Tiling of regular polygon with similar right triangles, II

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A tiling is a decomposition of a polygon into finitely many non-overlapping triangles. (Note that a tiling is not a triangulation, i.e. a vertex of a triangle can lie on a side of a triangle.)

Theorem 1. *If a regular n-gon, $n > 8$, $n \neq 12, 14, 20, 32, 44$, can be tiled with similar right triangles, then no angle of this triangle equals to $\frac{n-2}{3n}\pi$.*

Theorem 1 follows from [L20, Theorem 1] for all $n \geq 25$, $n \neq 32, 42$.

Corollary 2. *A regular n-gon, $n > 8$, $n \neq 12, 14, 20, 28, 32, 44$, can be tiled by similar right triangles then one of the angles of these triangles is in $\frac{\pi}{n}$ or $\frac{2\pi}{n}$.*

Theorem 1 and [V. Theorem 1] imply Corollary 2.
We denote $a_n = 2\frac{n-2}{3n}$ for integer n. Then $1 - a_n = \frac{n+4}{3n}$.
For all n: $a_n < 2/3$ and $1 - a_n > 1/3$.

Lemma 3. *If $3a_n = p(1 - a_n) + qa_n + r$, $n > 8$, $n \neq 14, 20, 32$, and p, q, r are non-negative integers, then $q = 3$.*

Proof. Since $\frac{r}{3} \leq a_n = 2\frac{n-2}{3n} < \frac{2}{3}$, it follows that $r < 2$. Since $qa_n \leq 3a_n$, it follows that $q \leq 3$. We also have $p = \frac{a_n(3-q)-r}{1-a_n}, 1 - a_n > \frac{1}{3}, a_n < \frac{2}{3}$ therefore $p < \frac{\frac{2}{3}(3-q)-r}{\frac{1}{3}} = 2(3 - q) - 3r$. We consider all possible values r, q and show that either $q = 3$ or the values of n that correspond to the integer values of p that do not suit to limitations of the lemma:

1. Let $r = 0$.
1.1 Let $q = 0$. This is only possible if $0 \leq p < 6$. Then $n \in \{2, 5, 8, 14, 32\}$.
1.2 Let $q = 1$. This is only possible if $0 \leq p < 4$. Then $n \in \{2, 4, 8, 20\}$.
1.3 Let $q = 2$. This is only possible if $0 \leq p < 2$. Then $n \in \{2, 8\}$.
1.4 Let $q = 3$. Then we get the statement of the lemma.
2. Let $r = 1$. Since $a_n < \frac{2}{3}$, it follows that $q \leq \frac{3a_n-1}{a_n} < 2$.
2.1 Let $q = 0$. This is only possible if $0 \leq p < 3$. Then $n \in \{4, 8, 20\}$.
2.2 Let $q = 1$. This is only possible if $0 \leq p < 1$. Then $n = 8$. □

Lemma 4. *If $b = 1, 2, 3, 4$ and $b = p(1 - a_n) + qa_n + r$ where $n > 8$, $n \neq 12, 14, 20, 32, 44$ and p, q, r are non-negative integers, then $q \geq p$.*

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Proof. We consider four cases: $b = 1, 2, 3, 4$ separately.

1. If $r = 1$ then $p, q = 0$. If $r = 0$ we receive $1 = p(1 - a_n) + qa_n + r$. Then $q = 0$ or 1 because $a_n \geq \frac{1}{2}$. Suppose that $q = 0$. Then $p = \frac{1}{1-a_n}$. But $\frac{1}{3} < 1 - a_n < \frac{1}{2}$. Hence p is not non-negative integer. Contradiction. Hence $q = 1$. Then $p = q = 1$.

2. If $r > 0$ we have the previous case. If $2 = p(1 - a_n) + qa_n + r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{2}{1-a_n} = \frac{6n}{n+4}$. Then $0 < \frac{6n}{n+4} < 7$ is non-negative integer. Hence $n = 20$. Hence $p, q = 2$.

3. If $r > 0$ we have one of the previous cases. If $3 = p(1-a_n)+qa_n+r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{3}{1-a_n} = \frac{9n}{n+4}$. Then $0 < \frac{9n}{n+4} < 10$ is non-negative integer. Hence $n \in \{14, 32\}$. Hence $p, q = 3$.

4. If $r > 0$ we have one of the previous cases. If $4 = p(1-a_n)+qa_n+r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{4}{1-a_n} = \frac{12n}{n+4}$. Then $0 < \frac{12n}{n+4} < 13$ is non-negative integer. Hence $n \in \{12, 20, 44\}$. Hence $p, q = 4$. □

Proof of Theorem 1. Assume that a regular n-gon, $n > 8$, $n \neq 12, 14, 20, 32, 44$, is tiled with similar right triangles of angles $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$, $\frac{\pi}{2} - \alpha$ and $\frac{\pi}{2}$, and $\alpha = \frac{n+4}{6n}\pi$.

Take a vertex of the n-gon. By p, q, r we denote the number of smaller acute angles α, bigger acute angles $\frac{\pi}{2} - \alpha$ and angles $\frac{\pi}{2}$ respectively at this vertex. We split each angle including angle at the vertex of $n-gon$ by $\frac{\pi}{2}$. Then $3a_n = p(1 - a_n) + qa_n + r$, where $\frac{1}{2} < 3a_n < \frac{2}{3}$. So $a_n = 2\frac{\frac{\pi}{2}-\alpha}{\pi}$, $1 - a_n = 2\frac{\alpha}{\pi}$, $r = 2\frac{\frac{\pi}{2}-\alpha}{\pi}$. Then by Lemma 3 we have $q = 3, p = 0$.

For the triangles that have same vertices inside of regular n-gon or that have same vertices on the side of n-gon or on the side of a triangle we denote p, q, r as in previous case. Then by Lemma 4 $p \leq q$.

Hence the number of bigger acute angles is greater than the number of smaller acute angles. Hence our tiling is not realizable.

□

[V] I.Vasenov, Tiling of regular polygon with similar right triangles, https://arxiv.org/abs/2010.05052

[L20] M.Laczkovich, Irregular tilings of regular polygons with similar triangles, https://arxiv.org/abs/2002.12013