A MINIMAL COUNTEREXAMPLE TO THE LANDO'S CONJECTURE

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Abstract. The following conjecture was proposed in 2010 by S. Lando.

Let M and N be two unions of the same number of disjoint circles in a sphere. Then there exist two spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as M in one sphere and as N in the other (union M of disjoint circles is situated in one sphere as union M of disjoint circles in the other sphere if there is a homeomorphism between these two spheres which maps M to M).

Avvakumov found special M_1 and N_1, each of which is a union of 9 circles in a sphere. And using Avvakumov theorem and brute force computer method he proved that there exists no pair of spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as M_1 in one sphere and as N_1 in the other. That was the first counterexample to Lando's Conjecture. Then he found special M_2 and N_2, each of which is a union of 7 circles in a sphere. The aim of this work is prove of that there exists no pair of spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as M_2 in one sphere and as N_2 in the other. Pair M_2, N_2 is the minimal counterexaple to Lando's Conjecture.

Definitions. Let p and q be two sets of edges of a tree Y.

The set p is *on the same side* of q (in this tree Y) if $p \bigcap q = \emptyset$ and for each two vertices of edges of p there is a path in the tree connecting these two points, and containing an even number of edges of q. Sets p and q are *unlinked* (in this tree) if p is on the same side of q and q is on the same side of p .

For vertice P of graph we denote as δP precisely all edges whose end is P.

Let K and K' be two trees with the same number of edges. Let h be a bijection (one-to-one correspondence) between their edges.

Then h is called realizable if $h(\delta A)$ and $h(\delta B)$ are unlinked for each two vertices A and B in K, such that the path connecting A and B contains even number of edges .

K and K' are friendly if such bijection exists.

Suppose that M is a union of disjoint circles in sphere S_2. Define ('dual to M') graph $G = G(S_2; M)$ as follows. The vertices are the connected components of $S_2 \setminus M$. Two vertices are connected by an edge if the corresponding connected components are neighbors.

Avvakumov Theorem. *Suppose that M is a union of disjoint circles in sphere S_2.Suppose that M_0 is a union of disjoint circles in sphere S_2. Then there exist two spheres in 3-space whose intersection is transversal and is a union of disjoint circles that is situated as M in one sphere (union X of disjoint circles is situated in one sphere as union Y of disjoint circles in the other sphere if there is a homeomorphism between these two spheres which maps X to Y) and as M_0 in the other if and only if the dual to M and M_0 graphs are friendly.*

You can see prove of this theorem in [A]

Graph G has vertices A,C_i,A',C'_i and edges $C_3C'_3$, AC_i, $A'C'_i$, $i = 1, 2, 3$. Graph H has vertices B,D,P_i,Q_i and edges BD,BP_i,P_iQ_i, $i = 1, 2, 3$. G and H are dual to M_2 and N_2.

Theorem. *Graphs G and H are unfriendly.*

Proof. Suppose that pair of graphs G and H is friendly. Then there is bijection between their edges with the above properties.

For edges e_1,e_2,e_3,...,e_n of graph G by $H(e_1, e_2, e_3, ..., e_n)$ we denote graph formed by images of e_1,e_2,e_3,...,e_n in graph H. For edges e_1,e_2,e_3,...,e_n of graph H by $G(e_1, e_2, e_3, ..., e_n)$ we denote graph formed by images of e_1,e_2,e_3,...,e_n in graph G(bijection inversed).

Proposition 1. *Both graphs $H_1 := (AC_1, AC_2, AC_3)$ and $H_2 := H(A'C'_1, A'C'_2, A'C'_3)$ are connected.*

