Games with modified dice

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Abstract

The article studies variations of a standard six-sided dice. In particular, we consider dice with other integer numbers written on their faces, but with the same sum of those numbers. The question is raised about how any two of those modified dice play against one another in a random toss.

Two new definitions are introduced after that: a collective of modified dice and a characterizing sequence of a given dice. Interesting observations are made, describing some properties of mentioned definitions. The way to calculate the number of dice in a collective is also described. Using those new objects a theorem is proved by partitioning the collective into non-intersecting subgroups, showing that there is no unbeatable six-sided modified dice, if we only allow non-negative integers on faces.

The obtained result is then generalized upon an arbitrary number of faces and an arbitrary sum of numbers on those faces. The main theorem, characterizing all possible unbeatable dice, is proved. Another auxiliary theorem is proved that gives description of all possible collectives, containing at least one unbeatable dice. It also shows that natural collectives do not contain a single unbeatable dice.

Next, we ask a question about changing the lower boundary of numbers on faces. We show that the same four main types of unbeatable dice remain the same for any lower boundary. A variant of an auxiliary theorem for strictly positive numbers on faces is provided. Also we show that every natural collective in that case contains exactly one natural unbeatable dice. It is worth mentioning that the same general algorithm could be used to describe collectives that contain at least one unbeatable dice for any lower boundary.

At the end of the article we provide several possible applications of obtained results. One of them is the new way of solving and compiling Mathematical Olympiad problems, two examples of which are given. Also we provide a simple model of competitive business environment, for which the proven theorems could be used to determine the optimal economical strategy. The list of open questions is also given at the end to encourage further research on the subject.

Outline of the problem

One can find the following problem in books on the Theory of Probability.

Problem. Two players play a game. They throw two fair six-sided dice on every round. Player, whose dice shows a bigger number, wins. Who will win more frequently in a very long game?

Despite the answer for the problem being obvious from symmetry considerations, we would like to provide a stricter argumentation by analyzing the table of all equally likely outcomes for dice I and II. “+” in the table means that dice II wins in that outcome, “-” means dice I wins, “0” means the outcome is a draw. We will call every “-” and “+” a *scored point* for the corresponding dice.

I\II	«1»	«2»	«3»	«4»	«5»	«6»
«1»	0	+	+	+	+	+
«2»	-	0	+	+	+	+
«3»	-	-	0	+	+	+
«4»	-	-	-	0	+	+
«5»	-	-	-	-	0	+
«6»	-	-	-	-	-	0

As we can see, amounts of “+” and “-” are the same, resulting in a draw in an infinitely long game.

It is interesting to enquire, however, what will happen if one of the dice will be slightly modified. What if instead of the standard numbers $(1, 2, 3, 4, 5, 6)$ it has $(0, 2, 3, 4, 5, 7)$, for example, written on its faces?

I\II	«0»	«2»	«3»	«4»	«5»	«7»
«1»	-	+	+	+	+	+
«2»	-	0	+	+	+	+
«3»	-	-	0	+	+	+
«4»	-	-	-	0	+	+
«5»	-	-	-	-	0	+
«6»	-	-	-	-	-	+

As we can see, wins and losses distribution changed a little, but dice are still playing in a draw. The question can be asked, does there exist a dice with modified numbers on faces such that if it is not winning every other dice out there (which is impossible, since it always plays in a draw against oneself), it is at least not losing to any other dice in an infinitely long game.

That is the question we will be trying to solve.

General definitions

We will not constrain ourselves by discussing only six-sided dice, as we want to consider as broad problem as possible. An object with five or eight faces formally cannot be called a dice (maybe a spinner), but we will still use that name for the ease of terminology.

One can notice that in the example with modified dice $(0, 2, 3, 4, 5, 7)$ the sum of numbers on its faces remained the same, i.e. $0 + 2 + 3 + 4 + 5 + 7 = 1 + 2 + 3 + 4 + 5 + 6 = 21$. If one compares two dice with equal numbers of faces but different sums of numbers on those faces, then the dice with greater sum receives an advantage. Indeed, it is quite easy to construct a dice with sum 22 that beats the given dice with sum 21, say $(0, 2, 3, 4, 5, 7)$. One can just increase any number by 1. As shown in the table, $(0, 2, 3, 4, 6, 7)$ wins. Because of this advantage, we will only compare dice with the same sums of numbers.

