A generalization of lemma on the Geometry of Triangle

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Abstract

In [1] N. I. Beluhov gave an elementary proof of well-known Lester's theorem, using as lemma an interesting triangle configuration. Here we present it's generalization and two different proofs. The first one uses Desargues Involution Theorem, and the second one is mostly connected with properties of rectangular hyperbolas. The main motivation for exploration of such a connection is a proof of generalized Lester's theorem, proposed by Dao Thanh Oai in [3].

After reading of [1] author of this paper was interested in more thorough exploration of configuration, provided there as a lemma. We consequentially introduce the statement of original fact, formulate our main result in Section 1, establish a necessary theoretics of projective involutions in Section 2 and give a proof in Section 3. Section 4 is devoted to discussion of some other issues on the configuration and it's relation with conics.

Theorem [1]. In $\triangle APQ$ ($AP \neq AQ$) point B is reflection of P over AQ and C is reflection of Q over AP. Tangent to $\odot(ABC)$ at A intersect PQ at U. Then reflection T of U over A lies on BC.

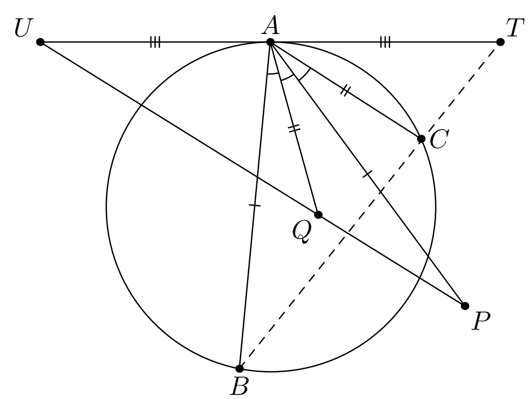

Figure 1: Original statement

The proof given in specified article was based on using of similarity and properties of cyclic quadrilaterals. Here we provide it's generalization with some new approaches.

1 Statement of main result

Definition 1.1 Division ratio $(XY;Z)$ of collinear points X,Y,Z is a real number r satisfying $r\overrightarrow{ZY}=\overrightarrow{ZX}$.

Main theorem. Given $\triangle ABC$ and points P,Q such that $ABP \stackrel{-}{\sim} ACQ$ (triangles are similar with opposite orientation). Denote by $\mathcal{H}$ homothety with center A and ratio $k=(CQ;AP\cap CQ)$. The tangent to $\odot(ABC)$ at A intersect PQ at U. Then point $T=\mathcal{H}(U)$ lies on BC.

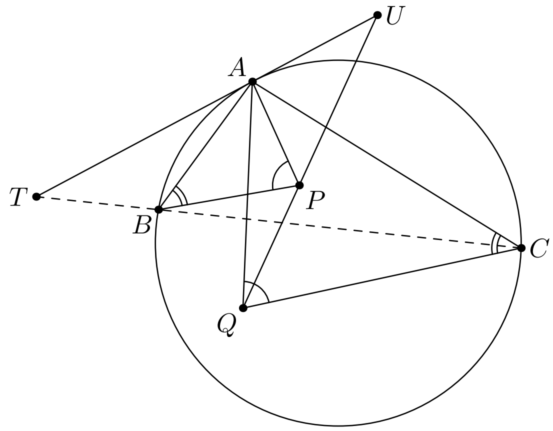

Figure 2: Generalization

We present two proofs for this claim. For the first proof we need to introduce some theoretical basis of projective geometry as follows.

2 Properties of projective involutions

Definition 2.1 Projective involution by line ℓ is a homography $f=f^{-1}:\ell\mapsto\ell$.

Definition 2.2 Pencil of conics is a set of all conic sections passing through a quad of fixed points A,B,C,D, including degenerate conics $\overline{AB}\cup\overline{CD}$, $\overline{AC}\cup\overline{BD}$, $\overline{AD}\cup\overline{BC}$.

Lemma 2.1 There exist unique involution by ℓ with two reciprocal pairs $(P,Q),(R,S)$.
Proof. The required involution is either central symmetry, if circles with diameters PQ and RS are concentric, or inversion with nonzero power with respect to intersection of ℓ and radical axis of circles. Uniqueness follows from the fact, that homography of line is uniquely defined by images of three points □

Desargues Involution Theorem. Line ℓ doesn't pass through points A,B,C,D. Then there exist an involution by ℓ, which swaps it's meet-points with each conic of pencil $\mathcal{L}$ through A,B,C,D.

