Short proofs of the Conway-Gordon-Sachs and Sachs Theorems

Arseny Zimin

Abstract

In this paper we present a short and apparently new proof of the Conway-Gordon-Sachs Theorem about the complete graph on 6 vertices and the the Sachs Theorem about the the complete bipartite graph each of whose sides has 4 vertices embedded in $\mathbb{R}^3$. We reduce these theorems to certain property of the complete graph on 5 vertices and the complete bipartite graph each of whose parts has 3 vertices mapped to the plane.

Points in 3-dimensional space are in *general position*, if no four of them are in one plane.

Two triangles in the 3-dimensional space whose six vertices are in general position are *linked* if the outline of the first triangle intersects the interior of the second triangle exactly at one point.

Rectilinear Conway-Gordon-Sachs Theorem. *Assume that six points in the 3-dimensional space are in general position. Then there exist two linked triangles with vertices at these points.*

Define a 2-*dimensional complex* as a set of triangles, segments and points in $\mathbb{R}^3$ that satisfies the following conditions:

• sides of any triangle from the complex are in the complex;

• endpoints of any segment from the complex are in the complex;

A 2-dimensional complex is called *local-euclidean* if for each vertex v of this complex all triangles containing this vertex form a chain va_1a_2, va_2a_3, ... $va_{n-1}a_n$ or va_1a_2,...,va_na_1.

Two non-intersecting closed broken lines a and b without self-intersections in 3-dimensional space are *linked*, if there exists a local-euclidean 2-dimensional complex, say A, embedded to $\mathbb{R}^3$, with boundary a such that the number of intersection points of A with b is odd and the vertices of the broken line b and the complex $A - a$ are in general position.

Denote by K_n the complete graph on n vertices. Denote by $K_{n,n}$ the complete bipartite graph each of whose sides has n vertices.

Conway-Gordon-Sachs Theorem. *Assume that the graph K_6 is piecewise-linearly embedded in the 3-dimensional space. Then there exist two linked cycles of length 3 in this graph.*

See another proof in [Sk03]. The idea of the previous proof of this theorem is to prove that if we move on of our six points then the parity of number of pairs of linked triangles does not change. Then there constructs an example when this number is odd. But in our proof we reduce this theorem to a planar problem and work only with planar objects.

Remark. The statement of the theorem is meaningful because any cycle of length 3 in this graph is a closed broken line.

Sachs Theorem. *Assume that the graph $K_{4,4}$ is piecewise-linear embedded in the 3-dimensional space. Then there exist two linked cycles of length 4 in this graph.*

First proof of the linear Conway-Gordon-Sachs Theorem. To prove the linear Conway-Gordon-Sachs Theorem we will use two following lemmas.

To state the first lemma we need the following definition.

Let a, b be segments in 3-dimensional space, S^2 be a sphere whose center is denoted by O. A segment a is *lower (higher)* than a segment b, if there exist a half-lane with the endpoint O that intersects segment a, say at the point A, and segment b, say at the point B, $A \neq B$ and $A \in [OB]$($B \in [OA]$).