I\II	«0»	«2»	«3»	«4»	«5»	«7»
«0»	0	+	+	+	+	+
«2»	-	0	+	+	+	+
«3»	-	-	0	+	+	+
«4»	-	-	-	0	+	+
«6»	-	-	-	-	-	+
«7»	-	-	-	-	-	0

On the other hand, dice with fewer faces gets an advantage when the sum is fixed. For example, to beat the dice $(1, 2, 3, 4, 5, 6)$ one can counter it with five-sided dice $(2, 3, 4, 5, 7)$, taking the smallest number out and increasing the largest to compensate the sum. Because of that, we will only compare dice with the same numbers of faces. Dice with just one face will not be considered.

I\II	«1»	«2»	«3»	«4»	«5»	«6»
«2»	-	0	+	+	+	+
«3»	-	-	0	+	+	+
«4»	-	-	-	0	+	+
«5»	-	-	-	-	0	+
«7»	-	-	-	-	-	-

We also need to constrain the modification of numbers on the faces. If we don't do that, then it becomes easy to find a counter-dice to any given one by simply decreasing the smallest number and increasing all of the others. E.g. $(1,2,3,4,5,6)$ completely loses to $(-4,3,4,5,6,7)$. We will put quite natural constraints, from our point view: 1) numbers on faces must be integers; 2) the lower boundary for numbers on faces must be 0. Clearly, we will not consider dice with the sum of numbers equal to zero in that case.

However, the method we will obtain later will allow us to find sets of unbeatable dice for other lower boundaries as well.

Definition. The set of dice with n faces and with the sum of numbers on those faces Σ (just *the sum* from now on), satisfying the discussed constraints, will be called a *collective* and denoted $[n, \Sigma]$. All the collectives $[n, \frac{n(n+1)}{2}]$ will be called *natural*. For example, collective $[6, 21]$ is natural.

Another remark should be made about the representation of every given dice. Since it does not matter which numbers are written on which faces, every dice will be written as a sequence of numbers in non-deceasing order. E.g. $(1,2,2,4,5,6)$, not $(1,2,4,2,6,5)$.

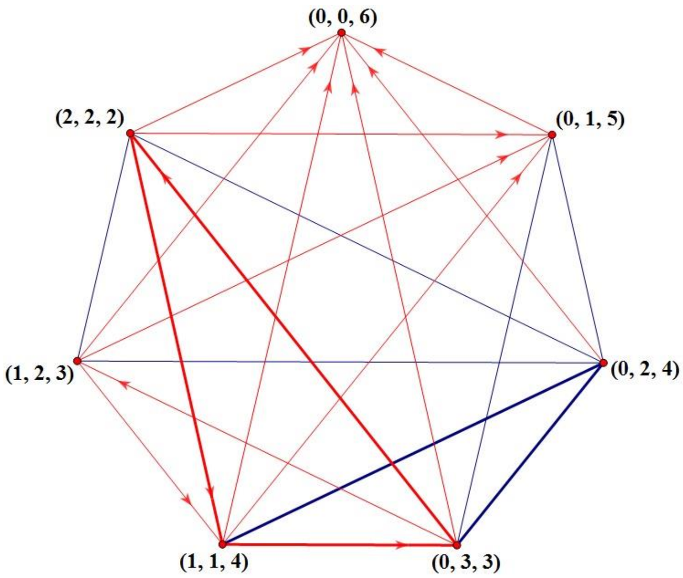
We will write $A \succ B$, if A beats B in an infinite game. If they play in a draw we will write $A \sim B$.

Interesting observations

We have $(1,2,3,4,5,6) \prec (0,3,3,4,5,6) \prec (1,1,4,4,4,7) \prec (1,2,3,4,5,6)$ (can be checked by using tables). All of those dice come from natural collective $[6, 21]$. That means the relationship $\prec$ is not transitive and the set of all the dice in a collective is not linearly or partially ordered.

