t_1 = t_0 + 10^{-37}.$$

The Chebyshev—Lagrange interpolation gives two polynomials f and g of degree 39 which satisfy

$$\sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-38} \text{ and } \inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-37}.$$

The equality (38) follows from Lemma 3.3.

Remark 4.19 (An elementary bound). As in the previous examples is not necessary to have a very precise knowledge of the value of $\dim_H(X_A)$ in order to establish that this is a counter example to the Hensley conjecture. It would be sufficient to know that $\dim_H(X_A) > \frac{1}{2}$. This can again be achieved using a simple calculation. We can consider instead the linear function $f(x) = 8 - 2x$. It is easy to compute its image under the transfer operator $\mathcal{L}_{0.5} f$:

$$\mathcal{L}_{0.5} f(x) = \left(\frac{8}{2+x} - \frac{2}{(2+x)^2}\right) + \left(\frac{8}{4+x} - \frac{2}{(4+x)^2}\right) + \left(\frac{8}{6+x} - \frac{2}{(6+x)^2}\right)$$
$$+ \left(\frac{8}{8+x} - \frac{2}{(8+x)^2}\right) + \left(\frac{8}{10+x} - \frac{2}{(10+x)^2}\right).$$

Clearly, this is a monotone decreasing function.

By taking derivatives or otherwise we can justify that

$$\inf_S \frac{\mathcal{L}_{0.5} f}{f} = \frac{1}{6} \mathcal{L}_{0.5} f(1) > 1.$$

The result now follows from Lemma 3.3.

Although the original Hensley conjecture is false, Moshchevitin and Shkredov recently proved an interesting modular version.

Theorem 4.20 ([43]). *Let $A \subset \mathbb{N}$ be a finite set for which $\dim_H(X_A) > \frac{1}{2}$. Then for any prime p there exist $q = 0(\bmod p)$, r (coprime to q) and $a_1, \cdots, a_n \in A$ such that*

$$\frac{r}{q} = [a_1, \cdots, a_n].$$

Thus there is some interest in knowing which examples of $A \subset \mathbb{N}$ satisfy $\dim_H(X_A) > \frac{1}{2}$ so that this result applies. For example, one can easily check that for $A_1 = \{1,4,9\}$ we have $\dim_H(X_{A_1}) = 0.5007902321\,42100396 \pm 10^{-18}$ or for $A_2 = \{2,3,6,9\}$ we have $\dim_H(X_{A_2}) = 0.5003228005\,96840463 \pm 10^{-18}$.

4.4 Primes as denominators

There is an interesting variation on Theorem 4.8 where we consider only the denominators which are prime numbers.

Theorem 4.21 (Bourgain—Kontorovich, Huang)**.** *There are infinitely many prime numbers q which have a primitive root*[9] *a* mod *q such that the partial quotients of $\frac{a}{q}$ are bounded by* 7*.*

This was originally proved by Bourgain and Kontorovich with the weaker conclusion that the partial quotients of $\frac{a}{q}$ are bounded by 51. The improvement of Huang was conditional on the Hausdorff dimension of limit set E_6 for the iterated function scheme $\left\{T_j(x) = \frac{1}{x+j} \mid 1 \leq j \leq 6\right\}$ satisfying $\dim_H E_6 > \frac{19}{22}$. In [26] it was rigorously shown using the periodic point method that

$$\dim_H E_6 = 0.86761915 \pm 10^{-8}.$$

Furthermore, there was a heuristic estimate of $\dim_H(E_6) = 0.8676191732401\ldots$ We can apply Chebyshev—Lagrange interpolation with $S = [0, 1]$, $m = 20$ to confirm this estimate.

Theorem 4.22.

$$\dim_H E_6 = 0.8676191732\,4015 \pm 10^{-13}.$$

Proof. By Chebyshev—Lagrange interpolation applied to the operators $\mathcal{L}_{t_0}$ and $\mathcal{L}_{t_1}$ we obtain two polynomials f and g of degree 19 which satisfy

$$\sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-13} \text{ and } \inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-14}.$$

The statement now follows from Lemma 3.3. □

Remark 4.23 (An elementary bound)*.* As in the previous two examples, it is not necessary to have a very precise knowledge of the value of $\dim_H(E_6)$ in order to establish the conditions necessary for Theorem 4.21. It would be sufficient to know that $\dim_H(E_6) > \frac{19}{22}$. This can again be achieved using a simplified choice of f, although it might be a slight exaggeration to say that this is entirely elementary. We can consider the degree 3 polynomial $f\colon [0, 1] \to \mathbb{R}^+$ defined by $f(x) = 0.67 - 0.57x + 0.35x^2 - 0.107x^3$. Letting $t = \frac{19}{22}$ we can consider the image under the transfer operator $\mathcal{L}_t$. In particular, one can readily check that

$$\inf_S \frac{\mathcal{L}_t f}{f} > 1 + 10^{-4}.$$

It follows from Lemma 3.3 that $P(t) > 0$ and thus we conclude that $\dim_H(E_6) > t = \frac{19}{22}$.

In the remainder of this section we will consider applications where the alphabets, and thus the number of contractions in the iterated function scheme, are infinite.

[9] In other words, there exists n such that $a^n = 1 \mod q$.

4.5 Modular results and countable iterated function schemes

Given $N \geq 2$ and $0 < r \leq N$, we want to consider a set

$$X_{r(N)} = \{[0; a_1, a_2, a_3, \cdots] \mid a_n \equiv r \,(\bmod N)\}$$

consisting of those numbers whose continued fraction expansion only has digits equal to r (mod N). This can be interpreted as a limit set for the countable family of contractions $T_i(x) = \frac{1}{x+r+Nk}$ $(k \geq 0)$. However, unlike the case of a finite iterated function scheme the limit set $X_{r(N)}$ is not a compact set.

Similarly to the case of a finite alphabet, a key ingredient in determining the Hausdorff dimension $\dim_H(X_{r(N)})$ is to consider a one-parameter family of transfer operators $\mathcal{L}_t : C^1([0,1] \to C^1([0,1])$ given by

$$(\mathcal{L}_t w)(x) = \sum_{k=0}^{\infty} (x + r + Nk)^{-2t} w\left((x + r + Nk)^{-1}\right). \tag{39}$$

It is well defined for $\Re(t) > \frac{1}{2}$. A common approach to the analysis of these operators is to truncate the series to a finite sum of first K, which contributes an error of $O\left(K^{-2t}\right)$ to the estimates of leading eigenvalue. Then this would require K to be chosen quite large for even moderate error bounds.

A more successful alternative approach in the present context is to employ the classical Hurwitz zeta function from analytic number theory.

Definition 4.24. *The Hurwitz zeta function is a complex analytic function on a half-plane $\Re x > 0$, $\Re s > 1$ defined by the series*

$$\zeta(x, s) = \sum_{k=0}^{\infty} (x + k)^{-s}.$$

It can be extended to a meromorphic function on $\mathbb{C}$ for $s \neq 1$. The famous Riemann zeta function is a particular case $\zeta(1, s)$.

We would like to consider monomials $w_n(x) := x^n$, $n \geq 0$ and to rewrite $\mathcal{L}_t w_n$ using the Hurwitz zeta function as follows

$$\begin{aligned}
\mathcal{L}_t w_n(x) &= \sum_{k=0}^{\infty} (x + kN + r)^{-2t} \cdot (x + kN + r)^{-n} \\
&= N^{-2t-n} \sum_{k=0}^{\infty} \left(\frac{x+r}{N} + k\right)^{-2t-n} \\
&= N^{-n-2t} \zeta\left(\frac{x+r}{N}, n + 2t\right).
\end{aligned} \tag{40}$$

From a computational viewpoint, the advantage we gain from expressing the transfer operator in terms of the Hurwitz zeta function stems from fact that there are very efficient algorithms for evaluation of $\zeta(x, s)$ to arbitrary numerical precision (cf. [27] and references therein). In particular, the Hurwitz zeta function is implemented both in Mathematica and within the Arb library.

We can now return to our usual strategy to estimate $\dim(X_{r(N)})$. We begin with the following simple result.

Lemma 4.25. *For $t > \frac{1}{2}$ the operator $\mathcal{L}_t$ has a simple maximal eigenvalue $e^{P(t)}$. The function P is real analytic and strictly decreasing on the interval $(\frac{1}{2}, +\infty)$. Moreover, there is a unique $t_0 \in (\frac{1}{2}, +\infty)$ such that $P(t_0) = 0$ and $t_0 = \dim(X_{r(N)})$.*

Proof. The analyticity comes from analytic perturbation theory and the simplicity of the maximal eigenvalue (see [37], Theorem 6.2.12). The strict monotonicity of the function P comes from the perturbation identity $P'(t) = -2\int(\log x)h(x)d\mu(x)/\int h d\mu < 0$ where $\mathcal{L}_t h = e^{P(t)}h$ and $\mathcal{L}_t^* \mu = e^{P(t)}\mu$. The interpretation of $\dim(X_{r(N)})$ in terms of the pressure was shown in [37], Theorem 4.2.13. □

Proceeding as in §3.2 we can fix m, compute the zeros of the m'th Chebyshev polynomials $\{y_k\}_{k=1}^m \in [0,1]$ and define Lagrange interpolation polynomials $lp_k(x)$ for $0 \le k \le m$ as before using (5). These polynomials can also be written in a standard form

$$lp_k(x) = \sum_{n=0}^{m-1} a_n^{(k)} x^n = \sum_{n=0}^{m-1} a_n^{(k)} w_n(x), \qquad \text{for } 0 \le x \le 1.$$

Since the operator $\mathcal{L}_t$ is linear, we can write using equation (40)

$$(\mathcal{L}_t lp_k)(x) = \sum_{n=0}^{m-1} a_n^{(k)} (\mathcal{L}_t w_n)(x) = \sum_{n=0}^{m-1} a_n^{(k)} N^{-n-2t} \zeta\left(\frac{x+r}{N}, n+2t\right). \qquad (41)$$

Since the Hurwitz zeta function can be evaluated to arbitrary precision, we can now compute the $m \times m$ matrix $(\mathcal{L}_t lp_k)(y_j)_{k,j=1}^m$, the associated left eigenvector $v = (v_1, \cdots, v_m)$ and construct a test function $f = \sum_{j=1}^m v_j lp_j$ to use in Lemma 3.3.