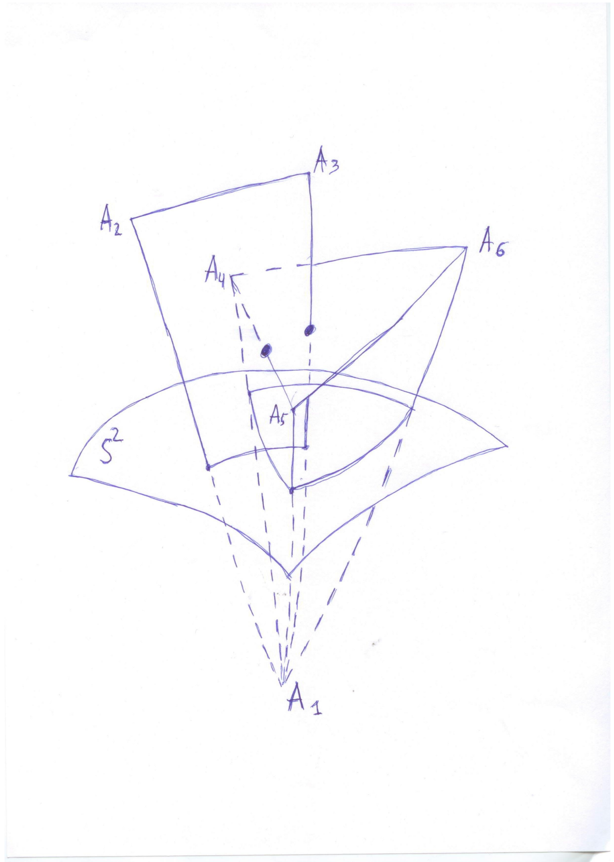

Figure 1: To Lemma 1

Lemma 1. *Assume that the vertices of two triangles are in general position. Denote by $A_1A_2A_3$ the first triangle. Denote by S^2 a sphere with the center A_1 and radius so small that all the vertices of the triangles except A_1 are outside S^2. If the number of the sides of the second triangle that are lower than A_2A_3 is odd then these two triangles are linked.*

Remark. The condition that the vertices of the triangles except A_1 are outside the sphere could be avoided at the price of some complications both in the statement and the proof.

Proof of Lemma 1.

Denote by A_4, A_5, A_6 the vertices of the second triangle. Let $f : \mathbb{R}^3 - \{A_1\} \to S^2$ be the central projection with the center A_1. Note that if a segment is higher than another segment then their projections to S^2 intersect each other. And note that if the projections of some two segments intersect each other then one of this segment is higher than another. By the assumption of the lemma there exists a side, say A_4A_5, of the triangle $A_4A_5A_6$ such that A_2A_3 is higher than A_4A_5. Then the point $f^{-1}(f(A_2A_3)) \cap A_4A_5$ is inside the 2-dimensional triangle $A_1A_2A_3$. Since $f(A_2A_3)$ is an ark of a circle on S^2 and $f(A_4A_5A_6)$ is a spherical triangle on S^2, $f(A_2A_3)$ intersects the outline of $f(A_4A_5A_6)$ in at most 2 points. So A_4A_5 is the unique side of the triangle $A_4A_5A_6$ that is lower than A_2A_3. This implies that the outline of the triangle $A_4A_5A_6$ intersects the interior of the triangle $A_1A_2A_3$ at a unique point $f^{-1}(f(A_2A_3)) \cap A_4A_5$. And the vertices of these two triangles are in general position. So these two triangles are linked. **Lemma 1 is proved.**

Lemma 2. *Assume that five points are in general position on the plane. Then the number of intersection points of all the unordered pairs of disjoint segments with the vertices at these five points is odd.*

This lemma is known, see, e.g., [Sk, §1].

Proof of Lemma 2.

Let a collection f of five points in general position in the plane be given. For any four distinct points A, B, C, D of the collection, the segments AB and CD either are disjoint or have a unique common point. Define $v(f)$ to be the parity of the number of intersection points of all the unordered pairs of disjoint segments with the vertices at the collection f.

$$v(f) := \sum \{|AB \cap CD| \ : \ \{A,B\},\{C,D\} \subset \binom{f}{2}, \ \{A,B\} \cap \{C,D\} = \emptyset\} \mod 2.$$

This lemma is implied by the following two assertions.

(a) For the collection f_0 of five vertices of a regular pentagon we have $v(f_0) = 1$.

(b) $v(f)$ does not depend on f.

Assertion (a) is clear. Let us prove (b). It suffices to prove that if we change the position of the first point keeping the remaining four fixed then the number $v(f)$ is not changed. Denote by K' the new position of the point K and by f' the obtained collection.

For each $A \in f - \{K\}$ denote by $\triangle_A$ the triangle with vertices from $f - \{A, K\}$. Then the assertion follows from

$$v(f') - v(f) = \sum_{A \in f - \{K\}} (|KA \cap \triangle_A| - |K'A \cap \triangle_A|) = \sum_{A \in f - \{K\}} |KK' \cap \triangle_A| = 0 \mod 2.$$

Here the first equality is clear. The second equality holds because $|KK'A \cap \triangle_A|$ is even for each $A \in f - \{K\}$ because the outlines of two triangles on the plane whose vertices are in general position intersect each other at an even number of points (see [BE82]). The last equality holds because for each unordered pair $\{A, B\} \subset f - \{K\}$ there exist exactly two triangles with vertices from $f - \{K\}$ containing the segment AB. So for each unordered pair $\{A, B\} \subset f - \{K\}$ the number $|KK' \cap AB|$ appears in the sum twice for two triangles $\triangle_A$, $\triangle_B$. **Lemma 2 is proved.**

Remark. We can also state the spherical analog of this lemma whose proof is analogous.