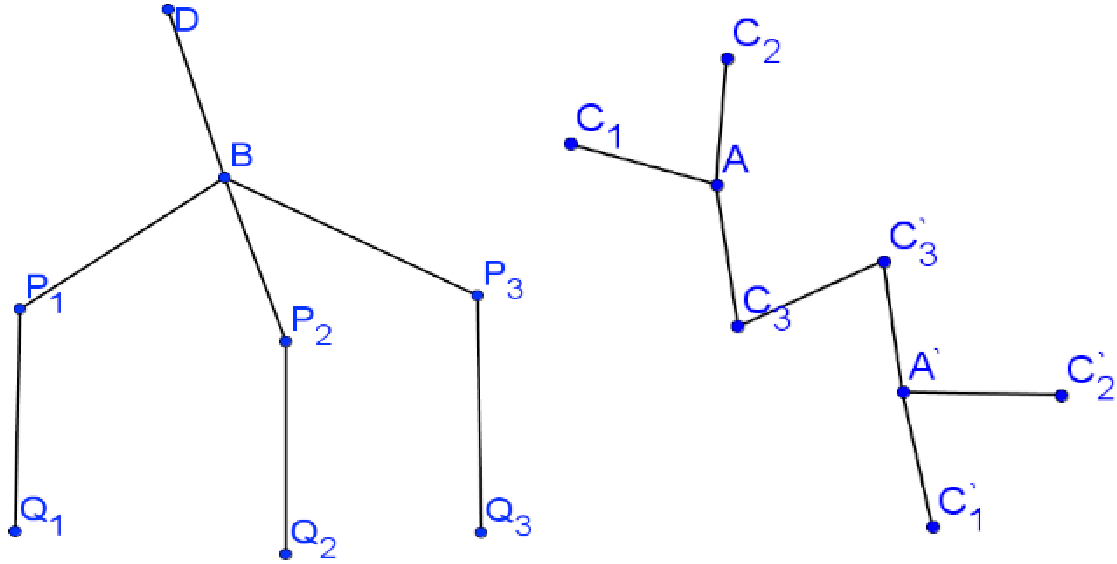

Figure 1: Graphs G and H.

Proof. Lets prove the proposition for H_1, and for H_2 the prove is analogically. Vertices A and C_1' are linked by path of even length. So $H(A'C_1')$ belongs to no one way, linking a pair of edges of graph H_1. Analogically $H(A'C_2')$ belongs to no one path, linking a pair of edges of graph H_1. Vertices A and C_3' are linked by path of even length too. Hence,

• *Case 1.* Neither $H(C_3C_3')$ nor $H(A'C_3')$ belong to any path, linking a pair of edges of graph H_1;

• *Case 2.* $H(C_3C_3')$ and $H(A'C_3')$ belong to path, linking a pair of edges J_1,J_2 of graph H_1.

In the first case the graph H_1 is connected.

In the second case J_1,$H(C_3C_3')$,$H(A'C_3')$,J_2 form a path of length 4. Without loss of generality let this path be $Q_1P_1BP_2Q_2$. Hence the path, linking edges J_1 and $H_1 - J_1 - J_2$ intersect only one of edges $H(C_3C_3')$,$H(A'C_3')$. Which is impossible.

QED

Proposition 2. *Vertex B is an endpoint of edge $H(C_3C_3')$.*

Proof. Vertices B and Q_i are linked by path of even length. Hence $G(BD, BP_1, BP_2, BP_3) = G(\delta B)$ is connected, because it is unlinked with any edge of $G(P_iQ_i)$ in G . There are only 2 connected subgraphs with 4 edges in G:

The first subgraph is called X. Its vertices are C_1, C_2, C_3, A, C_3' and edges are precisely the edges of G with both ends in X.

The second subgraph is called Y, whose vertices are C_1, A, C_3, C_3', A' and edges are precisely the edges of G with both ends in Y.

Since edge C_3C_3' is both in X and Y, one of edges BD, BP_1, BP_2, BP_3 is $H(C_3C_3')$.

QED

Completion of the proof. According to proposition 1 graph $H - H(C_3C_3')$ is a union of two connected graphs with 3 edges. Hence one of them has at least two of the edges P_iQ_i as his edge. Without loss of generality let P_1Q_1 and P_2Q_2 be in $H(AC_1, AC_2, AC_3) = H_1$. But then the length of the path linking Q_1, Q_2 is 4. Which is impossible since there are only 3 edges in H_1.

QED

References

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