Example 4.26. *To illustrate the efficiency of the approach, we apply this general construction in a number of cases, which we borrow from a recent work by Chousionis at al. [7], where the estimates are obtained using a combination of a truncation technique with the method of Falk and Nussbaum. We apply the bisection method with $m = 12$ and summarise our results in the Table 1 below. For comparison, we include estimates from [7].*

$r(N)$	New estimate $\dim_H(X_{r(N)})$	Old bounds	
		s_0	s_1
2 (2)	$0.7194980248\,366 \pm 3 \cdot 10^{-13}$	0.719360	0.719500
1 (2)	$0.8211764906\,5 \pm 3.5 \cdot 10^{-10}$	0.821160	0.821177
3 (3)	$0.6407253143\,83684 \pm 2 \cdot 10^{-15}$	0.639560	0.640730
3 (2)	$0.6654623380\,4075 \pm 2.5 \cdot 10^{-13}$	0.664900	0.665460
3 (1)	$0.7435862804\,5 \pm 2.5 \cdot 10^{-10}$	0.743520	0.743586
1 (8)	$0.6194381921\,5 \pm 1.5 \cdot 10^{-10}$	N/A	N/A

Table 1: The estimates for $\dim_H(X_{r(N)})$ based on the bisection method with Hurwitz function employed and the bounds on $\dim_H(X_{r(N)}) \in [s_0, s_1]$ from [7].

4.6 Lower bounds for deleted digits

We can also use the method in the previous section to address the following natural problem: *Given $N \geq 1$ give a uniform upper bound on the dimension $\dim_H(X_{\mathcal{A}})$ where $\mathcal{A}$ ranges over all families of symbols satisfying $\mathcal{A} \cap \{1, \cdots, N\} \neq \emptyset$.*

By the natural monotonicity (by inclusion) of $\mathcal{A} \mapsto \dim_H(X_{\mathcal{A}})$ we see that such an upper bound will be given by $\dim_H(X_{\mathcal{A}_{N+1}})$ where we let

$$\mathcal{A}_{N+1} = \{N+1, N+2, N+3, \cdots\}.$$

As in the preceding subsection we can write the transfer operator associated to the infinite alphabet $\mathcal{A}_{N+1}$ acting on w_n in terms of the Hurwitz zeta function:

$$\begin{aligned} \mathcal{L}_t w_n(x) &= \sum_{k=N+1}^{\infty} (x+k)^{-2t} \cdot (x+k)^{-n} \\ &= \sum_{k=0}^{\infty} (x+N+1+k)^{-2t-n} \\ &= \zeta\left(x+N+1, n+2t\right). \end{aligned} \tag{42}$$

We have the natural analogue of Lemma 4.25.

Lemma 4.27. *For $t > \frac{1}{2}$ the operator $\mathcal{L}_t$ has a simple maximal eigenvalue $e^{P(t)}$. The function P is real analytic and strictly decreasing on the interval $(\frac{1}{2}, +\infty)$. Moreover, there is a unique $t_0 \in (\frac{1}{2}, +\infty)$ such that $P(t_0) = 0$ and $t_0 = \dim(X_{\mathcal{A}_{N+1}})$.*

Again proceeding as in §3.2 we can fix m, compute the zeros of the m'th Chebyshev polynomials $\{y_k\}_{k=1}^m \in [0,1]$ and define Lagrange interpolation polynomials $lp_k(x)$ for $0 \leq k \leq m$ as before using (5). By analogy with (39) we can write using equation (42)

$$(\mathcal{L}_t lp_k)(x) = \sum_{n=0}^{m-1} a_n^{(k)} (\mathcal{L}_t w_n)(x) = \sum_{n=0}^{m-1} a_n^{(k)} \zeta\left(x+N+1, n+2t\right). \tag{43}$$

Since the Hurwitz zeta function can be evaluated to arbitrary precision, we can now compute the $m \times m$ matrix $(\mathcal{L}_t lp_k)(y_j)_{k,j=1}^m$, the associated left eigenvector $v = (v_1, \cdots, v_m)$ and construct a test function $f = \sum_{j=1}^m v_j lp_j$ to use in Lemma 3.3. The results are presented in Table 2.

N	Estimate on $\dim_H(X_{\mathcal{A}_{N+1}})$
1	0.840884586 ± 10^{-8}
2	0.785953471 ± 10^{-8}
3	0.757889122 ± 10^{-8}
4	0.757889122 ± 10^{-8}
5	0.728307126 ± 10^{-8}

Table 2: Estimates on $\dim_H(X_{\mathcal{A}_{N+1}})$ which give upper bounds on $\dim_H(X_{\mathcal{A}})$ for those alphabets $\mathcal{A}$ with $\mathcal{A} \cap \{1, \cdots, N\} = \emptyset$.

4.7 Local obstructions

In his thesis, Huang makes an interesting conjecture on local obstructions for the Zaremba conjecture (see [21], p. 18). More precisely, to every $m \in \mathbb{N}$ we can associate the "modulo m map" which we denote by

$$\pi_m : \mathbb{Z}_+ \to \mathbb{Z}_m = \{0, 1, 2, \cdots, m-1\} \text{ given by } \pi_m(q) = q \,(\bmod m).$$

We say that a finite set $A \subset \mathbb{N}$ has no local obstructions if for all $m \geq 1$,

$$\pi_m \left(\left\{ q \in \mathbb{N} \middle| \exists p \in \mathbb{N}, (p,q) = 1; a_1, \cdots, a_n \in \{1,2,3,4,5\} : \frac{p}{q} = [0; a_1, \cdots, a_n] \right\} \right) = \mathbb{Z}_m$$

i.e., for each m the map π_m is surjective.

Conjecture 4.28 (Huang, [21])**.** *If the limit set X_A of the alphabet $A \subset \mathbb{N}$ has dimension $\dim_H(X_A) > \frac{5}{6}$ then there are no local obstructions.*

Huang also reduced this to a statement about dimensions of specific limit sets.

Proposition 4.29 ([21], Theorem 1.3.11)**.** *In notation introduced above, a* finite *alphabet A has no local obstructions provided* $\dim_H X_A > \max(\dim_H(X_{2(2)}), \dim_H(X_{1(8)}))$.

In particular, in light of the use of Proposition 4.29 to establish Conjecture 4.28 it is sufficient to know that $\dim_H(X_{2(2)})$ and $\dim_H(X_{1(8)})$ both have dimension at most $\frac{5}{6}$. Fortunately, we have rigorously established these inequalities as Table 1 shows.

Proposition 4.30. *Conjecture 4.28 above is correct.*

4.7.1 Truncation method

The use of the Hurwitz zeta function works well for the modular examples considered above. For more general countable alphabets $A \subset \mathbb{N}$ we may have to resort to a cruder approximation argument. To illustrate this we can consider the restriction to a finite alphabet $A_N = A \cap \{1, 2, \cdots, N\}$. If we let $\mathcal{L}_{A,t} f(x) = \sum_{n \in A} f((x+n)^{-1})(x+n)^{-2t}$ and $\mathcal{L}_{A_N,t} f(x) = \sum_{n \in A_N} f((x+n)^{-1})(x+n)^{-2t}$. Given t we can pick an m and apply the Chebyshev—Lagrange interpolation with m nodes to find a polynomial test function

$$f_{N,m}(x) = \sum_{j=0}^{m-1} a_j x^j$$

approximating the eigenfunction for $\mathcal{L}_{A_N,t}$. But if we subsequently want to use this function in Lemma 3.3 to study $\mathcal{L}_{A,t}$ we also need to obtain an upper bound for the remainder

$$E(x) = \sum_{n \in A \setminus A_N} f_{N,m}\left(\frac{1}{x+n}\right) \frac{1}{(x+n)^{2t}} = \sum_{j=0}^{m} a_j \sum_{n \in A \setminus A_N} \frac{1}{(x+n)^{j+2t}}.$$

We can also bound

$$\sum_{n \in A \setminus A_N} (x+n)^{-j-2t} \leq \sum_{N+1}^{\infty} (x+n)^{-j-2t} \leq \int_N^{\infty} x^{-j-2t} dx = \frac{1}{(j+2t-1) \cdot N^{j+2t-1}}$$

and then

$$|E(x)| \leq \sum_{j=0}^{m} |a_j| \frac{1}{(j+2t-1)\cdot N^{j+2t-1}}.$$

This bound is only polynomial in N giving this approach limited use in precise estimates, in comparison with limits sets generated by a finite number of contractions.

However, this elementary approach leads to a simple direct proof of Conjecture 4.28 (without resorting to introducing the Hurwitz zeta function).

