Algebraic Topology From a Geometric Standpoint

A. Skopenkov

Abstract.

It is shown how main ideas, notions and methods of algebraic topology naturally appear in a solution of geometric problems. The main ideas are exposed in simple particular cases free of technical details. We keep algebraic language to a necessary minimum. So most of the book is accessible to beginners and non-specialists, although it contains beautiful non-trivial results. Part of the material is exposed as a sequence of problems, for which hints are provided. The book is intended for students, researchers, and teachers, who wish to know

• why what I learn or teach is interesting and useful?

• how the main idea of a result / proof / theory is exposed in simple terms?

• how is this idea elaborated to produce the result / proof / theory?

Here students could be undergraduate or postgraduate; with majors in mathematics, computer science or physics. All this would hopefully allow them to make their own useful discoveries (not necessarily in mathematics).

Thus the book is different from other textbooks on algebraic topology.

We start from important visual objects of mathematics: graphs and vector fields on surfaces, continuous maps and their deformations. In §§1,2,5 basic theory of graphs on surfaces is exposed in a simplified way. In later sections I carry such a 'non-specialist', or 'user' or 'computer science' approach to topology pretty far. The appearing instruments include homology groups, obstructions and invariants, characteristic classes.

The book is based on decades of teaching topology courses in leading mathematical centers of Moscow (Moscow State University, Independent University of Moscow, Moscow Institute of Physics and Technology).

General information.

This is an updated English translation of a book published in Russian in 2015, 2020 by MCCME, Moscow. This publicly available part of the preprint is for personal or private reading only. It comprises contents and most of sections 1, 2, 5, 6 (observe that sections 5 and 6 are essentially independent of sections 3 and 4).

Translated by I. Alexeev (§14), A. Balitskiy (§§5,6,12,13), M. Fedorov (§§10,11), D. Mamaev (§15), A. Nordskova (§16), A. Pratoussevitch (§§7,8,9), and N. Tsilevich (§§1,2); translation is edited by the author.

Early Russian versions of this book were available since 2008 at `https://arxiv.org/pdf/0808.1395.pdf` and `http://www.mccme.ru/circles/oim/obstruct.pdf` .

The translation is accepted for publication by 'Moscow Lecture Notes' of Springer in January, 2021. The translation was essentially rejected by Springer by sending an unacceptable publishing agreement, promising to make amends suggested by the author in May, 2021, and neither making amends nor informing the author that the amends are not accepted, by January, 2022 (in spite of the author's monthly reminders). Thus no contract was signed, and this book is no longer submitted to Springer. See some details in the last pages of this file; *they could be useful for other authors considering to publish with Springer.*

Updates will be presented here.

A. Skopenkov is supported in part by Russian Foundation for Basic Research Grant No. 19-01-00169.

Contents

§ 1. Graphs in the Plane

> Dass von diesem schwer lesbaren Buche noch vor Vollendung des ersten Jahrzehntes eine zweite Auflage notwendig geworden ist, verdanke ich nicht dem Interesse der Fachkreise...
>
> S. Freud. *Die Traumdeutung*, Vorwort zur zweiten Auflage[3]

1.1. Introduction and Main Results

In § 1.3 we prove basic results on graphs and map colorings in the plane, Assertions 1.1.1 and 1.3.2.

1.1.1. (a) A triangle is divided into finitely many convex polygons. They can be colored in six colors in such a way that any two polygons sharing a common boundary segment receive different colors.

(b)* The same for five colors.

(The famous Four Color Conjecture claims that four colors are enough, but its proof is much more involved.)

A graph is said to be *planar* (or *embeddable in the plane*) if it can be drawn in the plane without edges crossing. The basic notions of graph theory are recalled in § 1.2; a more rigorous definition of planarity is given in § 1.3.

Embeddability of graphs (or graphs with an additional structure) in the plane, torus, Möbius strip, and other surfaces (see § 2) is one of the main problems in topological graph theory [MT01].

Proposition 1.1.2. *There is an algorithm for deciding whether a graph is planar.* (See [Sk, footnote 4], [Sk18, footnote 7].)

One of the simplest (but slow) algorithms is constructed in §§ 1.5 and 1.6 (Assertion 1.1.2 follows from Assertions 1.6.1 (f) and 1.6.3 (a)). It is based on an important construction of *thickening*, which arises in many problems of topology and its applications (synonyms: graph with

---

[3]If within ten years of the publication of this book (which is very far from being an easy one to read) a second edition is called for, this is not due to the interest taken in it by the professional circles... (S. Freud. *The Interpretation of Dreams*. Preface to the second edition.)

rotations, dessin [Ha, LZ, MT01]). The algorithm uses no nontrivial results (such as Kuratowski's theorem or Fáry's theorem; for the statements, as well as for a polynomial-time algorithm, see [Sk, § 1.2 'Algorithmic results on graph planarity']).