Proof of the linear Conway-Gordon-Sachs Theorem. Suppose that points $A_1, A_2, A_3, A_4, A_5, A_6$ are in general position in 3-dimensional space. Consider the complete graph K_5 whose vertices are the points A_2, A_3, A_4, A_5, A_6 and edges are segments joining pairs of these points. Consider a sphere S^2 with center A_1. Let this sphere be small enough that points A_2, A_3, A_4, A_5, A_6 are outside the sphere. Consider the central projection $f : \mathbb{R}^3 - A_1 \to S^2$ with the center A_1. For each ordered pair (e, e') of edges $e, e' \in K_5$ denote

$$e \circ e' := \begin{cases} 1, & \text{if } e \text{ is higher than } e'; \\ 0, & \text{otherwise.} \end{cases}.$$

For any edge $e \in K_5$ define its *linking number*

$$S_e := \sum_{e' \in (K_5 - e)} e \circ e'.$$

Define $v(f)$ as the number of self-intersections of $f(K_5)$.

Then

$$\sum_{e \in K_5} S_e \equiv \sum_{(e,e'), e, e' \in K_5} e \circ e' \equiv v(f') \equiv 1 \mod 2.$$

Hence the linking number of some edge, say A_2A_3, is odd. Then Lemma 1 implies that triangles $A_1A_2A_3$ and $A_4A_5A_6$ are linked.

Here the first equality follows from definition of S_e. The second equality holds because

- for any two edges $e, e' \in K_5$ $|f(e) \cap f(e')| \le 1$ because vertices of K_5 are in general position
- if edges $e, e' \in K_5$ are nonadjacent and $f(e) \cap f(e') \ne \emptyset$ then $e \circ e' + e' \circ e = 1$

• if edges $e, e' \in K_5$ are adjacent or $f(e) \cap f(e') = \emptyset$ then $e \circ e' + e' \circ e = 0$.

The third equality follows from the spherical analogue of Lemma 2. **QED**

Second proof of Linear Conway-Gordon-Sachs Theorem.

Denote by $1, 2, 3, 4, 5, 6$ our six points. We may assume that there exist a plane π in general position such that $2 \in \pi$ and point 1 and points $3, 4, 5, 6$ are in different half-spaces with respect to π. Take a line $l \subset \pi$ such that point 2 and points $1i \cap \pi$, $i \in \{3, 4, 5, 6\}$, are in different half-spaces of π with respect to l.

Denote by (ijk) the plane containing points $i, j, k \in \mathbb{R}^3$ and by ijk the triangle with vertices $i, j, k \in \mathbb{R}^3$. Consider the graph K_6 with vertices at points $1, ..., 6$ and edges on segments joining any two of these points. For each $ij \in K_6 - \{1\}$ denote

$$S_{ij} := \sum_{km \in K_6 - \{1,i,j\}} km \cap 1ij$$

.

Lemma 3. *If for some edge $ij \in K_6 - \{1\}$ $S_{ij} \equiv 1 \mod 2$ then the triangle $1ij$ and the triangle with the vertices from $K_6 - \{1, i, j\}$ are linked.*

Proof of Lemma 3. Note that $S_{ij} \leq 2$. Since S_{ij} is odd it follows that $S_{ij} = 1$. Then there is a unique side of the triangle with vertices from $K_6 - \{1, i, j\}$ that intersects the triangle $1ij$. And that side intersects the interior of $1ij$ because our six points are in general position. **Lemma 3 is proved.**

Consider the projection $f : \mathbb{R}^3 \to l$, $f(X) = (12X) \cap l$. Denote $i' := f(i)$. Assume that points $3', 4', 5', 6'$ follow in order a', b', c', d' on the line l. For each unordered pair $\{i, km\}$ of vertex $i \in K_6 - \{1, 2\}$ and edge $km \in K_6 - \{1, 2, i\}$ denote

$$i \circ km := \begin{cases} 1, & \text{if } i' \in k'm' \text{ and segments } 1i, 12 \text{ are in different half-spaces with respect to } (1km); \\ 0, & \text{otherwise.} \end{cases}.$$