Example 4.31 (An elementary bound for $\dim_H X_{0(2)}$ revisited)**.** *We will use a more elementary proof based on the interpolation method. We can consider the transfer operator*

$$(\mathcal{L}_t f)(x) = \sum_{n=1}^{\infty} f\left(\frac{1}{x+2n}\right)\frac{1}{(x+2n)^{2t}}.$$

We can separate the first term of the infinite series and estimate the remaining sum by an integral. More precisely, we may write

$$(\mathcal{L}_t f)(x) = f\left(\frac{1}{x+2}\right)\frac{1}{(x+2)^{2t}} + \sum_{n=2}^{\infty} f\left(\frac{1}{x+2n}\right)\frac{1}{(x+2n)^{2t}}.$$

We can consider the trivial test function $f = \mathbb{1}_{[0,1]}$ and obtain an upper bound

$$\begin{aligned}|E(x)| :&= \sum_{n=2}^{\infty} f\left(\frac{1}{x+2n}\right)\frac{1}{(x+2n)^{2t}} \leq \|f\|_\infty \left(\sum_{n=2}^{\infty}\frac{1}{(x+2n)^{2t}}\right) \leq \int_1^\infty \frac{1}{(2u)^{2t}}du \\ &= \frac{1}{(2t-1)2^{2t}}.\end{aligned}$$

In particular, if we take $t = \frac{5}{6}$ then $|E(x)| \leq 3\cdot 2^{-7/3}$. Thus

$$\sup_{[0,1]} \frac{\mathcal{L}_t \mathbb{1}_{[0,1]}}{\mathbb{1}_{[0,1]}} \leq \sup_{[0,1]}\left(\frac{1}{(2+x)^{5/3}} + E(x)\right) \leq 2^{-5/3} + 3\cdot 2^{-7/3} < 0.95.$$

We conclude that $P\left(\frac{5}{6}\right) < 0$ and Lemma 3.3 implies that $\dim_H(X_{even}) < \frac{5}{6}$.

Example 4.32 (An elementary bound for $\dim_H(X_{1(8)})$ revisited)**.** *This time we can take $f(x) = 1 - \frac{x}{2}$ and consider the image under the transfer operator*

$$\mathcal{L}_t f(x) = f\left(\frac{1}{x+1}\right)\frac{1}{(x+1)^{2t}} + \sum_{n=2}^{\infty} f\left(\frac{1}{x+8n-7}\right)\frac{1}{(x+8n-7)^{2t}} \qquad (44)$$

We have an upper bound for the remainder term

$$\begin{aligned}|E(x)| :&= \sum_{n=2}^{\infty} f\left(\frac{1}{x+8n-7}\right)\frac{1}{(x+8n-7)^{2t}} \\ &= \sum_{n=2}^{\infty}\frac{1}{(x+8n-7)^{2t}}\left(1 - \frac{1}{2(x+8n-7)}\right) \\ &\leq \left(\int_1^\infty \frac{1}{(8u-7)^{2t}}du - \frac{1}{2}\int_2^\infty \frac{1}{(8u-7)^{2t+1}}du\right) \\ &= \frac{1}{8}\left(\frac{1}{2t-1} - \frac{1}{4t\cdot 9^{2t}}\right).\end{aligned}$$

Substituting $t = \frac{5}{6}$ we obtain

$$|E(x)| \le \frac{3}{16}\left(1 - \frac{1}{5\cdot 9^{5/3}}\right). \tag{45}$$

In particular, we can now estimate

$$\sup_{[0,1]} \frac{\mathcal{L}_t f(x)}{f(x)} = \sup_{[0,1]} \left(\frac{f\left(\frac{1}{x+1}\right)}{f(x)(x+1)^{5/3}} + \frac{E(x)}{f(x)} \right) \le \sup_{[0,1]} \frac{f\left(\frac{1}{x+1}\right)}{f(x)(x+1)^{5/3}} + \frac{\sup_{[0,1]} E(x)}{\inf_{[0,1]} f(x)}$$
$$= \sup_{[0,1]} \frac{1 - \frac{1}{2(x+1)}}{(1-\frac{x}{2})(x+1)^{5/3}} + \frac{3}{8}\left(1 - \frac{1}{5\cdot 9^{5/3}}\right) < 1.$$

The result follows from Lemma 3.3.

4.8 Symmetric Schottky group

We can represent the limit set $X_\Gamma \subset \{z \in \mathbb{C} \mid |z| = 1\}$ of a Fuchsian Schottky group as the limit set of an associated Markov iterated function scheme. To construct the contractions it is more convenient to use the alternative model for hyperbolic space consisting of the upper half-plane $\mathbb{H}^2 = \{x + iy \mid y > 0\}$ supplied with the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. Geodesics on the upper half plane $\mathbb{H}^2$ are either circular arcs which meet the boundary orthogonally or vertical lines. Applying the transformation $T : \mathbb{D} \to \mathbb{H}^2$ given by $T : z \mapsto -i\frac{z-1}{z+1}$ we obtain three geodesics on $\mathbb{H}^2$. The group generated by reflections with respect to three geodesics with end points $e^{\pi i(2j\pm1)/6}$, $j = 0, 1, 2$ in the unit disk when transferred to the half-plane becomes a group generated by reflections with respect to half-circles, see Figure 5.

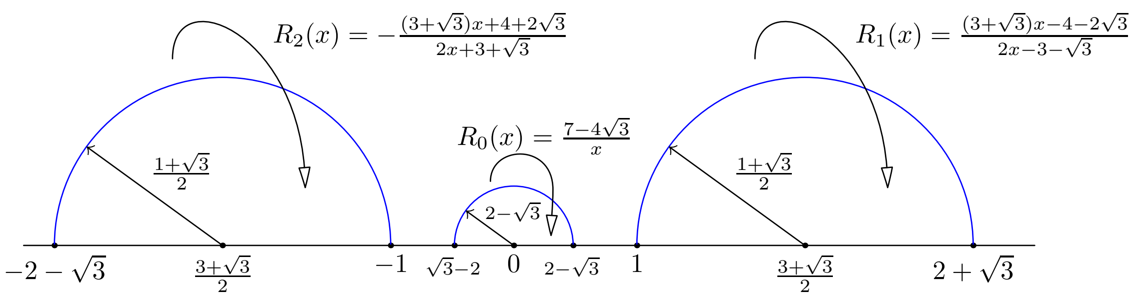

Figure 5: Group Γ generated by three reflections R_0, R_1, R_2.

A reflection with respect to a circle of radius r centred at c is given by the formula

$$R(z) = \frac{r^2}{(z-c)} + c. \tag{46}$$

To compute the radius and the centre of the circle of reflection, we calculate the end points by the formula $T(e^{i2\varphi}) = \tan\varphi$ and applying (46), we obtain

$$R_0(z) = \frac{7-4\sqrt{3}}{z}, \quad R_1(z) = \frac{(3+\sqrt{3})z - 4 - 2\sqrt{3}}{2z - 3 - \sqrt{3}}, \quad R_2(z) = -\frac{(3+\sqrt{3})z + 4 + 2\sqrt{3}}{2z + 3 + \sqrt{3}}. \tag{47}$$

Then the limit set $X_\Gamma \subset \cup_{j=0}^2 X_j \subset \mathbb{R}$ consists of accumulation points of the set

$$\{R_{j_1} R_{j_2} \cdots R_{j_n}(i) \mid j_1, j_2, \cdots, j_n \in \{0,1,2\} \text{ where } j_r \neq j_{r+1} \text{ for } 1 \le r \le n-1\}\,.$$

Since $T : \{z \colon |z| = 1\} \to \mathbb{R} \cup \{\infty\}$ is a conformal map we know that X_Γ has the same dimension as the corresponding limit set in the unit circle.

Proof of Theorem 1.9. In order to apply the technique developed in §3 we need to define a Markov iterated function scheme consisting of contractions whose limit set coincides with X_Γ. For instance we may consider the three intervals enclosed by the geodesics, more precisely, we define

$$X_0 := [-2+\sqrt{3}, 2-\sqrt{3}], \quad X_1 := [1, 2+\sqrt{3}], \quad X_2 := [-2-\sqrt{3}, -1]. \tag{48}$$

Then the limit set X_Γ for Γ can be identified with the limit set $X_A \subset \cup_{j=0}^2 X_j \subset \mathbb{R}$ of the Markov iterated function scheme with contractions $R_j \colon \cup_{k\neq j} X_k \to X_j$ for $j = 0, 1, 2$ with transition matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The associated transfer operator takes the form

$$\begin{aligned}
(\mathcal{L}_t \underline{f})_0(z) &= f_1(R_1(z)) \cdot |R_1'(z)|^t + f_2(R_2(z)) \cdot |R_2'(z)|^t, \quad z \in X_0 \\
(\mathcal{L}_t \underline{f})_1(z) &= f_0(R_0(z)) \cdot |R_0'(z)|^t + f_2(R_2(z)) \cdot |R_2'(z)|^t, \quad z \in X_1 \\
(\mathcal{L}_t \underline{f})_2(z) &= f_0(R_0(z)) \cdot |R_0'(z)|^t + f_1(R_1(z)) \cdot |R_1'(z)|^t, \quad z \in X_2.
\end{aligned}$$

Looking at the formulae (47) we may observe that $R_1(z) = -R_2(-z)$, $R_1'(z) = R_2'(-z)$, and $R_0(z) = -R_0(-z)$ and therefore $\mathcal{L}_t$ preserves the subspace

$$V_0 := \{(f_0, f_1, f_2) \in C^\alpha(S) \mid f_1(z) = f_2(-z), f_0(z) = f_0(-z)\}.$$

We can apply the bisection method to get rigorous estimates on the dimension of the limit set $X_\Gamma = X_A$ with the setting $S = \cup_{j=0}^2 X_j$, $m = 20$ and $\varepsilon = 10^{-7}$. We take the interpolation nodes to be zeros of Chebyshev polynomials transferred to each of the intervals X_j affinely. It gives

$$t_0 \colon = 0.29554647 < \dim_H X_A < 0.29554648 =: t_1. \tag{49}$$

Lagrange—Chebyshev interpolation gives test functions $\underline{f}, \underline{g} \in V_0$, whose components are presented in Figure 6.

We can estimate numerically

$$\sup_S \frac{\mathcal{L}_{t_1} \underline{f}}{\underline{f}} < 1 - 10^{-10}, \qquad \inf_S \frac{\mathcal{L}_{t_0} \underline{g}}{\underline{g}} > 1 + 10^{-8}.$$

The result follows from Lemma 3.3.

