REALIZABILITY OF HYPERGRAPHS [1]

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Содержание

0.1 Realizability in $\mathbb{R}^3$ 1
0.2 How to work with four-dimensional space? 2
0.3 Realizability in $\mathbb{R}^4$ 2

0.1 Realizability in $\mathbb{R}^3$

The cone over the graph K_4 embedded into the plane shows that there are 5 points $0,1,2,3,4$ in 3-space such that the set of all the triangles $0jk$, $1\le j<k\le 4$, is embedded. The Intersection Property refram-cone shows that no 6 points with analogous property exist.

(In more advanced language not necessary here the above remarks state that *neither a complete 2-complex on 6 vertices nor even the cone over K_5 is embeddable into the 3-space.*)

0.1. For each n there exist $2n$ points $A_1,\dots,A_n$, $B_1,\dots,B_n$ in 3-space such that the set of all the triangles

$$A_jB_jA_k \quad\text{and}\quad A_kB_kB_j,\quad 1\le j<k\le n,\qquad\text{is embedded.}$$

0.2 (Join)**.** For which l,m,n does there exist $l{+}m{+}n$ points $A_1,\dots,A_l$, $B_1,\dots,B_m$, $C_1\dots,C_n\in\mathbb{R}^3$ such that the set of all the triangles

$$A_iB_jC_k,\quad 1\le i\le l,\quad 1\le j\le m,\quad 1\le k\le n,$$

is embedded? (Start with the cases $(l,m,n)=(222),(223),(233)$!)

Hint: use the Intersection Property refram-cone. Answer: at most one of numbers l,m,n is greater than 2.

An alternative proof of the Product Theorem. Assume to the contrary that there exists a linear embedding $K_5\times K_3\to\mathbb{R}^3$. Then the vertex $A_{5,1}$ is joined

- to the vertex $A_{i,1}$ by the segment $A_{5,1}A_{i,1}$, for each $1\le i\le 4$;
- to the vertex $A_{i,j}$ by the broken line $A_{5,1}A_{5,j}A_{i,j}$, for each $1\le i\le 4$, $2\le j\le 3$.

Denote by $T_{4,3}$ the union of triangles of the linear embedding $K_4\times K_3\to\mathbb{R}^3$ formed by vertices A_{ij}, $i\in\{1,2,3,4\}$, $j\in\{1,2,3\}$. Since no vertex $A_{5,j}$ belongs to $T_{4,3}$, each of these segments and broken lines intersects X only at the endpoints. Denote by B the boundary of the connected component of $\mathbb{R}^3-T_{4,3}$ containing point $A_{5,1}$. Then $B\ni A_{i,j}$ for each $1\le i\le 4$ and $1\le j\le 3$, because $A_{i,j}$ is joined to $A_{5,1}$ by a segment or a broken line whose interior is disjoint with $T_{4,3}$. Thus $B\supset T_{4,3}$, so $B=T_{4,3}$. This is impossible. (It is not easy to prove the impossibility, cf. Problem 0.3.) QED

0.3. (a) Существует невыпуклый многогранник в пространстве и 3 его вершины A,B,C, для которых треугольник ABC не разбивает не внутренности, ни внешности многогранника.

(b) Describe outer-spatial 2-polyhedra.

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[1] This is a complement to §1 and §5 of *A. Skopenkov, Algorithms for recognition of the realizability of hypergraphs,* in Russian, www.mccme.ru/circles/oim/algor.pdf.

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0.2 How to work with four-dimensional space?

One can define
- the *line* as the set of all real numbers;
- the *plane* as the set of all ordered pairs (x, y) of real numbers x and y;
- *three-dimensional space* as the set of all ordered triples (x, y, z) of real numbers;
- *four-dimensional space* $\mathbb{R}^4$ as the set of all ordered quadruples (x, y, z, t) of real numbers.

Then one can 'analytically' define lines in a plane, lines and planes in three-dimensional space, lines, planes and (three-dimensional) hyperplanes in four-dimensional space. However, usually only the simplest properties of planar and spatial geometric objects are deduced from the analytic definition (or just accepted as axioms). More complicated properties can be deduced 'synthetically' from the simplest ones (i.e., as in school geometry, without using the analytic definition). Often it is convenient to reduce a planar problem to a linear one (i.e., to a problem in a line), and a spatial problem to a planar one. Similarly, the best approach to the following four-dimensional problems is an analogy or a reduction to spatial ones. While solving problems about $\mathbb{R}^4$, you can use without proof all rigorously formulated and correct facts about solutions of systems of linear equations.

Examples of simple arguments in four-dimensional space are presented as hints to problems, or below the problems.

0.4. (a) For each two points, which are not in the plane $x = y = 0$ in four-dimensional space, there exists a broken line which connects these points and does not intersect this plane.

(b) For each hyperplane in four-dimensional space, there exist two points not in this hyperplane such that each broken line connecting them intersects this hyperplane.

In Problems 0.5 and 0.6 below it suffices to give correct answers.

