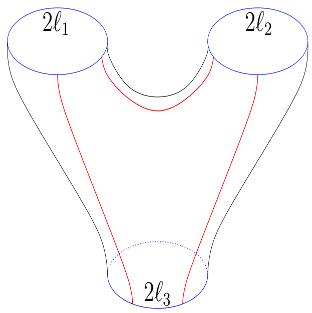
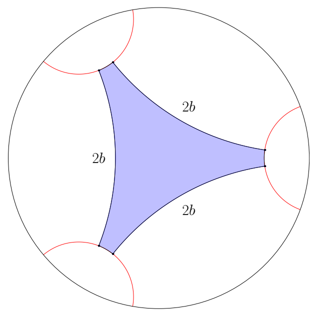
Zeros of the Selberg zeta function for non-compact surfaces

Polina Vytnova, ICERM & QMUL and Mark Pollicott, Warwick

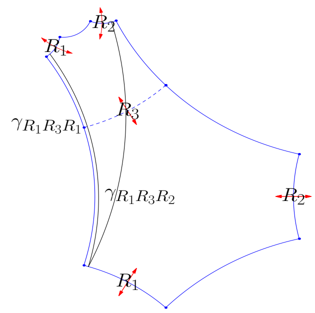
Closed geodesics on a pair of pants



A pair of pants is a three-punctured sphere. It is uniquely defined by the lengths of boundary geodesics: $2\ell_1$, $2\ell_2$, and $2\ell_3$. Cutting the pants along the red geodesics, we obtain a right-angled hyperbolic hexagon.



The pair of pants appears then as a factor-space $\mathbb{H}/\Gamma$, where the group Γ is generated by reflections with respect to red geodesics. The group Γ is uniquely defined by the pairwise distances ℓ_1, ℓ_2, and ℓ_3.



We are interested in the short closed (non-simple) geodesics. To any geodesics corresponds a reduced periodic word of period $2n$ in the alphabet of three symbols.

Selberg Zeta Function

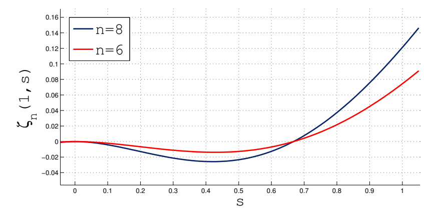
The Selberg zeta function for a hyperbolic surface is a complex analytic function defined in terms of lengths of closed geodesics $\lambda(\gamma)$:

$$Z(s) = \prod_n \prod_\gamma (1 - e^{-(s+n)\lambda(\gamma)}).$$

Any closed geodesics is uniquely defined by a sequence of reflections with respect to the red cuts. This allows us to define the dynamical zeta function as

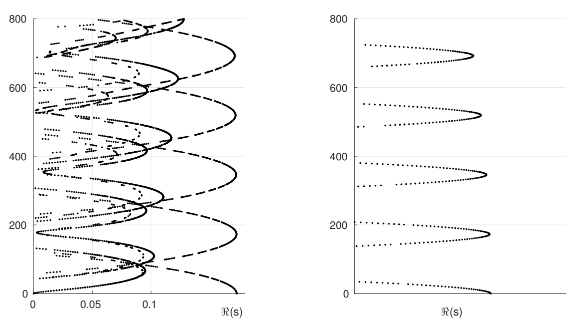
$$\zeta(z,s) = \exp\left(-\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{|\gamma|=m} \frac{e^{-s\lambda(\gamma)}}{1-e^{-\lambda(\gamma)}} \right)$$

The power series coefficients converge to zero superexponentially, and this gives us an efficient way of computing zeta function numerically, as $\zeta(1,s) = Z(s)$.



Typical graphs of a zeta function, computed using closed geodesics corresponding to words of periods $n = 6$ and $n = 8$.

Zeros of the Selberg Zeta Function



Zeros of the Selberg Zeta function associated to a symmetric pair of pants with boundary geodesics of the length 8.

Let the length of boundary geodesics be $\ell_j \equiv 2b > 8$.

- the vertical spacing of zeros is approximately $\frac{\pi}{b}$;
- the vertical apparent periodicity of the pattern of zeros is approximately πe^b;
- the zeros belong to small neighbourhoods of four distinct curves, which have common intersection point at $\frac{\delta}{2} + i\frac{\pi}{2}e^b$:

$$\mathcal{C}_1 = \left\{ \log\left|e^{2it}+1\right| + it \mid t \in \mathbb{R} \right\}$$
$$\mathcal{C}_2 = \left\{ \log\left|e^{2it}-1\right| + it \mid t \in \mathbb{R} \right\}$$
$$\mathcal{C}_3 = \left\{ \log\left|2 - e^{4it} - e^{2it}\sqrt{4e^{2it} - 3e^{4it}}\right| - \log 2 + it \mid t \in \mathbb{R} \right\}$$
$$\mathcal{C}_4 = \left\{ \log\left|2 - e^{4it} + e^{2it}\sqrt{4e^{2it} - 3e^{4it}}\right| - \log 2 + it \mid t \in \mathbb{R} \right\}$$

Acknowledgements, Credits, References

The conference *"Computation in Dynamics"* poster is based on the work in progress. Here are some references for the background.

1. For background in Fuchsian groups actions, see A. F. Beardon, The geometry of discrete groups. Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983. xii+337 pp.

2. Selberg zeta function has been introduced by Atle Selberg, in "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series", J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.

3. For relation between Selberg Zeta Function and Dynamical Zeta Function, see D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, 34 (1976), 231–242.

4. The error term estimates for Taylor series of the Dynamical Zeta Function, are due to A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.*, 16 (1955), 1–140.

5. The problem has been introduced to us by David Borthwick during the conference on Quantum Chaos in Roscoff, also see Borthwick, D. Distribution of resonances for hyperbolic surfaces. Exp. Math. 23 (2014), no. 1, 25–45.

Software used

1. The poster prepared with LaTeX, using beamerposter package.

2. The hyperbolic plane graphics is prepared with Asymptote, using hyperbolic geometry module by Raoul Bourquin.

3. Numerical data the zeros obtained with a help of `Matlab`©.