There is a simple connection between the dimension $\dim_H(X)$ of the limit set X and the smallest eigenvalue $\lambda_0 > 0$ of the Laplace—Beltrami operator on the non-compact surface $\mathbb{H}^2/\Gamma$ [51]. More precisely, $\lambda_0 = \dim_H(X_\Gamma)(1 - \dim_H(X_\Gamma))$. Applying the estimates (49), we obtain $\lambda_0 = 0.2081987565 \pm 2.5 \cdot 10^{-9}$. □

Remark 4.33. By increasing m it is an easy matter to get better estimates on the dimension of the limit set. For example, taking $m = 25$ we can improve the bounds to $\dim(X_\Gamma) = 0.2955464798\,845 \pm 4.5 \cdot 10^{-12}$ and $\lambda_0 = 0.208198758112 \pm 2.5 \cdot 10^{-13}$. The coefficients of the corresponding test functions are given in §A.3.

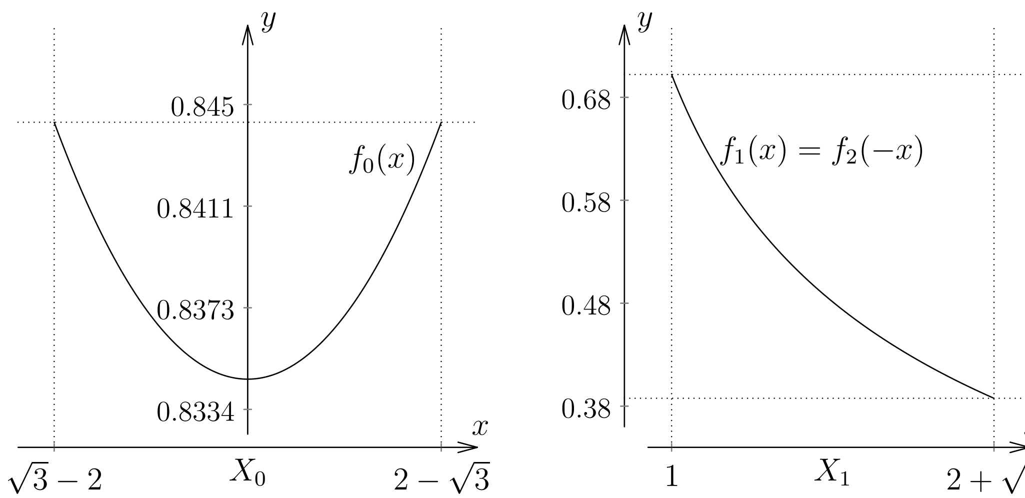

Figure 6: Theorem 1.9: The graphs of $f_0 : X_0 \to \mathbb{R}$ and $f_1 : X_1 \to \mathbb{R}$ (the plots of g_k, $k = 0, 1$ are similar). The plot of f_2 is the mirror image of f_1.

4.8.1 Other symmetric Schottky groups

More generally, McMullen [41] considered the Schottky group $\Gamma_\theta = \langle R_0, R_1, R_2 \rangle$ generated by reflections $R_0, R_1, R_2 : \mathbb{D}^2 \to \mathbb{D}^2$ in three symmetrically placed geodesics (with respect to the Poincaré metric) with six end points $e^{2\pi i j/3 \pm \theta/2}$, $j = 0, 1, 2$ on the unit circle (Figure 7).

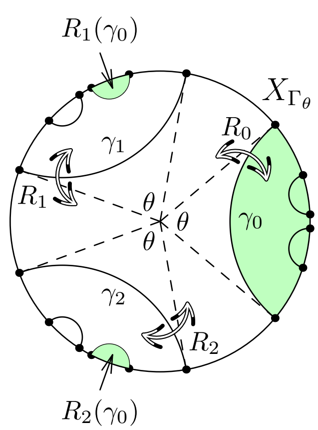

Figure 7: The group Γ_θ generated by three reflections in geodesics γ_0, γ_1, and γ_2.

Similarly to the special case that $\theta = \frac{\pi}{3}$ which we have already considered, one can transform the unit disk $\mathbb{D}^2$ to the upper half plane $\mathbb{H}^2$ and compute the centres and the radii of reflections

$$c_j = \frac{1}{2}\left(\tan\left(\frac{\pi j}{3} + \frac{\theta}{4}\right) + \tan\left(\frac{\pi j}{3} - \frac{\theta}{4}\right)\right), \text{ and}$$
$$r_j = \frac{1}{2}\left|\tan\left(\frac{\pi j}{3} + \frac{\theta}{4}\right) - \tan\left(\frac{\pi j}{3} - \frac{\theta}{4}\right)\right|, \ j = 0, 1, 2.$$

The limit set X_{Γ_θ} is again defined as the accumulation points of the orbit $\Gamma_\theta i$. We can introduce a corresponding Markov iterated function scheme whose limit set coincides with X_{Γ_θ}. We can consider two representative examples and estimate the Hausdorff dimension of the associated limit set.

Example 4.34 ($\theta = 2\pi/9$)**.** *In this case setting $m = 15$ we can obtain an estimate to 11 decimal places of the form*

$$\dim_H(X_{\Gamma_\theta}) = 0.2177658102\,55 \pm 5 \cdot 10^{-12}.$$

This agrees with McMullen's result (given to 8 decimal places).

Example 4.35 ($\theta = \pi/9$)**.** *In this case we can let $m = 12$ to deduce an estimate to 11 decimal places of the form*

$$\dim_H(X_{\Gamma_\theta}) = 0.1511836820\,35 \pm 5 \cdot 10^{-12}.$$

This agrees with McMullen's result (given to 8 decimal places).

4.9 Other iterated function schemes

To conclude we will collect together a number of other examples of iterated function schemes that have attracted attention of other authors and give estimates on the Hausdorff dimension of their limit sets.

4.9.1 Non-linear fractional example

So far we have been studying iterated function schemes generated by linear fractional transformations. Following [41], §6 we will consider a simple example of a map of $\mathbb{H}^2$ of a different nature. For any $0 < t \leq 1$ we can define

$$f_t(z) = \frac{z}{t} - \frac{1}{z}.$$

If $t = 1$ then the real line is f-invariant and there is no strictly smaller closed invariant set. If $0 < t < 1$ then there there exist a f-invariant Cantor $X_t \subset \mathbb{R}$ [49].

Example 4.36 ($t = \frac{1}{2}$)**.** *The map $f(z) = 2z - \frac{1}{z}$ has a limit set $X \subset [-1, 1]$ and there are two inverse branches $T_1, T_2 \colon [-1, 1] \to [-1, 1]$ given by*

$$\begin{aligned} T_1(x) &= \frac{1}{4}(x - \sqrt{8 + x^2}]) \\ T_2(x) &= \frac{1}{4}(x + \sqrt{8 + x^2}), \end{aligned} \tag{50}$$

which define a Bernoulli system on $[-1, 1]$. The transfer operator defined by (1) takes the form

$$(\mathcal{L}_t f)(x) = f(T_1(x))|T_1'(x)|^t + f(T_2(x))|T_2'(x)|^t.$$

We may observe that $T_1(x) = -T_2(-x)$ and $T_1'(x) = T_2'(-x)$. It follows that the transfer operator preserves subspaces consisiting of odd and even functions. Applying the bisection method with $S = [-1, 1]$, $m = 10$ we obtain that[10]

$$\dim_H X = 0.4934480908\,025 \pm 5 \cdot 10^{-13}.$$

[10] Table 14 in [41] gives an estimate 0.49344815, which is correct except for the last two significant figures. An inconsequential typographical mistake in [41] is that there is an incorrect sign in the equation in the caption to Table 14.

The corresponding test functions f *and* g *for* $t_0 = 0.493448088\,02$ *and* $t_1 = 0.4934480908\,03$, *respectively, turn out to be even and given by*

$$f(x) = \sum_{n=0}^{7} a_{2n}x^{2n} \qquad g(x) = \sum_{n=0}^{7} b_{2n}x^{2n},$$

which are plotted in Figure 8 and whose coefficients are given in §A.4.

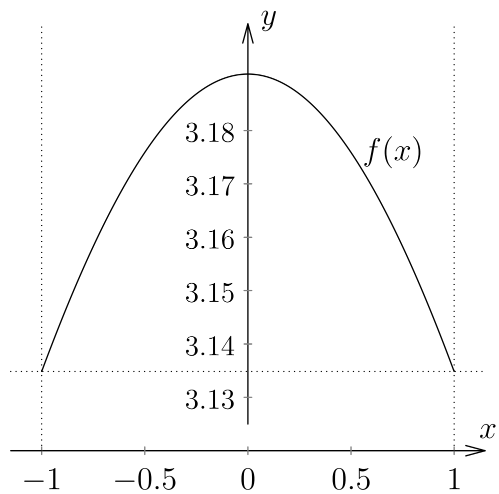

Figure 8: A plot of the function f for the system (50) (the function g being similar)

We can also compute

$$\inf_S \frac{\mathcal{L}_{t_0} f}{f} > 1 + 10^{-13} \qquad \sup_S \frac{\mathcal{L}_{t_1} g}{g} < 1 - 10^{-13}.$$

to justify the dimension estimates above.

Remark 4.37. These estimates can easily be improved by increasing the number of Chebyshev points. For example, letting $m = 20$ gives a better estimate

$$\dim_H(X) = 0.4934480908\,02613 \pm 10^{-15}.$$

4.9.2 Hensley Examples

In a well known article from 1992, Hensley [19] presented an algorithm for calculating the Hausdorff dimension of the limit sets for suitable iterated function schemes. In this article there was included a table containing estimates on selected examples which were good for the computational resources available at the time and continued to be quoted up to the present time. It is a simple matter to apply the method in §3 to improve these estimates. We give lower and upper bounds in Table 3.

In Table 1 in another paper by Hensley [20], there are a number of numerical results on Hausdorff dimension various of limit sets. Let us consider two typical examples from the list.