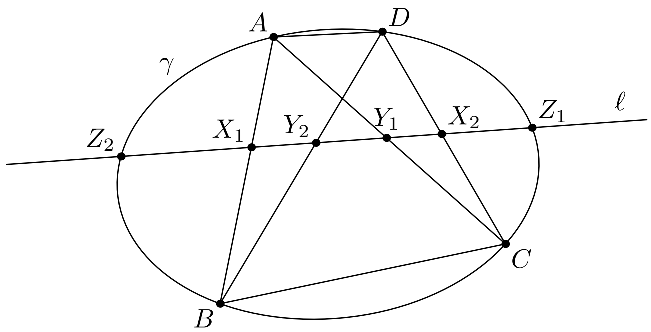

Figure 3: Desargues Involution Theorem

Proof. Consider the labeling accorded to the Figure 3, where γ denotes the arbitrary conic of $\mathcal{L}$. Denote by ψ projective involution on ℓ with reciprocal pairs $(X_1, X_2), (Z_1, Z_2)$. Projecting conic onto line and using well-known properties of cross-ratios we obtain that

$$(X_2\psi(Y_1)Z_1Z_2) \stackrel{\psi}{=} (X_1Y_1Z_2Z_1) \stackrel{A}{=} (BCZ_2Z_1)_\gamma \stackrel{D}{=}$$

$$\stackrel{D}{=} (Y_2X_2Z_2Z_1) = (X_2Y_2Z_1Z_2) \implies \psi(Y_1) = Y_2.$$

As ψ swaps in pairs $(X_1, X_2), (Y_1, Y_2)$ it doesn't depend on Z_1. The result follows $\square$
Remark. Analogously $\psi(\ell \cap AD) = \ell \cap BC$, so result holds for degenerate conics.

3 Proof of main result

First proof of main theorem. Throughout the proof all angles are supposed to be oriented. As ∞_ℓ we denote infinite point of line ℓ. Notice that $\measuredangle PAQ = -\measuredangle QAP$ yields $ABP \cup \overline{AQ} \bar{\sim} ACQ \cup \overline{AP}$ and in particular $(BP; AQ \cap BP) = k$, $\measuredangle BAX = -\measuredangle CAY$. Let $X = \mathcal{H}(P), Y = \mathcal{H}(Q)$, so $T \in XY$. By Thales theorem $AP \parallel CY$, $AQ \parallel BX$.

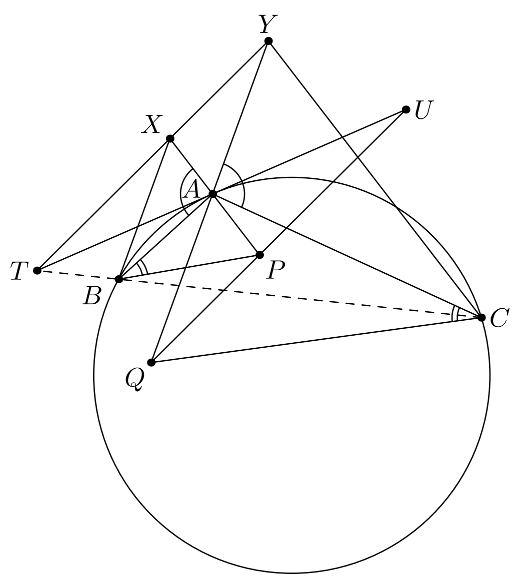

Figure 4: Main theorem

By Lemma 2.1 there exist unique involution i by BC with reciprocal pairs (B, C) and $(AX \cap BC, AY \cap BC)$, thus this involution coincide with reflection over bisector of angle

BAC followed by projection from A onto line BC. It's known that lines $\overline{AT}, \overline{A\infty_{BC}}$ are isogonals with respect to angle BAC, so by Desargues Involution Theorem on $X\infty_{AX}Y\infty_{AY}$ it follows that $AT \cap BC = i(\infty_{BC}) = BC \cap XY$. This finally yields the concurrency of lines AT, XY, BC and the collinearity of T, B, C ■

Corollary 3.1 Points $B, C, X, Y, \infty_{AX}, \infty_{AY}$ lie on one hyperbola.
Proof. Due to the Desargues Invloution Theorem hyperbola passing through points $B, X, Y, \infty_{AX}, \infty_{AY}$ meet BC again at point $i(B) = C$ □

4 Other issues on configuration

Here we discuss some facts from geometry of triangle, properties of conics (see [2]), and it's relation with presented construction. As a result, we give another proof of main theorem. Firstly, redefine T as a common point of BC and tangent to $\odot(ABC)$ at A.

Proposition 4.1 Circumscribed rectangular hyperbola γ of $\triangle ABC$ with pair of antipodal points B, C intersect infinite line by infinite points of both internal and external bisectors of angle BAC.
Proof. Let γ intersect infinite line at R, S. Since pole of BC with respect to γ lies on infinite line, it follows that $(AB, AC, AR, AS) = (BCRS)_\gamma = -1$. But from γ is a rectangular hyperbola it follows that $AR \perp AS$, which yields the conclusion □