Next, we have $(1,2,3,4,5,6) \sim (1,1,3,4,6,6) \sim (0,1,4,4,6,6) \succ (1,2,3,4,5,6)$, what implies that the relationship $\sim$ is not an equivalence.

We can conclude that oriented graph would be the best form to represent a collective of dice. One can use a directed edge between vertices A and B to show that $A \succ B$ and a not oriented one if $A \sim B$. You can see the example of such a graph for collective $[3, 6]$ below. It contains 7 dice.



The number of dice in the collective $[n, \Sigma]$ is equal to the number of partitions $P(n+\Sigma, n)$ of the number $n+\Sigma$ into n positive terms, written in non-decreasing order. It is known[2] that the number of those partitions does not have an explicit formula, but can be calculated using the recurrent relation $P(n,k) = P(n-1,k-1) + P(n-k,k)$. We have calculated that the number of dice in the collective $[6, 21]$ is equal 331.

Characterizing sequence

Unfortunately, it is not possible to determine if the given dice is unbeatable through simple tables analysis. We add a new object that will help us solve the above problem.

Definition. *Characterizing sequence* of an arbitrary dice is an infinite sequence, on the i-th place of which the number of faces with i written on them in the dice is set. We will start numerating places in the sequence from zero, since zero is the smallest number that could be seen on the face, according to our constraints.

To give an example, the characterizing sequence of the dice $(1,2,3,4,5,6)$ is $< 0,1,1,1,1,1,1,0,\ldots >$, since the dice contains 0 zeroes, 1 unit, 1 two, etc. In most case we will just be writing $< 0,1,1,1,1,1,1 >$, dropping down all the zeroes at the end (we will call it *short characterizing sequence*).

Characterizing sequence will be written in angular brackets, so that we do not confuse it with a dice. One may notice that the sum of all the numbers in a characterizing sequence is equal to the number of faces n.

Let us take two identical dice with fragment $< \ldots, A, B, \ldots >$ on k-th и $k+1$-th places of the characterizing sequence. We will modify the first dice into $< \ldots, A-1, B+1, \ldots >$, i.e. we will increase one of the k-s by one (calling it *shifting A to the right*). Before modification that face of dice I played against dice II as follows:

I\II	…	k	…	k	$k+1$	…	$k+1$	…
k	-	0	A раз	0	+	B раз	+	+

After modification it plays differently:

I\II	…	k	…	k	$k+1$	…	$k+1$	…
$k+1$	-	-	A раз	-	0	B раз	0	+

Instead of A zeroes that face receives A minuses, instead of B pluses it gets B zeroes. Thus, the modified dice wins $A+B$ points from unmodified one, since all the other rows in the table remained unchanged.

By applying the same logic, we can understand that the modification «to the left» (we will call it *shifting B to the left*), the dice $< \dots, A+1, B-1, \dots >$ loses $A+B$ points to the unmodified dice. Thus, the following theorem holds true.

Theorem. **When the dice is being modified by increasing a number $\boldsymbol{k}$ on one of the faces to $\boldsymbol{k+1}$, the resulting dice wins as many points from the unmodified predecessor as there were faces with numbers $\boldsymbol{k}$ and $\boldsymbol{k+1}$ combined. When a number $\boldsymbol{k}$ is decreased to $\boldsymbol{k-1}$ the resulting dice loses as many points to the predecessor as there were faces with $\boldsymbol{k}$ and $\boldsymbol{k-1}$ combined.**

By consecutive increases and decreases of numbers on faces one can obtain any dice from any given one. Since only one row in the table changes every time and since we keep comparing the resulting dice to the original one and not to the intermediary modifications, the number of won or lost points can be summed up algebraically. Thus, by looking only on the original dice's characterizing sequence and counting points after every modification, we can determine if the resulting dice wins or loses to the original one.

This theorem will allow us to construct a counter-dice to any given one or prove that such counter-dice does not exist.