The proofs of these results illustrate applications of Euler's Formula 1.3.3 (c). (So, they are better postponed until the reader becomes familiar with it.) This formula is proved in § 1.4, where we also explain, in the language of algorithms, the nontriviality of this result ignored in some expositions.

1.2. Glossary of Graph Theory

The reader is probably familiar with the notions introduced below, but we give clear-cut definitions in order to fix the terminology (which can be different in other books).

A *graph* $G = (V, E)$ is a finite set $V = V(G)$ together with a set $E = E(G)$ of two-element subsets (i.e., unordered pairs of distinct elements). (A more precise term for the notion we have introduced is *graph without loops or multiple edges*, or *simple graph*.) Elements of the set V are called *vertices*, elements of the set E are called *edges*. Although edges are unordered pairs, in graph theory they are traditionally denoted by parentheses. Given an edge (a, b), the vertices a and b are called its *endpoints*, or *vertices*.

When working with graphs, it is convenient to use their drawings, e.g., in the plane or in the space (or, in more technical terms, maps of their geometric realizations to the plane or to the space, cf. §5.1). See Figs. 1.3.1, 1.3.2, 1.7.2 below. Vertices are represented by points. Every edge is represented by a polygonal line joining its endpoints. (But only the endpoints of polygonal lines represent vertices of the graph.) The polygonal lines are allowed to intersect, but their intersection points (other than the common endpoints) are not vertices. Importantly, a graph and a drawing of this graph are not the same. For example, Figs. 1.3.2 (middle and right), 1.3.1 show different drawings of the same graph (more exactly, of isomorphic graphs). Sometimes, not all vertices are shown in a drawing, see Figs. 1.2.1 and 1.6.2 (left).

The *path* P_n is the graph with vertices $1, 2, \ldots, n$ and edges $(i, i + 1)$, $i = 1, 2, \ldots, n - 1$. The *cycle* C_n is the graph with vertices

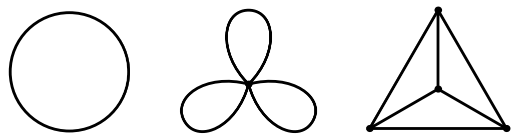


Figure 1.2.1. A cycle, a wedge of cycles, and the graph K_4

$1, 2, \ldots, n$ and edges $(1, n)$ and $(i, i+1)$, $i = 1, 2, \ldots, n-1$. (Do not confuse these graphs with a *path in a graph* and a *cycle in a graph*, which are defined below.)

The graph with n vertices any two of which are joined by an edge is called a *complete* graph and denoted by K_n. If the vertices of a graph can be partitioned into two sets so that no edge joins two vertices from the same set, then the graph is said to be *bipartite*, and the two sets of vertices are called its *parts*. By $K_{m,n}$ one denotes the bipartite graph with parts of size m and n that contains all the mn edges joining vertices from different parts. See Fig. 1.3.2.

Roughly speaking, a *subgraph* of a given graph is a part of this graph. Formally, a graph G is called a *subgraph* of a graph H if every vertex of G is a vertex of H and every edge of G is an edge of H. Note that two vertices of G joined by an edge in H are not necessarily joined by an edge in G.

A *path*[4] in a graph is a sequence $v_1e_1v_2e_2\ldots e_{n-1}v_n$ such that for every i the edge e_i joins the vertices v_i and v_{i+1}. (The edges $e_1, e_2, \ldots, e_{n-1}$ are not necessarily pairwise distinct.) A *cycle* is a sequence $v_1e_1v_2e_2\ldots e_{n-1}v_ne_n$ such that for every $i < n$ the edge e_i joins the vertices v_i and v_{i+1}, while the edge e_n joins the vertices v_n and v_1.

A graph is said to be *connected* if every pair of its vertices can be joined by a path, and *disconnected* otherwise. A graph is called a *tree* if it is connected and contains no simple cycles (i.e., cycles that do not pass twice through the same vertex). A *spanning tree* of a graph G is any subgraph of G that is a tree and contains all vertices of G. Clearly, every connected graph contains such a subgraph.