Alphabet A	Hensley Estimate	d
$1, 2$	0.5312805062 772051416	0.5312805062 7720514162 4
$1, 3$	0.4544827	0.4544890776 6182874384 5
$1, 4$	0.4111827	0.4111827247 7479177684 4
$2, 3$	0.337437	0.3374367808 0606363630 4
$2, 4$	0.306313	0.3063127680 5278403027 7
$3, 4$	0.263737	0.2637374828 9742655875 0
$1, 2, 3$	0.7056609080	0.7056609080 2873823060 7
$1, 2, 4$	0.66922149	0.6692214869 1028607643 2
$1, 3, 4$	0.6042422606 9111965	0.6042422577 5648956551 0
$2, 3, 4$	0.480696	0.4806962223 1757304132 2
$1, 2, 3, 4$	0.788946	0.7889455574 8315397254 0
$1, 2, 7$	0.6179036954 6338	0.6179036954 6337565066 2
$1, 3, 7$	0.55324225	0.5532422505 6731096881 6
$1, 4, 7$	0.51788376	0.5178837570 0691696528 4
$2, 3, 7$	0.43801241	0.4380124057 1403118230 1
$2, 4, 7$	0.410329	0.4103293158 3768700408 7
$3, 4, 7$	0.36757914	0.3675791395 9190093176 3
$1, 2, 3, 7$	0.75026306	0.7502630613 3714304325 2
$1, 2, 4, 7$	0.7185418875	0.7185418874 7036994981 8
$2, 3, 4, 7$	0.540036	0.5400358121 5475951902 6
$1, 2, 3, 4, 7$	0.820004	0.8200039471 2686900746 5
$10, 11$	0.146921	0.1469212353 9078346331 0
$100, 10000$	0.052247	0.0522465926 3865887865 1
$2, 7$	0.26022398	0.2602238774 2217867170 7
$1, 3, 4, 7$	0.66015538	0.6601553798 3237807776 6
$1, 7$	0.34623824	0.3462382435 3395787983 0
$4, 7$	0.2052533419 4	0.2052534193 6736493221 5
$3, 7$	0.2249239471 918	0.2249239471 9177898918 3
$1, 2, 3, 4, 5$	0.8368294437	0.8368294436 8120882244 2
$2, 3, 4, 5$	0.55963645	0.5596364501 6477671331 0
$2, 3, 5$	0.4616137	0.4616136840 1828922267 4
$1, 500$	0.1094760117 37	0.1094760117 3723275274 5

Table 3: Numerical data for Hensley examples from [19]; $\dim_H X_A = d \pm 10^{-20}$.

Alphabet A	Jenkinson Estimate	d
$1,3,8$	0.5438	0.5438505824 0696620129 10871
$1,3,6$	0.5652	0.5652752192 8623250537 07768
$1,3,5$	0.5813	0.5813668211 3469731449 44763
$1,2,10$	0.5951	0.5951117365 4560755184 18957
$1,3,4$	0.6042	0.6042422576 9541848140 50596
$1,2,7,40$	0.6265	0.6265741168 9229403866 4271
$1,2,5$	0.6460	0.6460620828 3482621991 26074
$1,2,5,40$	0.6532	0.6532480771 5774872727 88226
$1,2,4$	0.6692	0.6692214868 6131601289 10582
$1,2,4,40$	0.6754	0.6754204446 6970405658 86491
$1,2,4,15$	0.6899	0.6899117699 4923640369 39765
$1,2,4,6$	0.7275	0.7275240485 5844736070 17215
$1,2,4,5$	0.7400	0.7400268606 0207750663 59866
$1,2,3,6$	0.7588	0.7588596765 7522348478 34758
$1,2,3,5$	0.7709	0.7709149398 4418222560 66922
$1,2,3,4,10$	0.8081	0.8081711218 9508471948 06225
$1,2,3,4,6$	0.8269	0.8269084945 9163116837 24267
$1,2,3,4,5,9$	0.8541	0.8541484705 3932261542 70362
$1,2,3,4,5,7$	0.8616	0.8616561744 0626491056 99743
$1,2,3,4,5,6$	0.8676	0.8676191730 6718378091 2243
$1,2,3,4,5,6,8$	0.8851	0.8851175915 5644894823 12343
$1,2,3,4,5,6,7$	0.8889	0.8889553164 9195167843 64394
$1,2,\dots,8$	0.9045	0.9045526893 2916142728 20095
$1,2,\dots,9$	0.9164	0.9164211122 6835174040 64645
$1,2,\dots,10$	0.9257	0.9257375908 8754612367 25506
$1,2,\dots,13$	0.9445	0.9445341091 7126158776 76671
$1,2,\dots,18$	0.961	0.9611931848 1599230516 44346
$1,2,\dots,34$	0.980	0.9804196247 7958255969 58015

Table 4: Numerical data for Jenkinson examples from [24]; $\dim_H X_A = d \pm 2 \cdot 10^{-24}$.

(i). Let $X_{1,2,7} = \{[0; a_1, a_2, a_3, \cdots] \mid a_n \in \{1,2,7\}\}$. Hensley presents an estimate

$$\dim_H(X_{1,2,7}) = 0.6179036954\,6338,$$

accurate to 13 decimal places. The bisection method with $S = [0,1]$, $\varepsilon = 10^{-23}$ and $m = 30$ gives

$$\dim_H(X) = 0.6179036954\,6337565066\,3413 \pm 10^{-24},$$

with the corresponding test functions $\underline{f}$ and $\underline{g}$ satisfying

$$\inf_S \frac{\mathcal{L}_{t_0} g}{g} > 1 + 10^{-24}, \qquad \sup_S \frac{\mathcal{L}_{t_1} f}{f} < 1 - 10^{-24}.$$

(ii). Let $X_{1,3,4} = \{[0; a_1, a_2, a_3, \cdots] \mid a_n \in \{1,3,4\}\}$. Hensley presents an estimate $\dim_H(X_{1,3,5}) = 0.6042422606\,9111965$. However, this is only accurate to seven decimal places (there seeming to be a typographical error) and applying the bisection method with $S = [0,1]$, $\varepsilon = 10^{-23}$ and $m = 30$ we can correct the estimate as follows.

$$\dim_H(X) = 0.6042422577\,5648956551\,0773 \pm 10^{-24}.$$

4.9.3 Other limit sets

In [42] Moreira considered a limit set X for the IFS

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+\frac{1}{2+x}}.$$

After Theorem 3.4 therein he gives a rigorous estimate $0.353 < \dim_H(X) < 0.3572$[11]. Applying the bisection method with $\varepsilon = 10^{-30}$ and $m = 40$ we obtain

$$\dim_H X = 0.3554004768\,3384079791\,6306289490\,45 \pm 5 \cdot 10^{-32}.$$

There are also additional examples studied by Jenkinson, in connection with his numerical investigation of the Texan conjecture. It is a simple matter to apply the bisection method to compute intervals $[t_0, t_1]$ containing the actual values and these are presented in Table 4.

A Coefficients for polynomials

For completeness, we collect together the coefficients of the polynomials which appear in the proofs of the theorems. The exceptions to this is Theorem 1.5 where the polynomials are of degree 200. However, in all of these examples the reader may easily reconstruct these polynomials using the method described in §3.2 and §3.3.

Coefficients given in this section are exact rational numbers.

A.1 Estimates of $\dim_H \mathcal{M} \setminus \mathcal{L}$

A.1.1 Part 1: $(\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{5}, \sqrt{13})$

We present coefficients of the test functions $\underline{f} = (f_1, f_2, f_3, f_4)$ and $\underline{g} = (g_1, g_2, g_3, g_4)$ used in (17).

$$f_j = \sum_{k=0}^{7} a_k^j x^k \quad g_j = \sum_{k=0}^{7} b_k^j x^k, \qquad j = 1, 2, 3, 4.$$

Straightforward calculation shows that the functions f_j and g_j are monotone decreasing on $[0, 1]$ and achieve their minima at 1. For convenience, we give a lower bound $s_j < \min(f_j(1), g_j(1))$.

f_1		f_2		f_3		f_4	
a_0^1	0.9719420630	a_0^2	0.6996881504	a_0^3	0.5151068706	a_0^4	1.0035153909
a_1^1	−0.4083622154	a_1^2	−0.3257856180	a_1^3	−0.1503102708	a_1^4	−0.3675009146
a_2^1	0.2105627503	a_2^2	0.1808409982	a_2^3	0.0521402712	a_2^4	0.1667939317
a_3^1	−0.1166905024	a_3^2	−0.1054480473	a_3^3	−0.0190434708	a_3^4	−0.0824212096
a_4^1	0.0649256757	a_4^2	0.0606297675	a_4^3	0.0070462566	a_4^4	0.0416454207
a_5^1	−0.0321113965	a_5^2	−0.0305675512	a_5^3	−0.0024652813	a_5^4	−0.0191792606
a_6^1	0.0114531320	a_6^2	0.0110148596	a_6^3	0.0006866462	a_6^4	0.0065321353
a_7^1	−0.0020311821	a_7^2	−0.0019639179	a_7^3	−0.0001041463	a_7^4	−0.0011267545
s_1	0.6	s_2	0.4	s_3	0.4	s_4	0.7

[11] Their application was superseded by other work, so the interest in this bound is mainly academic.

g_1		g_2		g_3		g_4	
b_0^1	0.9719420489	b_0^2	0.6996881913	b_0^3	0.5151068139	b_0^4	1.0035153929
b_1^1	−0.4083624405	b_1^2	−0.3257858144	b_1^3	−0.1503103363	b_1^4	−0.3675011259
b_2^1	0.2105629174	b_2^2	0.1808411480	b_2^3	0.0521403058	b_2^4	0.1667940700
b_3^1	−0.1166906125	b_3^2	−0.1054481493	b_3^3	−0.0190434862	b_3^4	−0.0824212916
b_4^1	0.0649257436	b_4^2	0.0606298318	b_4^3	0.0070462630	b_4^4	0.0416454670
b_5^1	−0.0321114321	b_5^2	−0.0305675854	b_5^3	−0.0024652837	b_5^4	−0.0191792835
b_6^1	0.0114531451	b_6^2	0.0110148723	b_6^3	0.0006866469	b_6^4	0.0065321434
b_7^1	−0.0020311845	b_7^2	−0.0019639202	b_7^3	−0.0001041464	b_7^4	−0.0011267559
s_1	0.6	s_2	0.4	s_3	0.4	s_4	0.7