Lemma 4. *a)For each $3 \leq i \leq 6$ we have $i \circ km \equiv 1 \mod 2 \iff 2i$ intersects the interior of $1km$ or km intersects the interior of $12i$.*

b) $b \circ ad + c \circ ad \equiv 1 \mod 2 \iff ad$ intersects the interior $1bc$ or bc intersects the interior of $1ad$.

c) $b \circ ac + c \circ bd \equiv 0 \mod 2 \iff ac$ intersects the interior of $1bd$ or bd intersects the interior of $1ac$.

Proof of Lemma 4.

Proposition. The points k and m are in different half-spaces with respect to $(12i) \iff i' \in k'm'$.

Proof of the Proposition. Denote $a_i := 1i \cap \pi$. Since the point 2 and points $1i \cap \pi$, $i \in \{3, 4, 5, 6\}$, are in different half-spaces of π with respect to l we have that each segment $2a_i$, $3 \leq i \leq 6$, intersects the line l. Then we have that for each $3 \leq i \leq 6$ the triangle $12i$ intersects l. Then $i' \in k'm' \iff$ the triangle $12i$ is between the planes $(12k)$ and $(12m) \iff$ the points k and m are in different half-spaces with respect to $(12i)$. QED

Proof of (a). 1) If $i \circ km \equiv 1 \mod 2$ then $i' \in k'm'$. Then the points k and m are in different half-spaces with respect to $(12i)$ and the segments 12 and $1i$ are in different half-spaces with respect to the plane $1km$. Then triangles $12i$ and $1km$ intersect each other. Then $2i$ intersects the interior of $1km$ or km intersects the interior of $12i$ because our points are in general position.

2) Assume that $2i$ intersects $1km$ or km intersects $12i$. Then we have that the triangles $12i$ and $1km$ intersect each other. Then the segment $2i$ intersects the plane $(1km)$ and the segment km intersects the plane $(12i)$. Then the points 2 and i are in different half-spaces with respect to the plane $(1km)$. And the points k and m are in different half-spaces with respect to the plane $(12i)$ that implies that $i' \in k'm'$ by Proposition. Then $i \circ km = 1$.

Proof of (b).

1) Assume that $b \circ ad + c \circ ad \equiv 1 \mod 2$. Without loss of generality we may assume that $b \circ ad = 0$, $c \circ ad = 1$. Then it follows that the points b and c are in different half-spaces with respect to the plane $(1ad)$. And the Proposition implies that the points a and d are in different half-spaces with respect to $(1bc)$. Hence the triangle $1bc$ intersects the triangle $1ad$. Then ad intersects the interior $1bc$ or bc intersects the interior of $1ad$ because our six points are in general position.

2) Assume that ad intersects $1bc$ or bc intersects $1ad$. Then we have that the triangles $1bc$ and $1ad$ intersect each other. Then it follows that the points b and c are in different half-spaces with respect to the plane $(1ad)$. And since $b', c' \in a'd'$ it follows that $b \circ ad + c \circ ad \equiv 1 \mod 2$.

Proof of (c). 1) Assume $b \circ ac + c \circ bd \equiv 0 \mod 2$. Without loss of generality we may assume that $b \circ ac = c \circ bd = 1$. Denote $K := (12c) \cap bd$. Note that $K' = c'$ and $K \circ ac = 0$. Then $K \circ ac + b \circ ac = 1$ and $K', b' \in a'c'$. Then the assertion follows from the item (b).

2) Assume that ac intersects $1bd$ or bd intersects $1ac$. Denote $K := (12c) \cap bd$. Note that $K' = c'$ and $K \circ ac \neq c \circ bd$. Then $K', b \in a'c'$ and Kb intersects the interior of $1ac$ or ac intersects the interior of $1Kb$. Then the item (b) implies that $K \circ ac + b \circ ac \equiv 1 \mod 2$ then $c \circ bd + b \circ ac \equiv 0 \mod 2$

Lemma 4 is proved.