The definition of the operations of deleting an edge and deleting a vertex is clear from Fig. 1.2.2. The operation of *contracting an edge* (Fig. 1.2.2) deletes this edge from the graph, replaces its endpoints A and B with a vertex D, and replaces each edge from A or B to

[4]In graph theory, as opposed to topology, the term 'walk' is used.

a vertex C with an edge from D to C. (In contrast to the case of contracting an edge in a multigraph, each resulting edge of multiplicity greater than 1 is replaced with an edge of multiplicity 1.) For example, if the graph is a cycle with four vertices, then contracting any its edge results in a cycle with three vertices.

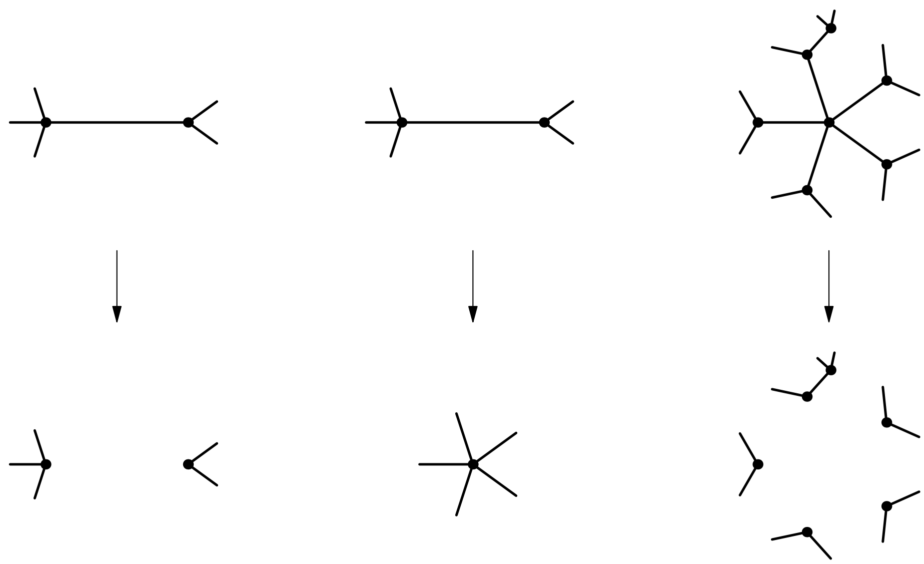


Figure 1.2.2. Deleting an edge $G - e$, contracting an edge G/e, and deleting a vertex $G - x$

In most of this book, one can use the notion of graph without loops or multiple edges. However, everything we have said is valid for the following generalization, which is even indispensable in some cases. A *multigraph* (or a *graph with loops and multiple edges*) is a square array (matrix) of nonnegative integers symmetric with respect to the main diagonal. The integer at the intersection of the ith row and jth column is interpreted as the number of edges (or the *multiplicity of the edge*) between the vertices i and j if $i \neq j$, and as the number of loops at the vertex i if $i = j$. An edge is said to be *multiple* if its multiplicity is greater than 1.

1.3. Graphs and Map Colorings in the Plane

A **plane graph** is a finite collection of non-self-intersecting polygonal lines in the plane such that any two of them meet only at their common endpoints (in particular, those with no common endpoints are disjoint).

The endpoints of the polygonal lines are called the *vertices* of the plane graph, and the polygonal lines themselves are its *edges*. Thus, to a plane graph there corresponds a graph (in the sense of § 1.2) for which the plane graph is a *plane drawing*. Sometimes, a plane graph is called just a graph, but this is not exactly correct, because one and the same graph can be drawn in the plane in different ways (if it can be drawn at all), see Fig. 1.3.1.

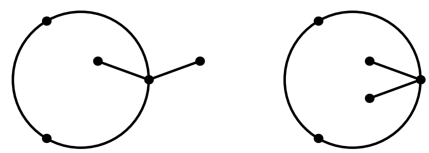


Figure 1.3.1. Different plane drawings of a graph

A graph is said to be **planar** if it can be represented by a plane graph.

1.3.1. The following graphs are planar:
(a) the graph K_5 without one edge (Fig. 1.7.2); (b) any tree;
(c) the graph of any convex polyhedron.

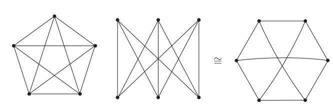


Figure 1.3.2. The nonplanar graphs K_5 and $K_{3,3}$

1.3.2. (a) The graph K_5 is not planar. (b) The graph $K_{3,3}$ is not planar.

(c) For every plane connected graph with V vertices and $E > 1$ edges, $E \leqslant 3V - 6$.

(d) Every plane graph contains a vertex with at most 5 incident edges.