Example. Let us look at dice $(1,2,3,4,5,6)$ and $(0,2,3,4,5,7)$. They have characterizing sequences $< 0, \overleftarrow{1}, 1, 1, 1, 1, \overrightarrow{1} >$ and $< 1, 0, 1, 1, 1, 1, 0, 1 >$. The second one can be obtained from the first one by moving $\overleftarrow{1}$ to the left and $\overrightarrow{1}$ to the right. Score is changed the following way $-1-0+1+0=0$ (can be determined by looking only on the initial dice's characterizing sequence), which implies that those two dice play in a draw. That complies with the conclusion, we have drawn from the table of outcomes.

Remark. When modifying a dice the total number of right shifts must be equal to the total number of left shifts. The newly created dice will have a different sum otherwise, which contradicts our desire to only compare dice with equal sums.

Non-existence of unbeatable dice in the collective $[\boldsymbol{6}, \boldsymbol{21}]$

We will assume that there exists an unbeatable dice in the collective $[6, 21]$, separately going through several cases, and disproving that conjecture in every case .

Case 1*. There are no zeroes in the short characterizing sequence.

In that case the dice looks as follows: $< a_0, a_1, \dots, a_k >$, where $a_i \neq 0$ and $a_0 + a_1 + \dots + a_k = 6$. We shift several (at least once) times to the right to obtain the dice

$< 0, 1, 1, 1, 1, 1, 1 >$. Since all the shifts were to the right, the resulting dice's sum is greater, implying that the original dice does not belong to $[6, 21]$.

Case 2*. There are zeroes in the short characterizing sequence of the dice and one goes after the first zero that is not followed by a zero.

In that case there cannot be any other number A in the characterizing sequence, save 0 and 1. Indeed, let us assume, without loss of generality, that characterizing sequence looks like $< \dots 0, \boxed{1}, \dots A, B, \dots >$. By shifting $\boxed{1}$ to the left and A to the right, we will get $-1 - 0 + A + B \geq A - 1 > 0$ in score, meaning that the original dice was not unbeatable.

After any 1 in the sequence, except the one following the mentioned 0, cannot stand another 1, because for $< \dots 0, \boxed{1}, \dots 1, 1, \dots >$ we get can similarly get $-1 - 0 + 1 + 1 > 0$. We can always find another 1 going after 0, since the number of ones is greater than four*, which implies that the boxed $\boxed{1}$ cannot be followed by 1 either.

Thus, all the ones are delimited by zeroes. If any 1 is preceded by at least two zeroes, like $< \dots, 0, 0, 1, \dots, 1, 0, \dots, 1, 0, \dots >$, we can shift this 1 two positions to the left and two other ones one step to the right, gaining $-1 - 0 + 1 + 1 > 0$ points.

It means the dice looks like $< A, 0, A, 0, \dots, A >$ or $< 0, A, 0, A, \dots, A >$ ($A = 1$ in that case). Later on we will show that all such dice do not belong to $[6, 21]$.

Case 3. There are zeroes in the short characterizing sequence of the dice and $A \geq 2$ goes after the first zero that is not followed by a zero. We will prove that the dice must look like $< \boxed{\dots}, 0, A, 0, A, \dots, 0, A >$, where the first $\boxed{\dots}$ can be any subsequence.

Let us assume the sequence contains the following part $< \boxed{\dots}, 0, A, B, \dots >$. By shifting one face from A to the left and one to the right we get $-A + A + B = B \geq 0$ points, therefore $B = 0$.

It means the sequence looks like that then: $< \boxed{\dots}, 0, A, 0, \dots, 0, B, C, \dots >$. C must be equal to 0 similarly to the previous reasoning, since $B \neq 1$ according to 2. By shifting A to the left and B to the right we get $-A + B \leq 0$ in score, because we assumed the dice is unbeatable. Therefore, $A \geq B$. On the other hand, if in the original dice we shift two times from A to the right by one position ($A \geq 2$) and one time from B to the left by two positions we will get $+A + A - B - 0 > 0$. It means that after A there could not be more than one zero, if it is not the end of the sequence, of course. Finally, if we move B to the left and A to the right in the original dice we will get $B \geq A$, implying that $B = A$. Thus, we got $< \boxed{\dots}, A, 0, A, 0, \dots >$.