A.1.2 Part 2: $(\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{13}, 3.84)$

We present coefficients of the test functions $\underline{f} = (f_1, f_2, f_3)$ and $\underline{g} = (g_1, g_2, g_3)$ used in (20).

$$f_j = \sum_{k=0}^{7} a_k^j x^k \quad g_j = \sum_{k=0}^{7} b_k^j x^k, \quad j = 1, 2, 3.$$

Straightforward calculation shows that the functions f_j and g_j are monotone decreasing on $[0, 1]$ and achieve their minima at 1. For convenience, we give a lower bound $s_j < \min(f_j(1), g_j(1))$.

f_1		f_2		f_3	
a_0^1	0.8909247279	a_0^2	1.0057862651	a_0^3	0.4637543240
a_1^1	−0.5666958388	a_1^2	−0.6057959903	a_1^3	−0.1989455013
a_2^1	0.3563184411	a_2^2	0.3687733983	a_2^3	0.0812067365
a_3^1	−0.2268317558	a_3^2	−0.2307067719	a_3^3	−0.0328166697
a_4^1	0.1399500506	a_4^2	0.1411336470	a_4^3	0.0130694788
a_5^1	−0.0741972949	a_5^2	−0.0745402261	a_5^3	−0.0048273012
a_6^1	0.0275558152	a_6^2	0.0276379657	a_6^3	0.0013941859
a_7^1	−0.0049924560	a_7^2	−0.0050037264	a_7^3	−0.0002161328
s_1	0.5	s_2	0.6	s_3	0.3

g_1		g_2		g_3	
b_0^1	0.8909246136	b_0^2	1.0057862594	b_0^3	0.4637545188
b_1^1	−0.5666952424	b_1^2	−0.6057953968	b_1^3	−0.1989454076
b_2^1	0.3563178857	b_2^2	0.3687728381	b_2^3	0.0812066594
b_3^1	−0.2268313254	b_3^2	−0.2307063388	b_3^3	−0.0328166280
b_4^1	0.1399497528	b_4^2	0.1411333481	b_4^3	0.0130694591
b_5^1	−0.0741971263	b_5^2	−0.0745400571	b_5^3	−0.0048272932
b_6^1	0.0275557503	b_6^2	0.0276379006	b_6^3	0.0013941834
b_7^1	−0.0049924440	b_7^2	−0.0050037144	b_7^3	−0.0002161324
s_1	0.5	s_2	0.6	s_3	0.3

A.1.3 Part 3: $(\mathcal{M} \setminus \mathcal{L}) \cap (3.84, 3.92)$

We present coefficients of the polynomial test functions $\underline{f} = (f_1, \ldots, f_7)$ and $\underline{g} = (g_1, \ldots, g_7)$ used in (23). It follows from the equality between columns of the transition matrix M that certain components are identical.

$$f_1 = f_2 = \sum_{k=0}^{7} a_k^1 x^k, \qquad g_1 = g_2 = \sum_{k=0}^{7} b_k^1 x^k;$$

$$f_4 = f_5 = f_6 = \sum_{k=0}^{7} a_k^4 x^k, \qquad g_4 = g_5 = g_6 = \sum_{k=0}^{7} b_k^4 x^k;$$

$$f_j = \sum_{k=0}^{7} a_k^j x^k, \qquad g_j = \sum_{k=0}^{7} b_k^j x^k; \quad j = 3, 7, 8, 9.$$

Straightforward calculation shows that the functions f_j and g_j are monotone decreasing on $[0, 1]$ and achieve their minima at 1. For convenience, we give a lower bound $s_j < \min(f_j(1), g_j(1))$.

	f_1		f_3		f_4
a_0^1	0.8752491446	a_0^3	0.8242977486	a_0^4	0.9435673129
a_1^1	−0.5862938673	a_1^3	−0.5673413247	a_1^4	−0.6108763701
a_2^1	0.3817921234	a_2^3	0.3753845810	a_2^4	0.3898319394
a_3^1	−0.2509102043	a_3^3	−0.2488176120	a_3^4	−0.2534505061
a_4^1	0.1594107381	a_4^3	0.1587442151	a_4^4	0.1601941335
a_5^1	−0.0866513835	a_5^3	−0.0864513792	a_5^4	−0.0868796783
a_6^1	0.0328162711	a_6^3	0.0327671074	a_6^4	0.0328711315
a_7^1	−0.0060331410	a_7^3	−0.0060262852	a_7^4	−0.0060406787
s_1	0.4	s_3	0.4	s_4	0.5

	f_7		f_8		f_9
a_0^7	0.2183470684	a_0^8	0.5232653503	a_0^9	1.0058222185
a_1^7	−0.0767355580	a_1^8	−0.2253827130	a_1^9	−0.6095554508
a_2^7	0.0245616045	a_2^8	0.0906682129	a_2^9	0.3705808500
a_3^7	−0.0076106356	a_3^8	−0.0360797100	a_3^9	−0.2326605853
a_4^7	0.0023068479	a_4^8	0.0141972990	a_4^9	0.1434620328
a_5^7	−0.0006627640	a_5^8	−0.0052036970	a_5^9	−0.0765436603
a_6^7	0.0001576564	a_6^8	0.0014967921	a_6^9	0.0286722918
a_7^7	−0.0000215279	a_7^8	−0.0002316038	a_7^9	−0.0052379012
s_7	0.1	s_8	0.3	s_9	0.5

	g_1		g_3		g_4
b_0^1	0.8752491756	b_0^3	0.8242978103	b_0^4	0.9435673145
b_1^1	−0.5862941999	b_1^3	−0.5673416594	b_1^4	−0.6108767042
b_2^1	0.3817924531	b_2^3	0.3753849098	b_2^4	0.3898322718
b_3^1	−0.2509104702	b_3^3	−0.2488178772	b_3^4	−0.2534507734
b_4^1	0.1594109280	b_4^3	0.1587444046	b_4^4	0.1601943239
b_5^1	−0.0866514938	b_5^3	−0.0864514894	b_5^4	−0.0868797888
b_6^1	0.0328163145	b_6^3	0.0327671507	b_6^4	0.0328711749
b_7^1	−0.0060331491	b_7^3	−0.0060262933	b_7^4	−0.0060406868
s_1	0.4	s_3	0.4	s_4	0.5

g_7		g_8		g_9	
b_0^7	0.2183469977	b_0^8	0.5232652393	b_0^9	1.0058222217
b_1^7	−0.0767355702	b_1^8	−0.2253827769	b_1^9	−0.6095557922
b_2^7	0.0245616149	b_2^8	0.0906682634	b_2^9	0.3705811764
b_3^7	−0.0076106401	b_3^8	−0.0360797369	b_3^9	−0.2326608405
b_4^7	0.0023068496	b_4^8	0.0141973115	b_4^9	0.1434622111
b_5^7	−0.0006627646	b_5^8	−0.0052037021	b_5^9	−0.0765437625
b_6^7	0.0001576566	b_6^8	0.0014967937	b_6^9	0.0286723317
b_7^7	−0.0000215279	b_7^8	−0.0002316041	b_7^9	−0.0052379086
s_7	0.1	s_8	0.3	s_9	0.5

A.1.4 Part 4: $(\mathcal{M} \setminus \mathcal{L}) \cap (3.92, 4.01)$

It follows from the equalities between columns of the transition matrix M (25), that components of the test function $\underline{f} = (f_{111}, \ldots, f_{333})$ and $\underline{g} = (g_{111}, \ldots, g_{333})$ used in (27) satisfy the following identities.

$$f_{111} = f_{112} = f_{113} = \sum_{k=0}^{7} a_k^{111} x^k \qquad g_{111} = g_{112} = g_{113} = \sum_{k=0}^{7} b_k^{111} x^k,$$

$$f_{211} = f_{2rs} = \sum_{k=0}^{7} a_k^{211} x^k \qquad g_{211} = g_{2rs} = \sum_{k=0}^{7} b_k^{211} x^k, \ 1 \le r, s \le 3,$$

$$f_{121} = f_{122} = f_{123} = \sum_{k=0}^{7} a_k^{121} x^k \qquad g_{121} = g_{122} = g_{123} = \sum_{k=0}^{7} b_k^{121} x^k,$$

$$f_{321} = f_{322} = f_{323} = \sum_{k=0}^{7} a_k^{123} x^k \qquad g_{321} = g_{322} = g_{323} = \sum_{k=0}^{7} b_k^{123} x^k,$$

$$f_{331} = f_{332} = f_{333} = \sum_{k=0}^{7} a_k^{133} x^k \qquad g_{331} = g_{332} = g_{333} = \sum_{k=0}^{7} b_k^{133} x^k.$$

Straightforward calculation shows that the functions f_j and g_j are monotone decreasing on $[0, 1]$ and achieve their minima at 1. For convenience, we give a lower bound $s_{qrs} < \min(f_{qrs}(1), g_{qrs}(1))$.

