Tiling of regular polygon with similar right triangles, II

Vigdorchik Leonid*

A tiling is a decomposition of a polygon into finitely many non-overlapping triangles. (Note that a tiling is not a triangulation, i.e. a vertex of a triangle can lie on a side of a triangle.)

Theorem 1. *If a regular n-gon, $n > 8$, $n \neq 12, 14, 20, 32, 44$, can be tiled with similar right triangles, then no angle of this triangle equals to $\frac{n-2}{3n}\pi$.*

Theorem 1 follows from [L21, Theorem 1]. I show alternative proof for part of cases from [L21, Theorem 1].

Lemma 2. *If $3a_n = p(1 - a_n) + qa_n + r$, $n > 8$, $n \neq 14, 20, 32$, and p, q, r are non-negative integers, then $q = 3$.*

Lemma 3. *If $b = 1, 2, 3, 4$ and $b = p(1 - a_n) + qa_n + r$ where $n > 8$, $n \neq 12, 14, 20, 32, 44$ and p, q, r are non-negative integers, then $q \geq p$.*

Lemma 2 and Lemma 3 will be proved after illustration of the proof of a particular case in the following corollary.

Corollary 4. *If a regular 12-gon can be tiled with similar right triangles, then no angle of this triangle equals to $\frac{5}{18}\pi$.*

Proof. A vertex of 12-gon is equal to $\frac{5}{6}\pi$. By p, q, r we denote the number of smaller acute angles $\frac{4}{18}\pi$, bigger acute angles $\frac{5}{18}\pi$ and angles $\frac{\pi}{2}$ respectively at this vertex. Then $\frac{5}{6}\pi = p\frac{4}{18}\pi + q\frac{5}{18}\pi + r\frac{\pi}{2}$. We divide this equality by $\frac{\pi}{2}$ and get $\frac{5}{3} = \frac{4}{9}p + \frac{5}{9}q + r$. Then by Lemma 2 $q = 3$. Hence at each vertex of the polygon the number of bigger acute angles is greater than the number of smaller acute angles.

For the triangles that have same vertices inside of our polygon or that have same vertices on the side of polygon or on the side of a triangle sum of the angles at this vertex is $\frac{\pi}{2}$ or π or $\frac{3\pi}{2}$ or 2π. Then for $b = 1, 2, 3, 4$: $\frac{\pi}{2}b = p\frac{4}{18}\pi + q\frac{5}{18}\pi + r\frac{\pi}{2}$. We divide this equality by $\frac{\pi}{2}$ as in previous case and get $b = \frac{4}{9}p + \frac{5}{9}q + r$. Then by Lemma 3 $q \geq p$. Hence also inside the polygon the number of bigger acute angles is greater than the number of smaller acute angles.

Therefore the number of smaller and bigger acute angles of similar right triangles are not equal. Hence our tiling is not realizable. □

*This work was prepared in frame of math circle "Math and Olympiades".

We denote $a_n = 2\frac{n-2}{3n}$ for integer n. Then $1 - a_n = \frac{n+4}{3n}$.

For all n: $a_n < \frac{2}{3}$ and $1 - a_n > \frac{1}{3}$.

Proof of Lemma 2. Since $\frac{r}{3} \leq a_n = 2\frac{n-2}{3n} < \frac{2}{3}$, it follows that $r < 2$. Since $qa_n \leq 3a_n$, it follows that $q \leq 3$. We also have $p = \frac{a_n(3-q)-r}{1-a_n}, 1-a_n > \frac{1}{3}, a_n < \frac{2}{3}$ therefore $p < \frac{\frac{2}{3}(3-q)-r}{\frac{1}{3}} = 2(3-q) - 3r$. We consider all possible values r, q and show that either $q = 3$ or the values of n that correspond to the integer values of p that do not suit to limitations of the lemma:

1. Let $r = 0$.

1.1 Let $q = 0$. This is only possible if $0 \leq p < 6$. Then $n \in \{2, 5, 8, 14, 32\}$.

1.2 Let $q = 1$. This is only possible if $0 \leq p < 4$. Then $n \in \{2, 4, 8, 20\}$.

1.3 Let $q = 2$. This is only possible if $0 \leq p < 2$. Then $n \in \{2, 8\}$.

1.4 Let $q = 3$. Then we get the statement of the lemma.

2. Let $r = 1$. Since $a_n < \frac{2}{3}$, it follows that $q \leq \frac{3a_n - 1}{a_n} < 2$.

2.1 Let $q = 0$. This is only possible if $0 \leq p < 3$. Then $n \in \{4, 8, 20\}$.

2.2 Let $q = 1$. This is only possible if $0 \leq p < 1$. Then $n = 8$. $\square$

Proof of Lemma 3. We consider four cases: $b = 1, 2, 3, 4$ separately.

1. If $r = 1$ then $p, q = 0$. If $r = 0$ we receive $1 = p(1 - a_n) + qa_n + r$. Then $q = 0$ or 1 because $a_n \geq \frac{1}{2}$. Suppose that $q = 0$. Then $p = \frac{1}{1-a_n}$. But $\frac{1}{3} < 1 - a_n < \frac{1}{2}$. Hence p is not non-negative integer. Contradiction. Hence $q = 1$. Then $p = q = 1$.

2. If $r > 0$ we have the previous case. If $2 = p(1 - a_n) + qa_n + r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{2}{1-a_n} = \frac{6n}{n+4}$. Then $0 < \frac{6n}{n+4} < 7$ is non-negative integer. Hence $n = 20$. Hence $p, q = 2$.

3. If $r > 0$ we have one of the previous cases. If $3 = p(1-a_n)+qa_n+r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{3}{1-a_n} = \frac{9n}{n+4}$. Then $0 < \frac{9n}{n+4} < 10$ is non-negative integer. Hence $n \in \{14, 32\}$. Hence $p, q = 3$.

4. If $r > 0$ we have one of the previous cases. If $4 = p(1-a_n)+qa_n+r$, then if $q \geq 1$ we have the previous case. Hence $p = \frac{4}{1-a_n} = \frac{12n}{n+4}$. Then $0 < \frac{12n}{n+4} < 13$ is non-negative integer. Hence $n \in \{12, 20, 44\}$. Hence $p, q = 4$. $\square$

Proof of Theorem 1. Assume that a regular n-gon, $n > 8$, $n \neq 12, 14, 20, 32, 44$, is tiled with similar right triangles of angles $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$, $\frac{\pi}{2} - \alpha$ and $\frac{\pi}{2}$, and $\alpha = \frac{n+4}{6n}\pi$.

Take a vertex of the n-gon. By p, q, r we denote the number of smaller acute angles α, bigger acute angles $\frac{\pi}{2} - \alpha$ and angles $\frac{\pi}{2}$ respectively at this vertex. We split each angle including angle at the vertex of n-gon by $\frac{\pi}{2}$. Then $3a_n = p(1 - a_n) + qa_n + r$, where $\frac{1}{2} < 3a_n < \frac{2}{3}$. So $a_n = 2\frac{\frac{\pi}{2}-\alpha}{\pi}$, $1 - a_n = 2\frac{\alpha}{\pi}$, $r = 2\frac{\frac{\pi}{2}-\alpha}{\pi}$. Then by Lemma 2 we have $q = 3, p = 0$.

For the triangles that have same vertices inside of regular n-gon or that have same vertices on the side of n-gon or on the side of a triangle we denote p, q, r as in previous case. Then by Lemma 3 $p \leq q$.

Hence the number of bigger acute angles is greater than the number of smaller acute angles. Hence our tiling is not realizable.

$\square$

[L21] M.Laczkovich and I.Vasenov, Tiling of regular polygons with similar right triangles, https://arxiv.org/abs/2109.07817