We can prove again in exactly the same fashion that this pattern continues on.

Case 4. The pattern $< \dots, B, C, 0, \dots, 0, A, 0, \dots >$, where $B, C > 0$, can be found in the short sequence.

Let us assume first that we have at least two zeroes in the middle. Shifting C to the left and A to the right we get $B + C \geq A$. By shifting from B two steps to the right and from A two steps to the left we obtain $+B + C + C - A - 0 \geq C > 0$ points. Thus, there could only be one middle zero.

Let us consider $< \ldots, B, C, 0, A, 0, \ldots >$. We shift one unity from A one step to the right, one unity from A two steps to the left and from B one step to the right. We get $+A - A - \mathrm{C} + B + C = B > 0$, concluding that there is no unbeatable dice in such form.

Case 5. There is a pattern $< C, 0, \ldots, 0, A, 0, \ldots >$ in the beginning of the short sequence.

If we have at least two zeroes in the middle we shift one step from A to the right, two steps from A to the left and one step from C to the right, obtaining $+A - A - 0 + C > 0$. It means there is only one zero.

Let us consider $< C, 0, \boxed{A}, 0, A, \ldots >$. By shifting one face from $\boxed{A}$ two positions to the left and one face from $\boxed{A}$ two positions to the right we get $-A - C + A + A \leq 0$, implying $C \geq A$. If we shift C to the right and A to the left in the original dice, we will get $A \geq C$. Thus, the dice has a form $< A, 0, A, 0, \ldots >$, as in case 2.

If the dice looks as follows $< C, 0, A >$, we can easily understand again that $A \geq C$. If $A \geq 3$, we will shift two faces from A one position to the right and one face from A two positions to the left obtaining $+A + A - A - C \geq 0$. It means that $A = C$, what we have seen above. If $A = 2$ and $C \neq 2$ then we have $< 1, 0, 2 >$, which is a potential candidate for the collective $[3, 4]$.

Case 6. The dice has a form $< 0, \ldots, 0, \boxed{A}, 0, A, \ldots, 0, A >$.

If there is only one A and $A \geq 3$ then, similarly to the previous case, there could not be more than one zero in the beginning of the sequence. If $A = 2$ then $< 0, \ldots, 0, 2 >$ is a potential candidate for a collective with two faces.

If we have several A then we can shift from $\boxed{A}$ two steps to the left and from $\boxed{A}$ two steps to the right, gaining $-A + A + A > 0$.

In conclusion, there is only one variant $< 0, A, 0, A, \ldots >$ left, as in case 2.

Case 7*. Dice $< 0, A, 0, A, \ldots >$.

Let $A = 1$. We must shift everything to the left to obtain $< 0, 1, 1, 1, 1, 1, 1 >$, which means the original dice has the sum greater than 21.

If $A \geq 2$, we must shift everything to the right, i.e. the original's sum is less than 21.

Case 8*. Dice $< A, 0, A, 0, \ldots >$.

If $A = 1$, we must shift first 1 to the right and five remaining to the left, which gives the sum greater than 21.

If $A \geq 2$, we must shift everything to the right to get $< 0, 1, 1, 1, 1, 1, 1 >$, implying that the sum of the original dice is less than 21.

***Theorem.* Cases 1-8 cover the whole variety of dice in the collective $[\mathbf{6}, \mathbf{21}]$. It means that $[\mathbf{6}, \mathbf{21}]$ does not contain a single unbeatable dice.**

Remark. Cases marked with * directly used dimensions of $[6, 21]$. We will remember that for the future.

Construction of unbeatable dice for an arbitrary collective

We will only consider cases marked with * in the previous part, as the only relying on the dimensions of [6,21].

Case 1. There are no zeroes in the short sequence. We will break this case into subcases.