For next results we will prove the following property of rectangular hyperbolas.
Lemma 4.1 [2]. Hyperbola γ is the locus of points D satisfying $\measuredangle ABD = -\measuredangle ACD$.
Proof. Notice that by Proposition 4.1 $D \in \gamma$ is equivalent to

$$\measuredangle RBD = -\measuredangle RCD \iff$$

$$\iff \measuredangle ABD = \measuredangle ABR + \measuredangle RBD = -\measuredangle ACR - \measuredangle RCD = -\measuredangle ACD \ \square$$

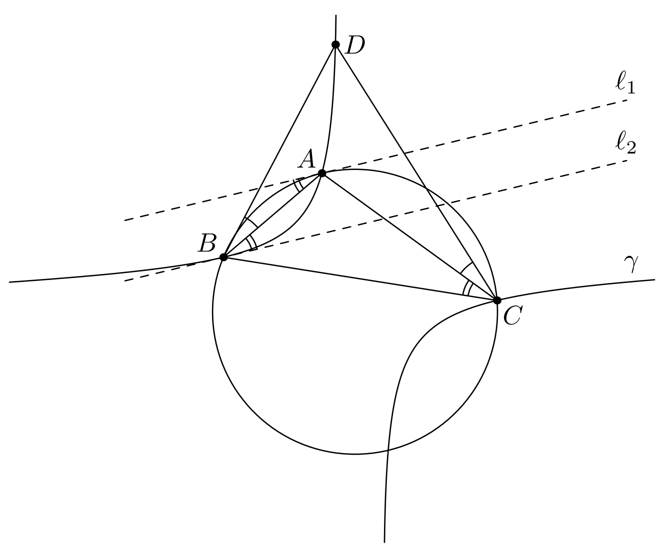

Figure 5: The tangent to the circumscribed rectangular hyperbola

Proposition 4.2 The image V of A under reflection over T lies on γ.
Proof. It's suffice to prove that tangent ℓ_1 to $\odot(ABC)$ at A and tangent ℓ_2 to γ at

B respectively are parallel. Indeed, taking the limiting case $D \to B$ of Lemma 4.1 we obtain $\angle(\ell_1, AB) = \angle(BC, AC) = \angle(\ell_2, AB)$ (see Figure 5) $\square$

Due to the Lemma 4.1 point $Z = BX \cap CY$ lies on γ. Let γ meets AP, AQ again at points P', Q' respectively.

Proposition 4.3 Points P', Q' are antipodal on γ.
Proof. Observe that central symmetry with respect to center of γ induces projective involution by hyperbola, which preserves points R, S. Therefore it coincides with reflection over AR, followed by projection from A onto γ, and so swaps P', Q' $\square$

Corollary 4.1 The $A-$symmedian of $\triangle ABC$ is tangent to γ.
Proof. Consider the limiting case $Q' \to A$ of Proposition 4.3. Then $ABP'C$ is a parallelogram, and tangent to γ at A is the isogonal of AP' with respect to angle BAC $\square$

Second proof of main theorem. Construct point $A \in \gamma$ such that $AA' \parallel XY$. Obviously $AXZY$ is a parallelogram, so XY bisects segment AZ. By Pascal theorem on hexagon $AP'CZBQ'$ and Proposition 4.3 lines BQ', CP', XY are parallel, so $(BCAV)_\gamma = -1 = (Q'P'A'Z)_\gamma$ yields $VZ \parallel XY$. Therefore XY bisects AV $\blacksquare$

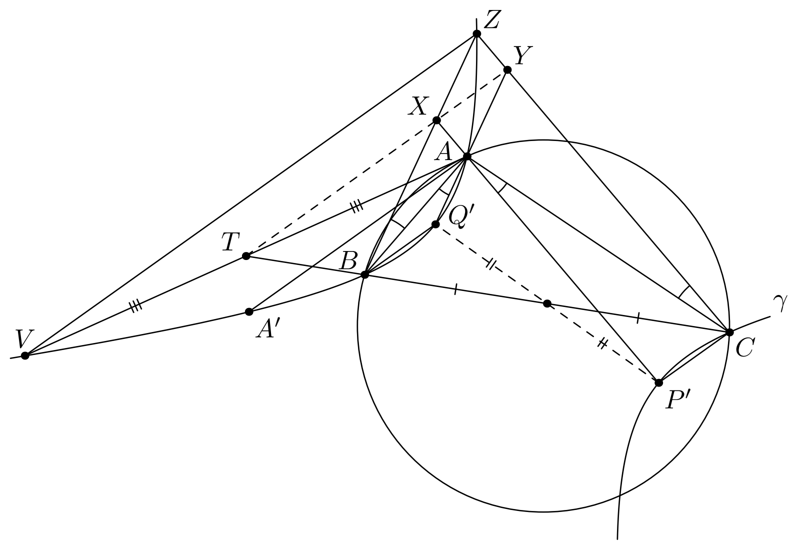

Figure 6: Configuration with rectangular hyperbola

References

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[3] Dao Thanh Oai. A Simple Proof of Gibert's Generalization of the Lester Circle. Forum Geometricorum, vol. 14, 123–125, 2014.

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