A plane graph divides the plane into regions called its *faces*. Here is a rigorous definition.

A subset of the plane is said to be **connected** if any two its points can be joined by a polygonal line inside this set. (Caution: for subsets more general than those we consider here, the definition of connectedness is different!)

A **face** of a plane graph G is any of the connected parts into which the plane $\mathbb{R}^2$ is divided by the cuts along all the polygonal lines (= edges) of G, i.e., any maximal connected subset of $\mathbb{R}^2 - G$. Note that one of these parts is 'infinite'.

1.3.3. (a) Draw a plane graph G that has a face whose boundary contains three pairwise disjoint cycles.

(b) For every plane graph with $E > 1$ edges and F faces, $3F \leqslant 2E$.

(c)* **Euler's Formula.** *For every connected plane graph with V vertices, E edges, and F faces, $V - E + F = 2$.*

(d) Find a version of Euler's Formula for a plane graph with s connected components.

As to part (b), think about how many faces an edge belongs to and what is the smallest number of edges bounding a face.

The proof of Euler's Formula is given below. First, using this formula without proof, solve Problems 1.1.1 and 1.3.2.

1.4. Rigorous Proof of Euler's Formula

1.4.1. (a) We are given a non-closed non-self-intersecting polygonal line L in the plane and two points outside it. There is an algorithm for constructing a polygonal line that joins these points and does not intersect L.

(b) The same for a tree L in the plane whose edges are segments.

(c) If two segments are disjoint, then the distance between them is positive.

Hint. To construct the algorithms, use induction (or recursion). The induction step is based on deleting a pendant vertex. Cf. the construction of the regular neighborhood of a tree, see Fig. 1.6.3 (left) and the definition near this figure, [BE82, § 6], [CR, pp. 293—294]. Part (c) can be proved by looking at the possible relative positions of the segments.

The nontriviality of the algorithms from Problems 1.4.1 illustrates the nontriviality of the following assertions. (A similar remark applies to Assertion 1.4.3 (a) and Jordan's Theorem 1.4.3 (b).)

1.4.2. (a) Any non-closed non-self-intersecting polygonal line L in the plane $\mathbb{R}^2$ does not separate the plane, i.e., $\mathbb{R}^2 - L$ is connected.

(b) No tree in the plane separates the plane.

(c) Deleting an edge in a plane graph decreases the number of faces at most by 1.

(d) For any connected plane graph with V vertices, E edges, and F faces, $V - E + F \leqslant 2$.

Hint. Use the ideas from the solution of Problem 1.4.1.

1.4.3. (a) There is an algorithm that, given a closed non-self-intersecting polygonal line L in the plane and two points outside L, decides whether these points can be joined by a polygonal line that does not intersect L.

(The same is true even if a part of the given polygonal line outside some square containing the given points is deleted.)

(b) **Jordan's Theorem.** *Any closed non-self-intersecting polygonal line L in the plane $\mathbb{R}^2$ divides the plane into exactly two connected parts, i.e., $\mathbb{R}^2 - L$ is disconnected and is a union of two connected sets.*

Usually, by Jordan's Theorem one means a version of Theorem 1.4.3 (b) for *continuous curves* L, whose proof is much more involved [An03, Ch99]. While Theorem 1.4.3 (b) is sometimes called the *Piecewise Linear Jordan Theorem.*

A simple proof of Jordan's Theorem 1.4.3 (b) is given in [CR, pp. 292—295], see Remark 1.4.8. We present a similar, but slightly more complicated, proof. In return, it involves an interesting Intersection Lemma 1.4.4 and demonstrates the parity and general position techniques (Lemmas 1.4.5 and 1.4.6) useful for what follows.

Sketch of the proof of Jordan's Theorem 1.4.3 (b). The claim that the number of parts is at most 2 is simpler; it follows from Assertions 1.4.2 (b, c). Cf. [BE82, § 6], [CR, pp. 293—294].

The claim that the number of parts is greater than 1 is more difficult. To prove it, pick two points that are sufficiently close to a segment of the polygonal line L and symmetric with respect to this segment. Then

($*$) *it is these points that cannot be joined by a polygonal line that does not intersect L.*

This is implied by the following Intersection Lemma 1.4.4. □

Lemma 1.4.4 (intersection)**.** *Any two polygonal lines in a square joining different pairs of opposite vertices must intersect.*

The Intersection Lemma can be deduced from the following Parity Lemma 1.4.5 and Approximation Lemma 1.4.6 (a, b).