Case 1a. In the beginning of the characterizing sequence there stands a pattern $< A, B, C, ... >$, where $B > 1$.

Shifting A to the right and C to the left gives us $+A + B - C - B \leq 0$, $C \geq A$. Shifting one face from B to the right and one to the left gives us $A = C$.

If there is D going after C, i.e. $< A, B, A, D, ... >$, we can similarly prove that $D \geq B$. If we shift B to the left now and the second A to the right, we will get $B = D$.

Following the same logic for next numbers we can show that the dice has one of the following forms: $< A, B, A, B, ... A >$ or $< A, B, A, B, ... B >$. Further ahead we will prove that all such dice are unbeatable.

Case 1b. There is a pattern $< A, 1, C, D, ... >$ in the beginning of the short sequence.

Similarly to previous case $C \geq A, D \geq 1$. When we shift 1 to the left and C to the right we get $-A - 1 + C + D \geq 0$. Thus, $C = A, D = 1$.

Again, we got sequences $< A, B, A, B, ... A >$ or $< A, B, A, B, ... B >$.

Case 1c. Short sequence has a form $< A, 1, C >$.

If $C = A$ we get the previously seen case. If $C \geq A + 1$ by shifting C to the right, 1 to the left we get $+C - 1 - A \geq 0$, resulting in $C = A + 1$.

Later on we will show that all the dice $< A, 1, A + 1 >$ are unbeatable.

Case 2. There are zeroes present in the short sequence and one goes after the first zero that is not followed by a zero.

We only need to cover the case with not more than three ones. At the same time, all the two-sided dice will be covered later. Thus, the only dice that is left is the one with three ones. If there is no zero on the first place in the sequence the reasoning for $[6, 21]$ is applicable, yielding the dice $< 0, 1, 0, 1, 0, 1 >$, which will be covered later.

If we have 1 on the first place, there are two possible dice $< 1, 0, 1, 0, 1 >$ and $< 1, 0, 1, 1 >$, according to the reasoning for $[6, 21]$.

Case 9. Dice $< 1, 0, 2 >$.

It is unbeatable, because the only shift to the right giving $+2$ must be compensated by the shift to the left from the same position.

Case 10. All the two-sided dice.

They have one of two forms $< \dots, 0, 1, 0, \dots, 0, 1, 0, \dots >$ or $< \dots, 0, 2, 0, \dots >$. Any shift of one face to the right must be compensated by the shift of the second face to the left. We will always get $-1-0-\cdots+1+0+\cdots=0$ in score. It means all the two-sided dice are unbeatable.

Case 11. All the dice of the forms $< A, B, A, B, \dots A >$ or $< A, B, A, B, \dots B >$.

A shift from any position to the left gives $-A-B$ in score, any shift to the right gives either $+A+B$ or $+A$ or $+B$. It means that in any combination of shifts the sum of won and lost points will be ≤ 0. Thus, all such dice are unbeatable in their respective collectives.

Case 12. Dice $< 1, 0, 1, 1 >$.

This dice is unbeatable. Shift to the right can give $+2$ only from the position of the second 1, all the other positions give $+1$. The only shift to the left, that will compensate the sum, can be done from the third 1. Those two shifts cancel each other. It means all available shifts to the right give $+1$ at maximum and the dice is unbeatable.

Case 13. Dice of the form $< A, 1, A+1 >$.

All of them are unbeatable as well. Any shift to the left gives $-A-1$ or $-A-2$. The only shift to the right giving $+A+2$ is from 1. The shift, compensating the sum, can only be done from $A+1$, cancelling the previous shift to the right.

Thus, we have proved the following theorem.

Theorem. **Every unbeatable dice has one of the following forms:**

1) any two-sided;

2) $\boldsymbol{< 1, 0, 1, 1 >}$ **or** $\boldsymbol{< 1, 0, 2 >}$**;**

3) $\boldsymbol{< A, 1, A+1 >, A \geq 1}$**;**

4) $\boldsymbol{< A, B, A, B, \dots B >}$ **or** $\boldsymbol{< A, B, A, B, \dots A >}$**,** $\boldsymbol{A, B}$ **– any non-negative integers.**

Thus, we have classified all unbeatable dice in our constraints.