f_{111}		f_{121}		f_{132}	
a_0^{111}	0.9884535079	a_0^{121}	0.9249363898	a_0^{132}	0.5745876575
a_1^{111}	−0.6902235127	a_1^{121}	−0.6661893079	a_1^{132}	−0.4908751097
a_2^{111}	0.4629840472	a_2^{121}	0.4549027221	a_2^{132}	0.3761466270
a_3^{111}	−0.3124708511	a_3^{121}	−0.3098685263	a_3^{132}	−0.2757231992
a_4^{111}	0.2030567394	a_4^{121}	0.2022427698	a_4^{132}	0.1878939498
a_5^{111}	−0.1122167788	a_5^{121}	−0.1119770597	a_5^{132}	−0.1064665719
a_6^{111}	0.0429529760	a_6^{121}	0.0428949582	a_6^{132}	0.0412628171
a_7^{111}	−0.0079480141	a_7^{121}	−0.0079400087	a_7^{132}	−0.0076832222
s_{111}	0.5	s_{121}	0.4	s_{132}	0.2

f_{133}		f_{211}		f_{311}	
a_0^{133}	0.8813943456	a_0^{211}	1.0003583962	a_0^{311}	0.5081203394
a_1^{133}	−0.6491441142	a_1^{211}	−0.6654322134	a_1^{311}	−0.2326160640
a_2^{133}	0.4489736082	a_2^{211}	0.4295333392	a_2^{311}	0.0972736889
a_3^{133}	−0.3078936043	a_3^{211}	−0.2812396746	a_3^{311}	−0.0398841238
a_4^{133}	0.2016042974	a_4^{211}	0.1785878079	a_4^{311}	0.0160797615
a_5^{133}	−0.1117833249	a_5^{211}	−0.0971302423	a_5^{311}	−0.0060041586
a_6^{133}	0.0428469743	a_6^{211}	0.0368077301	a_6^{311}	0.0017487411
a_7^{133}	−0.0079332871	a_7^{211}	−0.0067699556	a_7^{311}	−0.0002726247
s_{133}	0.4	s_{211}	0.5	s_{311}	0.3

f_{312}		f_{321}		f_{331}	
a_0^{312}	0.2013136513	a_0^{321}	0.8072631514	a_0^{331}	1.0061848844
a_1^{312}	−0.0743470595	a_1^{321}	−0.4648188145	a_1^{331}	−0.6476491585
a_2^{312}	0.0244467077	a_2^{321}	0.2575884636	a_2^{331}	0.4067501349
a_3^{312}	−0.0077137186	a_3^{321}	−0.1452652325	a_3^{331}	−0.2603687665
a_4^{312}	0.0023694138	a_4^{321}	0.0809083385	a_4^{331}	0.1624071150
a_5^{312}	−0.0006874054	a_5^{321}	−0.0398683005	a_5^{331}	−0.0872158384
a_6^{312}	0.0001645838	a_6^{321}	0.0141619841	a_6^{331}	0.0327839671
a_7^{312}	−0.0000225598	a_7^{321}	−0.0025036826	a_7^{331}	−0.0060002449
s_{312}	0.1	s_{321}	0.4	s_{331}	0.5

g_{111}		g_{121}		g_{132}	
b_0^{111}	0.9884535050	b_0^{121}	0.9249364233	b_0^{132}	0.5745877791
b_1^{111}	−0.6902239809	b_1^{121}	−0.6661897749	b_1^{132}	−0.4908755105
b_2^{111}	0.4629845398	b_2^{121}	0.4549032114	b_2^{132}	0.3761470588
b_3^{111}	−0.3124712630	b_3^{121}	−0.3098689364	b_3^{132}	−0.2757235761
b_4^{111}	0.2030570414	b_4^{121}	0.2022430710	b_4^{132}	0.1878942344
b_5^{111}	−0.1122169575	b_5^{121}	−0.1119772381	b_5^{132}	−0.1064667431
b_6^{111}	0.0429530470	b_6^{121}	0.0428950291	b_6^{132}	0.0412628857
b_7^{111}	−0.0079480275	b_7^{121}	−0.0079400220	b_7^{132}	−0.0076832353
s_{111}	0.5	s_{121}	0.4	s_{132}	0.2

g_{133}		g_{211}		g_{311}	
b_0^{133}	0.8813944143	b_0^{211}	1.0003583991	b_0^{311}	0.5081201992
b_1^{133}	−0.6491445844	b_1^{211}	−0.6654326742	b_1^{311}	−0.2326161465
b_2^{133}	0.4489740965	b_2^{211}	0.4295338070	b_2^{311}	0.0972737581
b_3^{133}	−0.3078940138	b_3^{211}	−0.2812400555	b_3^{311}	−0.0398841619
b_4^{133}	0.2016045985	b_4^{211}	0.1785880816	b_4^{311}	0.0160797798
b_5^{133}	−0.1117835035	b_5^{211}	−0.0971304019	b_5^{311}	−0.0060041662
b_6^{133}	0.0428470454	b_6^{211}	0.0368077929	b_6^{311}	0.0017487435
b_7^{133}	−0.0079333006	b_7^{211}	−0.0067699674	b_7^{311}	−0.0002726251
s_{133}	0.4	s_{211}	0.5	s_{311}	0.3

g_{312}		g_{321}		g_{331}	
b_0^{312}	0.2013135639	b_0^{321}	0.8072631281	b_0^{331}	1.0061848887
b_1^{312}	−0.0743470726	b_1^{321}	−0.4648191338	b_1^{331}	−0.6476496175
b_2^{312}	0.0244467204	b_2^{321}	0.2575887534	b_2^{331}	0.4067505913
b_3^{312}	−0.0077137243	b_3^{321}	−0.1452654416	b_3^{331}	−0.2603691317
b_4^{312}	0.0023694160	b_4^{321}	0.0809084732	b_4^{331}	0.1624073740
b_5^{312}	−0.0006874061	b_5^{321}	−0.0398683727	b_5^{331}	−0.0872159881
b_6^{312}	0.0001645840	b_6^{321}	0.0141620110	b_6^{331}	0.0327840257
b_7^{312}	−0.0000225598	b_7^{321}	−0.0025036874	b_7^{331}	−0.0060002559
s_{312}	0.1	s_{321}	0.4	s_{331}	0.5

A.1.5 Part 5: $(\mathcal{M} \setminus \mathcal{L}) \cap (\sqrt{20}, \sqrt{21})$

We present coefficients of the polynomial components of test functions $\underline{f} = (f_1, f_2, f_3, f_4)$ and $\underline{g} = (g_1, g_2, g_3, g_4)$ used in (30). It follows from the equality between the first and second columns of the matrix M that $f_1 = f_2$ and $g_1 = g_2$.

$$f_j = \sum_{k=0}^{9} a_k^j x^k \quad g_j = \sum_{k=0}^{9} b_k^j x^k, \quad j = 1, 2, 3, 4.$$

Straightforward calculation shows that the functions f_j and g_j are monotone decreasing on $[0, 1]$ and achieve their minima at 1. For convenience, we give a lower bound $s_j < \min(f_j(1), g_j(1))$.

f_1		f_3		f_4	
a_0^1	0.9799928531	a_0^3	1.0045893915	a_0^4	0.1934516264
a_1^1	−0.7406258897	a_1^3	−0.7487821988	a_1^4	−0.0765314787
a_2^1	0.5273926877	a_2^3	0.5296982401	a_2^4	0.0259667329
a_3^1	−0.3800229054	a_3^3	−0.3806370202	a_3^4	−0.0083489461
a_4^1	0.2763248479	a_4^3	0.2764834085	a_4^4	0.0026152698
a_5^1	−0.1955487122	a_5^3	−0.1955888334	a_5^4	−0.0008056462
a_6^1	0.1241717000	a_6^3	0.1241816277	a_6^4	0.0002419581
a_7^1	−0.0622409671	a_7^3	−0.0622432782	a_7^4	−0.0000670179
a_8^1	0.0206092394	a_8^3	0.0206096817	a_8^4	0.0000146636
a_9^1	−0.0032442178	a_9^3	−0.0032442666	a_9^4	−0.0000017681
s_1	0.4	s_3	0.4	s_4	0.05

g_1		g_3		g_4	
b_0^1	0.9799928382	b_0^3	1.0045893900	b_0^4	0.1934516765
b_1^1	−0.7406256446	b_1^3	−0.7487819559	b_1^4	−0.0765314768
b_2^1	0.5273924149	b_2^3	0.5296979674	b_2^4	0.0259667280
b_3^1	−0.3800226607	b_3^3	−0.3806367755	b_3^4	−0.0083489435
b_4^1	0.2763246450	b_4^3	0.2764832054	b_4^4	0.0026152687
b_5^1	−0.1955485561	b_5^3	−0.1955886769	b_5^4	−0.0008056458
b_6^1	0.1241715958	b_6^3	0.1241815227	b_6^4	0.0002419578
b_7^1	−0.0622409134	b_7^3	−0.0622432239	b_7^4	−0.0000670177
b_8^1	0.0206092213	b_8^3	0.0206096634	b_8^4	0.0000146636
b_9^1	−0.0032442150	b_9^3	−0.0032442637	b_9^4	−0.0000017681
s_1	0.4	s_3	0.4	s_4	0.05

A.2 Zaremba theory

A.2.1 $\dim_H(E_5)$

We present coefficients of the polynomial test functions f and g used in (36).

$$f = \sum_{k=0}^{15} a_k x^k, \quad g = \sum_{k=0}^{15} b_k x^k.$$

Similarly to the previous examples, the functions f and g are monotone and they can be bounded from below by their value at 1. In particular, we have $f(1), g(1) \geq 0.4$.

f		g	
a_0	1.002075775192587	b_0	1.002075775192580
a_1	−0.863832791554195	b_1	−0.863832791551160
a_2	0.694904605500679	b_2	0.694904605500778
a_3	−0.563982609501401	b_3	−0.563982609938672
a_4	0.462936795376894	b_4	0.462936803438964
a_5	−0.382787786278096	b_5	−0.382787860060200
a_6	0.317798375235142	b_6	0.317798788806115
a_7	−0.263717774563333	b_7	−0.263719322983292
a_8	0.216187532791877	b_8	0.216191581202906
a_9	−0.170354610076857	b_9	−0.170362167218400
a_{10}	0.123046705346496	b_{10}	0.123056842712685
a_{11}	−0.076412879097916	b_{11}	−0.076422574631579
a_{12}	0.037844499485800	b_{12}	0.037850946915568
a_{13}	−0.013650349050295	b_{13}	−0.013653178641107
a_{14}	0.003133011516183	b_{14}	0.003133747261017
a_{15}	−0.000339760445059	b_{15}	−0.000339846126736