0.5. What is the intersection of the *3-dimensional sphere*

$$S^3 := \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$$

with the following sets:

(a) the line $x = y = z = 0$, containing the center of the sphere;

(b) the plane $x = y = 0$, containing the center of the sphere;

(c) the (3-dimensional) hyperplane $x = 0$, containing the center of the sphere;

(d) the intersection of the positive sixteenth of $\mathbb{R}^4$ and the union of the 2-dimensional coordinate planes, i.e.

$$\{(x, y, z, t) \in \mathbb{R}^4 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ t \geq 0 \text{ and two of four numbers } x, y, z, t \text{ are zeros}\}.$$

0.6. Eight points 1,2,3,4,5,6,7,8 in general position in four-dimensional space are given. What is the intersection of:

(a) the line 12 and the hyperplane 5678? (b) the line 12 and the plane 567?

(c) the plane 123 and the hyperplane 5678? (d) the hyperplanes 1234 and 5678?

(e) the planes 123 and 567? (e') the triangles 123 and 567?

Answers. (a), (e), (e') A point or the empty set. (b) The empty set.

(c) A line or the empty set. (d) A plane or the empty set.

0.3 Realizability in $\mathbb{R}^4$

Let us present a more complicated (but still simple) proof of the van Kampen-Flores Theorem ref0-ne4 which illustrates a generalization to arbitrary graphs.

Proof of the van Kampen-Flores Theorem ref0-ne4. For a set $f \subset \mathbb{R}^4$ of seven general position points denote by $v(f)$ the modulo 2 residue in question. For the set f_0 of seven points from Example

0.7.d we have $v(f_0) = 1$ because by general position $|\mathrm{conv}\Delta \cap \mathrm{conv}\Delta'| = 1$. Hence it suffices to prove that if we change one point keeping the remaining six fixed, so that new seven points are in general position, then $v(f)$ is not changed. Assume that $K \in f$, $K' \notin f$ and $f' := (f - \{K\}) \cup \{K'\}$ is a general position set.

Proof that $v(f) = v(f')$ when $f \cup \{K'\}$ is a general position set. Denote $s := f - \{K\}$. For each segment e with the endpoints from $f - \{K\}$ denote by

- E_1, E_2 the endpoints of e;
- T_e the surface of the tetrahedron with the vertices at four points from s other E_1, E_2;
- Ke the triangle with the vertices at K, E_1, E_2.

From now on in any sum, if the limits of the summation are not written, the sum is over segments e with the endpoints from s. We have

$$v(f) - v(f') \overset{1}{=} \sum (|Ke \cap T_e| - |K'e \cap T_e|) \overset{2}{=}$$

$$= \sum |(KK'E_1 \cup KK'E_2) \cap T_e| \overset{3}{=} \sum |KK'E_1 \cap T_e| + \sum |KK'E_2 \cap T_e| \overset{4}{=} 0 \mod 2.$$

- The first equality is clear.
- The second equality follows by the Parity Lemma ref0-evens.
- The third equality holds because $f \cup \{K'\}$ is a general position set.
- Let us prove the last equality. For any point $E_1 \in s$ we have $|KK'E_1 \cap T_e| = \sum |KK'E_1 \cap PQR|$, where the sum is over triangles PQR from T_e. For any three distinct points $P, Q, R \in s - \{E_1\}$, the triangle PQR is contained in exactly two tetrahedra with the vertices from $s - \{E_1\}$. So the number $|KK'E_1 \cap PQR|$ 'appears twice' in the sum $\sum |KK'E_1 \cap T_e|$. Therefore this sum is equal to 0. Analogously $\sum |KK'E_1 \cap T_e| = 0$.

Proof that $v(f) = v(f')$ in general. There exists a point K'' such that both $f \cup \{K''\}$ and $f' \cup \{K''\}$ are general position sets. Then $v(f) = v((f - \{K\}) \cup \{K''\}) = v(f')$ by the previous case. QED

Proof of Proposition ref0-ne4j. Analogously to the beginning of the above proof, Proposition ref0-ne4j is reduced to the case of general position points. (However, in order to prove the general position case we would consider non-general position points satisfying certain condition.). Let f_1, f_2, f_3 be three-element subsets of $\mathbb{R}^4$ such that for any six distinct points $A_k, B_k \in f_k$, $k = 1, 2, 3$, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect in at most one point. Let $v(f_1, f_2, f_3) \in \mathbb{Z}_2$ be the modulo 2 reduction of the number of intersection points (in $\mathbb{R}^4$) of the interiors of such triangles. Analogously to the above proof, $v(f_1, f_2, f_3)$ does not depend on f_1, f_2, f_3 if $f_1 \cup f_2 \cup f_3$ is a general position set. So, it remains to prove that there exist three-element subsets f_1, f_2, f_3 of $\mathbb{R}^4$ such that $f_1 \cup f_2 \cup f_3$ is a general position set and $v(f_1, f_2, f_3) = 1$. Analogously to Example 0.7.a, it suffices to prove this assertion when $f_1 \cup f_2 \cup f_3$ is not a general position set.