Classifying collectives by the presence of an unbeatable dice

Now we can describe every collective that has an unbeatable dice.

Theorem. **The following collectives contain at least one unbeatable dice:**

1) $\boldsymbol{[2, \Sigma]}$**,** $\boldsymbol{\Sigma}$ **– arbitrary positive integer;**

2) $\boldsymbol{[3, 5]}$ **and** $\boldsymbol{[3, 4]}$**;**

3) $\boldsymbol{[2A+2, 2A+3]}$**,** $\boldsymbol{A \geq 1}$**;**

4) $\boldsymbol{[k \times A + k \times B, (k-1)k \times A + k^2 \times B]}$ **or** $\boldsymbol{[(k+1) \times A + k \times B, k(k+1) \times A + k^2 \times B]}$**,** $\boldsymbol{k \geq 1;\ A, B}$ **– arbitrary non-negative integers.**

Proof.

The main difficulty is presented by the last part. Let A, B repeat k times. Then for $< A, B, A, B, \dots B >$ $n = kA + kB$ and $\Sigma = A(0 + 2 + \cdots + 2k - 2) + B(1 + 3 + \cdots 2k - 1) = (k-1)k \times A + k^2 \times B$.

In case of $< A, B, A, B, \dots A >$ $n = (k+1)A + kB$ and $\Sigma = (0 + 2 + \cdots + 2k) + B(1 + 3 + \cdots 2k - 1) = k(k+1) \times A + k^2 \times B$.

The following theorem is also true.

Theorem. **Natural collectives with more than three faces do not contain unbeatable dice.**

Proof.

First of all, it is obvious that there exists two-sided unbeatable dice. For the collective $[3, 6]$ there is $< 1, 0, 1, 0, 1 >$, which is unbeatable.

We will analyze all groups of collectives that contain an unbeatable dice in a row. $[2, \Sigma]$ was already discussed, $[3, 5]$ and $[3, 4]$ are three-sided.

For $n = 2A + 2$ the natural $\Sigma = \frac{(2A+2)(2A+3)}{2} = (A+1)(2A+3) > 2A + 3$.

For $< A, B, A, B, \dots A >$ and $< A, B, A, B, \dots B >$ if $A \geq 2$ and $B \geq 2$ or $A = B = 1$, we will have to shift everything to the right to obtain the natural $< 0, 1, 1 \dots, 1 >$, implying that the sum of the original dice is less than that of a natural collective with the same number of faces. If $B = 1, A = 0$, then we must shift everything to the left and the original's sum is less than the natural one. If $B = 0, A = 1$ and the number of faces greater than three, we will have to shift one face to the right and more than one to the left.

The theorem is proved.

Changing the lower boundary

One may notice that most of the argumentations provided above never mention if the numeration of places in the characterizing sequence starts from zero or any other number. It means that we can change the lower boundary arbitrarily, retaining the same main forms of unbeatable dice. Of course, the set of collectives, containing unbeatable dice, will change.

We will only consider one special case when the lower boundary for our numbers on faces is 1 (also a very natural constraint). In that case the characterizing sequence for the dice $(1, 2, 3, 4, 5, 6)$ will be written as $< 1, 1, 1, 1, 1, 1 >$.

Theorem. **Every unbeatable dice with the lower boundary 1 belongs to one of the following categories:**

1) any two-sided;
2) $< \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1} >$ or $< \mathbf{1}, \mathbf{0}, \mathbf{2} >$;
3) $< \boldsymbol{A}, \mathbf{1}, \boldsymbol{A} + \mathbf{1} >, \boldsymbol{A} \geq \mathbf{1}$;

4) $\boldsymbol{< A, B, A, B, \dots B >}$ **or** $\boldsymbol{< A, B, A, B, \dots A >}$**,** $\boldsymbol{A, B}$ **– arbitrary non-negative integers.**

Theorem. **The following collectives contain at least one unbeatable dice, when the lower boundary is 1:**