A.2.2 $\dim_H(E_4)$

Coefficients of the polynomial test functions f and g used in (37).

$$f = \sum_{k=0}^{15} a_k x^k, \quad g = \sum_{k=0}^{15} b_k x^k.$$

Similar to the previous examples, the functions f and g are monotone and they can be bounded from below by their value at 1. In particular, we have $f(1), g(1) \geq 0.4$.

	f		g
a_0	1.001981557057916	b_0	1.001981557057906
a_1	−0.824549641777407	b_1	−0.824549641773357
a_2	0.632377740413372	b_2	0.632377740394589
a_3	−0.489751892709023	b_3	−0.489751892404770
a_4	0.384644919648888	b_4	0.384644915987248
a_5	−0.305093638471734	b_5	−0.305093610668140
a_6	0.243481281464752	b_6	0.243481142895873
a_7	−0.194635007255869	b_7	−0.194634539676458
a_8	0.154196605135780	b_8	0.154195521320957
a_9	−0.118018014953805	b_9	−0.118016296613633
a_{10}	0.083325052299187	b_{10}	0.083323249134991
a_{11}	−0.050893608677143	b_{11}	−0.050892477178423
a_{12}	0.024910421954700	b_{12}	0.024910155247198
a_{13}	−0.008908424701076	b_{13}	−0.008908569288906
a_{14}	0.002031148876994	b_{14}	0.002031269948929
a_{15}	−0.000219053588808	b_{15}	−0.000219079665840

A.2.3 $\dim_H(E_{1235})$

Coefficients of the polynomial test functions f and g corresponding to $t_1 = 0.7709149399\,36$ and $t_0 = t_1 + 3 \cdot 10^{-12}$ respectively.

$$f = \sum_{k=0}^{15} a_k x^k, \quad g = \sum_{k=0}^{15} b_k x^k.$$

Similar to the previous examples, the functions f and g are monotone and they can be bounded from below by their value at 1. In particular, we have $f(1), g(1) \geq 0.4$.

	f		g
a_0	1.001943825796940	b_0	1.001943825796930
a_1	−0.808837588421526	b_1	−0.808837588417559
a_2	0.615494218966991	b_2	0.615494218954915
a_3	−0.474372511899657	b_3	−0.474372511806688
a_4	0.371576736077143	b_4	0.371576735828591
a_5	−0.294687522275668	b_5	−0.294687526007311
a_6	0.235767134415525	b_6	0.235767178162078
a_7	−0.189403864881314	b_7	−0.189404097267897
a_8	0.151071322536495	b_8	0.151072090158663
a_9	−0.116493763140853	b_9	−0.116495485397309
a_{10}	0.082822264062997	b_{10}	0.082824975856055
a_{11}	−0.050871691397334	b_{11}	−0.050874703061709
a_{12}	0.025002624053740	b_{12}	0.025004941184306
a_{13}	−0.008966855413747	b_{13}	−0.008968032838312
a_{14}	0.002048316877335	b_{14}	0.002048672642558
a_{15}	−0.000221169553698	b_{15}	−0.000221217982471

A.3 Fuchsian Schottky groups

Coefficients of the polynomial test functions $\underline{f} = (f_0, f_1, f_2)$ and $\underline{g} = (g_0, g_1, g_2)$, where each f_j and g_j for $j = 0, 1, 2$ is defined on the interval X_j, respectively.

$$f_j = \sum_{k=0}^{24} a_k^j x^k, \quad g_j = \sum_{k=0}^{24} b_k^j x^k, \quad j = 0, 1, 2.$$

The functions f_0 and g_0 are even and therefore their odd coefficients vanish: $a_{2k+1}^0 = b_{2k+1}^0 \equiv 0$. Moreover, $f_1(x) = f_2(-x)$ and $g_1(x) = g_2(-x)$ therefore their even coefficients agree and their odd coefficients have the opposite signs: $a_{2k}^1 = a_{2k}^2$, $a_{2k+1}^1 = -a_{2k+1}^2$; $b_{2k}^1 = b_{2k}^2$, $b_{2k+1}^1 = -b_{2k+1}^2$.

f_1		g_1	
a_0^1	2.8770704827462130787849251492030	b_0^1	2.877070483741637235339396793709
a_1^1	$-11.4401515732168029128293331165380$	b_1^1	$-11.440151580102508283298777329005$
a_2^1	41.2783414403998127337761764215520	b_2^1	41.278341467154591279141888564443
a_3^1	$-111.0688407519945991015658074607160$	b_3^1	$-111.068840826093040583068331287781$
a_4^1	229.2359826830719504436183231429020	b_4^1	229.235982838419674330781933540686
a_5^1	$-374.1839317918421651508580498351190$	b_5^1	$-374.183932047938578510658766660908$
a_6^1	494.24861828912318666860550011193190	b_6^1	494.248618629707611522216515713345
a_7^1	$-537.1550718818002043450127874148360$	b_7^1	$-537.155072253799253891555878102640$
a_8^1	486.1852872370976132985936371604760	b_8^1	486.185287575077537070728753265866
a_9^1	$-369.6791357119376158018979254258860$	b_9^1	$-369.679135969693151986947450814820$
a_{10}^1	237.5656093180881359651809836821330	b_{10}^1	237.565609484126480000044027316791
a_{11}^1	$-129.5193450984147750120850142699940$	b_{11}^1	$-129.519345189116330520934728232633$
a_{12}^1	60.0196876244694040557343257845630	b_{12}^1	60.019687666569911991602610502470
a_{13}^1	$-23.6421320534647605005954912805860$	b_{13}^1	$-23.642132070071466367919223736091$
a_{14}^1	7.9021374699139922100387246571970	b_{14}^1	7.902137475471208082330326553248
a_{15}^1	$-2.2327574635963721814894503262740$	b_{15}^1	$-2.232757465168180629595168192690$
a_{16}^1	0.5301045742621054501101376372390	b_{16}^1	0.530104574635619914117715916573
a_{17}^1	$-0.1048257433528365630840057396770$	b_{17}^1	$-0.104825743426755206788900861842$
a_{18}^1	0.0170498220734007394058868515300	b_{18}^1	0.017049822085431863890249313609
a_{19}^1	$-0.0022409149968744747201392201870$	b_{19}^1	$-0.002240914998456736922493194695$
a_{20}^1	0.0002320048619867762156560576330	b_{20}^1	0.000232004862150679716335333676
a_{21}^1	$-0.0000182076188328122444526479340$	b_{21}^1	$-0.000018207618845681630026751529$
a_{22}^1	$1.017711979517475899435180 \cdot 10^{-6}$	b_{22}^1	$1.0177119802371267732818260 \cdot 10^{-6}$
a_{23}^1	$-3.60864141942206682494120 \cdot 10^{-8}$	b_{23}^1	$-3.60864142197485386059250 \cdot 10^{-8}$
a_{24}^1	$6.09941161252684275320 \cdot 10^{-10}$	b_{24}^1	$6.099411616843192081320 \cdot 10^{-10}$
f_0		g_0	
a_0^0	0.988509120501286981726492299097	b_0^0	0.988509120496506332431826934199
a_2^0	0.157542850847455232142931716289	b_2^0	0.157542850912623683440552637873
a_4^0	0.042904006074342082007020639894	b_4^0	0.042904006097255254396958615314
a_6^0	0.013311224077669383683707536427	b_6^0	0.013311224085738465758389603856
a_8^0	0.004429850096065885316293787561	b_8^0	0.004429850098979983540304963280
a_{10}^0	0.001543440394063281474032350116	b_{10}^0	0.001543440395140704805733043742
a_{12}^0	0.000555351173373256765356036270	b_{12}^0	0.000555351173779220956618567711
a_{14}^0	0.000204517408718030083947390234	b_{14}^0	0.000204517408873232993380371066
a_{16}^0	0.000076609680108249015238001359	b_{16}^0	0.000076609680168235804587352199
a_{18}^0	0.000029061155169646923244681761	b_{18}^0	0.000029061155193021143267002387
a_{20}^0	0.000011137276040685233840494967	b_{20}^0	0.000011137276049855568024571829
a_{22}^0	$4.2275802352717104321010360 \cdot 10^{-6}$	b_{22}^0	$4.2275802388232323523285170 \cdot 10^{-6}$
a_{24}^0	$1.9876322730511567993854040 \cdot 10^{-6}$	b_{24}^0	$1.9876322747582711067418060 \cdot 10^{-6}$

A.4 Non-linear example

Coefficients of the polynomial test functions f and g corresponding to $t_0 = 0.4934480908\,02$ and $t_1 = t_0 + 10^{-12}$ respectively.

$$f = \sum_{k=0}^{7} a_{2k} x^{2k}, \quad g = \sum_{k=0}^{7} b_{2k} x^{2k}.$$

f		g	
a_0	0.260509445190371	b_0	0.260509445190371
a_2	$-4.88560887219182 \cdot 10^{-3}$	b_2	$-4.88560887219092 \cdot 10^{-3}$
a_4	$3.65942002704976 \cdot 10^{-4}$	b_4	$3.65942002704768 \cdot 10^{-4}$
a_6	$-3.48521776330216 \cdot 10^{-5}$	b_6	$-3.48521776329928 \cdot 10^{-5}$
a_8	$3.61101766657243 \cdot 10^{-6}$	b_8	$3.61101766656879 \cdot 10^{-6}$
a_{10}	$-3.88502731592718 \cdot 10^{-7}$	b_{10}	$-3.88502731592271 \cdot 10^{-7}$
a_{12}	$4.09696977259504 \cdot 10^{-8}$	b_{12}	$4.09696977258989 \cdot 10^{-8}$
a_{14}	$-3.23465524617897 \cdot 10^{-9}$	b_{14}	$-3.23465524617468 \cdot 10^{-9}$

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