For this denote the vertices of one part of the graph $K_{3,3}$ by A_1, A_2, A_3 and of the other part by B_1, B_2, B_3. Embed this graph in $\mathbb{R}^3$ in such a way that $A_1B_1A_2B_2$ is a square and points A_3 and B_3 are in different half-spaces of $\mathbb{R}^3$ w.r.t. the plane $A_1B_1A_2$. Let C_1 and C_2 be in different half-spaces of $\mathbb{R}^4$ w.r.t. $\mathbb{R}^3$. Finally, take a point C_3 inside the pyramid $C_1A_1B_1A_2B_2$ with the vertex C_1. One can check that $v(\{A_1, A_2, A_3\}, \{B_1, B_2, B_3\}, \{C_1, C_2, C_3\}) = 1$. QED

0.7. (a) There exist 7 points in $\mathbb{R}^4$ such that only for one non-ordered pair Δ_1, Δ_2 of two 3-element subsets among all such pairs we have $\mathrm{conv}\Delta_1 \cap \mathrm{conv}\Delta_2 \neq \mathrm{conv}(\Delta_1 \cap \Delta_2)$, and for such pair $\Delta_1 \cap \Delta_2 = \emptyset$. (By $\mathrm{conv}V$ we denote the convex hull of a set V.)

(b) Same for general position points.

Proof of Example 0.7.a. Let $ABCD$ be a regular tetrahedron in $\mathbb{R}^3$ and let E be the center of $ABCD$. Let I be a point in the interior of the tetrahedron $ABCE$ such that the points A, B, C, D, E, I are in general position in $\mathbb{R}^3$. Let l be a line $\mathbb{R}^4$ perpendicular to $\mathbb{R}^3$ and intersecting $\mathbb{R}^3$ at I. Finally, choose points F, G on l which are on opposite sides with respect to I.

Clearly, if a triangle whose vertices are from $f_0 := \{A, B, C, D, E, F, G\}$ and the triangle DFG have a common vertex or a common side, then the intersection of these triangles is this vertex or this side. Thus in order to show that the set f_0 is as required, it suffices to prove the following assertions.

(1) For any two 3-element subsets $\Delta_1, \Delta_2 \neq \{D, F, G\}$ of f_0 we have $\mathrm{conv}\Delta_1 \cap \mathrm{conv}\Delta_2 = \mathrm{conv}(\Delta_1 \cap \Delta_2)$ (this means that the set of all the triangles with the vertices from f_0, except the triangle DFG, is embedded, so this is sufficient for the simplified version);

(2) There is exactly one 3-element subset $\Delta_1 \subset f_0$ such that $\Delta_1 \cap \{D, F, G\} = \emptyset$ and $\mathrm{conv}\Delta_1 \cap DFG \neq \mathrm{conv}(\Delta_1 \cap \{D, F, G\})$.

Proof of (1). There are three types of triangles with the vertices from f_0:

$$(1) \quad XFG, \qquad (2) \quad XYF \text{ or } XYG, \qquad (3) \quad XYZ,$$

where $X, Y, Z \in \{A, B, C, D, E\}$. Clearly, the set of triangles of each type is embedded. The triangle XFG intersects a triangle of type 2 either at a common vertex F or G, or at a common edge XF or XG. A triangle of type 2 intersects a triangle of type 3 either at a common vertex X or at a common edge XY. The triangle XFG intersects $\mathbb{R}^3$ by the segment XI. If $X \neq D$, then the segment XI lies inside $ABCE$. Then XI intersects any triangle of type 1 in at most a common vertex. QED

Proof of (2). We have $DFG \cap ABCD = DI$. (By a tetrahedron we mean the convex hull of its vertices.) Since I is inside the tetrahedron $ABCE$ and D is outside it, it follows that the segment DI intersects the surface of the tetrahedron $ABCE$ at a unique point. So the triangle FGD intersects exactly one of the triangles with the vertices at the other points, more precisely, the triangle EXY, where $X, Y \in \{A, B, C\}$. So (2) holds for $\Delta_1 = \{E, X, Y\}$. QED

Proof of Example 0.7.b. Take the set f_0 of 7 points in $\mathbb{R}^4$ given by (a). Denote by d the minimum of the distances between unordered pairs of disjoint triangles with the vertices from f_0. If we replace each point $K \in f_0$ by a point $K' \in \mathbb{R}^4$ such that $|KK_1| < d/2$, then the set of all triangles with the vertices at shifted points, except the shifted triangle corresponding to $\mathrm{conv}\Delta_1$, is embedded. By Example 0.7.c there exists a number $d' > 0$ such that, if we replace each point $K \in f_0$ by a point $K' \in \mathbb{R}^4$ such that $|KK'| < d'$, then $\mathrm{conv}\Delta_1' \cap \mathrm{conv}\Delta_2' \neq \mathrm{conv}(\Delta_1' \cap \Delta_2')$, where Δ_1' and Δ_2' are shifted sets corresponding to Δ_1 and Δ_2, respectively. So if we replace each point $K \in f_0$ by a point $K' \in \mathbb{R}^4$ such that $|KK'| < \min\{d/2, d'\}$ and the shifted set is general position, then the shifted set is as required. QED