1) $\boldsymbol{[2, \Sigma]}$**,** $\boldsymbol{\Sigma}$ **– arbitrary positive integer;**

2) $\boldsymbol{[3, 8]}$ **and** $\boldsymbol{[3, 7]}$**;**

3) $\boldsymbol{[2A + 2, 4A + 5]}$**,** $\boldsymbol{A \geq 1}$**;**

4) $\boldsymbol{[k \times A + k \times B, k^2 \times A + k(k+1) \times B]}$ **or** $\boldsymbol{[(k+1) \times A + k \times B, (k+1)^2 \times A + k(k+1)^2 \times B]}$**,** $\boldsymbol{k \geq 1}$**;** $\boldsymbol{A, B}$ **– arbitrary non-negative integers.**

Theorem. **Every natural collective with** $\boldsymbol{n}$ **faces has exactly one unbeatable dice, namely** $\boldsymbol{(1, 2, 3, \dots, n)}$**.**

These theorem can be proved in the same way we have proved their previous versions. The same method can be used to obtain the set of collectives containing unbeatable dice for any lower boundary.

Possible applications and open questions

Obtained results can be used to solve and create Mathematical Olympiad problems.

Problem 1. We have coins worth from 1 to 10 tugriks, enough of every kind. Alice and Bob put 10 coins into their hats in such a way that the overall value in a hat is equal to 55 tugriks. Then they randomly take one coin each out of their hats. Person, taking the coin that is worth more, wins. Bob noticed that Alice put one coin of every kind into her hat. He is now sure he can choose a set of coins that has more chances to succeed. Is he right?

Solution. According to the theorem for the lower boundary 1, Alice's set is unbeatable. It means Bob is wrong.

Problem 2. Two players play the following game. Everyone has a regular deck of 52 cards. A player chooses 10 cards from their deck in such a way that the sum of their values is equal to 60 (jack, queen, king, ace are worth 11, 12, 13, 14 points respectively). Every round then consists of each player choosing a random card from their set. The card that is worth more wins the round. Can one choose a set to expect to win not less rounds than one loses in a very long game?

Solution. We can apply our theorem for the $[10, 60]$ with the lower boundary 2. According to the theorem there is an unbeatable set $(2, 2, 4, 4, 6, 6, 8, 8, 10, 10)$, because its characterizing sequence is $< 2, 0, 2, 0, 2, 0, 2, 0, 2 >$. This unbeatable deck will win at least half of the non-draw rounds in a very long game. We can also notice that the deck does not use cards that are worth more than ten.

Our theorems can not only be used in problems solving, but can also be applied in finding optimal strategies in simple economic models. Let us assume that some players on the market have (financial or other) resources denoted by the number Σ. Every player can invest their resources into projects. There exist projects that cost $x, x+1, x+2, \dots$ Every player must have a portfolio of n projects (not necessarily different). At the end of the timeframe one of the projects chosen at random of every player pays dividends equal to the cost of that project. The player who earned more wins this timeframe. We may need to determine the portfolio that wins with the maximum probability if the strategies of other players are unknown. Or we may need to find a counter-portfolio to defeat the portfolio of some other player.

The first problem can be solved by finding the unbeatable dice for the collective $[n, \Sigma]$ with the lower boundary x. The second problem can be solved by finding the counter-portfolio, using the characterizing sequence and the algorithm described in the theorem.

Several questions, related to games with modified dice, remain open. We will provide a short list.

1) Is it possible to solve the same problem if one would allow non-integer numbers on faces?

2) Is it possible to solve the same problem with infinite number of faces, but finite sum, allowing non-integer numbers on faces?

3) Does there exist an algorithm of finding the ideal counter-dice for any given one (the dice that wins the given one with the maximum score advantage)?

4) Is it possible to determine the number of dice that win, or the number of dice that lose to the given dice?

5) Does there exist the set of dice with the maximum algebraic sum of points that they win or lose to all the other dice in a collective? In other words, do there exist minimax dice that win on average against all the other dice collectively?

Any of those open questions can become a topic for further research.

References

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