The problem follows from

$$\sum_{ij \in K_6 - \{1\}} S_{ij} = \sum_{mn, ij \in K_6 - \{1\}} mn \cap 1ij \equiv$$

$$\equiv \sum_{\{ij,n\} \subset K_6 - \{1,2\}} (2n \cap 1ij + ij \cap 12n) + \sum_{\{ij,mn\} \in K_6 - \{1,2\}} (ij \cap 1mn + mn \cap 1ij) \equiv$$

$$\equiv \sum_{\{i,mn\} \subset K_6 - \{1,2\}} (i \circ mn) + b \circ ac + c \circ bd + 1 + b \circ ad + c \circ ad \equiv 2 \sum_{\{i,mn\} \subset K_6 - \{1,2\}} i \circ mn + 1 \equiv 1 \mod 2$$

Then Lemma 3 implies that some two triangles with vertices at our six points are linked.

The first and second equalities are clear. The third equality follows from Lemma 2 and because $a'b'$ do not intersect $c'd'$. The forth equality holds because for each $i \in \{b, c\}$ and $n, m \in K_6 - \{1, 2\}$ number $i \circ km$ appears once in the sum $\sum\limits_{\{i,mn\} \subset K_6 - \{1,2\}} (i \circ mn)$ and once in the sum $b \circ ac + c \circ bd + 1 + b \circ ad + c \circ ad$ and because for each $ij \in K_6 - \{1, 2, a, d\}$ we have $a \circ ij = d \circ ij = 0$. The last equality is clear. **QED.**

To prove the Conway-Gordon-Sachs Theorem we will use two following lemmas.

Consider a plane in general position. Define what it means for a segment a to be *higher* than a segment b analogously to the definition in the linear case but replacing 'sphere' with 'plane in general position' and 'central projection' with 'orthogonal projection'.

Lemma 5. *Consider two closed broken lines, denote by A, B, and a general position plane. Assume that the number of ordered pairs (a, b) of sides $a \in A$, $b \in B$ such that b is higher than a is odd. Then these two broken lines are linked.*

This lemma is known.

Lemma 6. *For any general position piecewise-linear map $f : K_5 \to \pi$ the number of self-intersections of $f(K_5)$ is odd.*

This lemma is the generalization of Lemma 2, see [Sk, §1]. But we will give the proof of this lemma here.

Proof of the Lemma 6. For each general position piecewise-linear map $f : K_5 \to \pi$ denote $v(f)$ as the number of self-intersections of $f(K_5)$.

The map $f:\mathbb{R}^3\to\pi$ is a general position map because π is a general position plane.

Proposition. The number of intersection points of two closed broken lines on the plane whose vertices are in general position is even.

This Proposition is known, see e.g [BE,§1].

Assume that $f', f: K_5\to\mathbb{R}^2$ are general position piecewise-linear maps. The case when these maps differ exactly at one vertex of graph K_5 is considered in the proof of Lemma 2. Suppose that these maps do not differ at any vertex of the graph K_5.

Then it suffices to prove that if f and f' differ exactly at the interior of some edge, say e, of graph K_5, then $v(f)=v(f')$. Let us prove this fact. Denote by C the cycle of edges of the graph K_5 nonadjacent to e. For any two sets A,B denote $A\triangle B:=(A\cup B)-A\cap B$. Then lemma follows from

$$v(f)-v(f')=(|f(e)\cap f(C)|-|f'(e)\cap f(C)|)=|(f(e)\triangle f'(e))\cap f(C)|=0 \mod 2$$

Here the last equality holds because $f(e)\triangle f'(e)$ and $f(C)$ are closed broken lines on the plane whose vertices are in general position. Then the Proposition 2 implies that the number of intersection points of this broken lines is even. **Lemma 6 is proved.**

Proof of the Conway-Gordon-Sachs Theorem.

For any ordered pair of broken lines (A,B) in the 3-dimensional space denote

$$A\circ B:=\begin{cases}1, & \text{if the number of ordered pairs } (a,b) \text{ of sides } a\in A,\ b\in B \text{ such that } a \text{ is higher than } b \text{ is odd;}\\ 0, & \text{otherwise.}\end{cases}.$$

Denote by a one of the vertices of graph K_6. Denote by C_{ij} the cycle in the graph $K_6-\{a\}$ that does not contain the edge ij. Define $v(f)$ as the number of self-intersections of $f(K_6-\{a\})$. Then the problem follows from

$$\sum_{bc\in K_6-\{a\}} abc\circ C_{bc}\equiv\sum_{bc\in K_6-\{a\}} ab\circ C_{bc}+\sum_{bc\in K_6-\{a\}} ac\circ C_{bc}+\sum_{bc\in K_6-\{a\}} bc\circ C_{bc}\equiv$$
$$\equiv\sum_{bc\in K_6-\{a\}} bc\circ C_{bc}\equiv v(f)\equiv 1\mod 2,$$

Hence for some two cycles abc, C_{bc} of graph K_6 the number $abc\circ C_{bc}$ is equal to 1 and Lemma 5 implies that these cycles are linked.

Proof of the second equality.

Note that $ab\circ C_{bc}=\sum\limits_{e\in C_{bc}} ab\circ e$

For each $b\in K_6-\{a\}$ and for each edge $e\in K_6-\{a,b\}$ there exist exactly two 3-length cycles in $K_6-\{a,b\}$ containing this edge. So for each $b\in K_6-\{a\}$ and each edge $e\in K_6-\{a,b\}$ the number $ab\circ e$ appears twice in the sum $\sum\limits_{bc\in K_6-\{a\}} ab\circ C_{bc}$. Therefore this sum is even. **The second equality is proved.**

The proof of the third equality is the same to the *proof of the second equality* in the linear case.

The last equality follows from Lemma 6 because the graph $K_6-\{a\}$ is the graph K_5. **QED**

To prove the Sachs Theorem we will use the following lemma.

Lemma 7. *For any general position piecewise-linear map $f: K_{3,3}\to\pi$ the number of self-intersections of $f(K_{3,3})$ is odd.*

This proof of this lemma is analogous to the proofs of Lemma 2 and Lemma 6.

Proof of the Sachs Theorem. Consider a plane π in general position. Consider the orthogonal projection $f : \mathbb{R}^3 \to \pi$. Denote by a, b two vertices of graph $K_{4,4}$ from different sides. Denote by C_{ij} the cycle of edges of the graph $K_{4,4} - \{a, b\}$ that do not contain $ij \in K_{4,4} - \{a, b\}$. Define $v(f)$ as the number of self-intersections of $f(K_{4,4} - \{a, b\})$. Then the problem follows from

$$\sum_{ij\in K_{4,4}-\{a,b\}} abij \circ C_{ij} \equiv \sum_{ij\in K_{4,4}-\{a,b\}} ab \circ C_{ij} + \sum_{ij\in K_{4,4}-\{a,b\}} aj \circ C_{ij} + \sum_{ij\in K_{4,4}-\{a,b\}} bi \circ C_{ij} +$$

$$+ \sum_{ij\in K_{4,4}-\{a,b\}} ij \circ C_{ij} \equiv \sum_{ij\in K_{4,4}-\{a,b\}} ij \circ C_{ij} \equiv v(f) \equiv 1 \mod 2$$

Then Lemma 5 implies the theorem. Here the first equality is clear. **Proof of the second equality.** The second equality holds because for each $j \in K_{4,4} - \{a, b\}$ and for each edge $e \in K_{4,4} - \{a, b, j\}$ there exist four 4-length cycles containing edge e. So for each edge $e \in K_{4,4} - \{a, b, j\}$ the number $aj \circ e$ appears four times in the sum $\sum\limits_{ij\in K_{4,4}-\{a,b\}} aj \circ C_{ij}$. Hence this sum is even. **The second equality is proved.**

The proof of the third equality is the same as the *proof of the second equality* in the proof of the linear Conway-Gordon-Sachs Theorem.

The last equality follows from Lemma 7 because the graph $K_{4,4} - \{a, b\}$ is the graph $K_{3,3}$. **QED**

The author is grateful to Arkady Skopenkov and Mikhail Skopenkov for productive discussions.

References

[Sk] A. Skopenkov, Algorithms for recognition of the realizability of hypergraphs, in Russian, http://www.mccme.ru/circles/oim/algor.pdf

[BE] V. Boltyansky, V. Efremovich, Visual topology, in Russian, http://ilib.mccme.ru/djvu/geometry/boltiansky-nagl-topo.htm

[Sk03] M. Skopenkov. Embedding products of graphs into Euclidean spaces // Fund. Math. 2003. 179. P. 191-198.