Several points in the plane are said to be **in general position** if no three of them lie on the same line and no three segments between them share a common interior point.

Lemma 1.4.5 (parity)**.** *If the vertices of two closed plane polygonal lines are in general position, then the polygonal lines meet in an even number of points.*

Cf. the comments and proof in [Sk, § 1.3 'The intersection number for polygonal lines in the plane'].

A polygonal line $A_0 \dots A_n$ is said to be *vertex-wise ε-close* to a polygonal line $B_0 \dots B_m$ if $m = n$ and $|A_i - B_i| < \varepsilon$ for every $i = 0, 1, \dots, n$.

Lemma 1.4.6 (approximation)**.** (a) *For every $\varepsilon > 0$ and any polygonal lines L_1, L_2 in a square joining different pairs of opposite vertices there exist polygonal lines L_1', L_2' in the square joining different pairs of opposite vertices such that L_1', L_2' are vertex-wise ε-close to L_1, L_2 and the vertices of L_1', L_2' are in general position.*

(b') *For every pair of disjoint segments XY and ZT there is $\alpha > 0$ such that for any points X', Y', Z', T' in the plane, the inequalities $|X - X'|, |Y - Y'|, |Z - Z'|, |T - T'| < \alpha$ imply that the segments $X'Y'$ and $Z'T'$ are disjoint.*

(b) *If two polygonal lines L_1, L_2 do not intersect, then there exists $\varepsilon > 0$ such that any polygonal lines L_1', L_2' that are vertex-wise ε-close to L_1, L_2 do not intersect either.*

Sketch of the proof of Euler's Formula 1.3.3 (c)*.* Induction on the number of edges outside a spanning tree. The induction base is Assertion 1.4.2 (b). The induction step follows from the fact that

($**$) *if deleting an edge from a plane graph results in a connected graph, then the number of faces decreases at least by* 1.

This can be proved analogously to the difficult part of Jordan's Theorem 1.4.3 (b) using the Intersection Lemma 1.4.4. □

The Intersection Lemma 1.4.4 is also useful for other results. It is often (e.g. in the following problem) more convenient to apply it instead of Jordan's Theorem 1.4.3 (b).

1.4.7. (a) Two bikers start at the same point moving northward and eastward, respectively. Both return (for the first time) to the initial point from south and west, respectively.

(b) Three bikers start at the same point moving westward, northward, and eastward, respectively. All of them arrive at another point from west, north, and east, respectively.

(a, b) Show that one of the bikers has crossed the track of another one. (See the middle pictures at Figs. 1.5.2 and 1.6.2 (left); the starting point is not counted as an intersection point of tracks; you may assume that the paths of the bikers are polygonal lines.)

Remark 1.4.8. (a) (on the proof of Jordan's Theorem 1.4.3 (b)) Jordan's Theorem is the special case of Euler's Formula 1.3.3 (c) for a graph that is a cycle. So deducing Jordan's Theorem from Euler's Formula would create a vicious circle.

The idea of the proof of claim $(*)$ is given in [CR, pp. 293—294], though the claim itself (i.e., the fact that $B \neq \varnothing$) is neither stated nor proved there. The argument uses simplified versions of the Parity Lemma (in the fifth paragraph at p. 293). At the beginning of the argument, one must pick a direction that is not parallel to any line passing through two vertices of the polygon (including nonadjacent ones); otherwise, in the fifth paragraph at p. 293, there arise more than two cases, contrary to what is stated.

The proof of claim $(*)$ given in [BE82, §6] uses the Parity Lemma 1.4.5.

The proof of Jordan's Theorem in [Pr14′, pp. 19—20] is incomplete, because it uses without proof nontrivial facts similar to the Parity Lemma. More specifically, for the reader not familiar with Jordan's Theorem, the claim (given without proof) from the second proposition at p. 20 (as well as the fact from the first proposition at p. 20 that the parity changes continuously) seems to be more complicated than Jordan's Theorem itself, whose proof uses this claim.

(b) (on the proof of Euler's Formula 1.3.3 (c)) In a beginners' course, it is reasonable not to prove the above assertion $(**)$, which is geometrically obvious. One should only draw the reader's attention to the fact that this assertion is not proved, to algorithmic problems illustrating its nontriviality (cf. Problems 1.4.1 and 1.4.3 (a)), and to the remark about ‘vicious circle’ given in the solution of Problem 1.3.2 (a). Unfortunately, this assertion is not proved, and even not commented upon, in some expositions which claim to be rigorous[5]. This might give the wrong idea that the proof of Euler’s Theorem does not use results close to Jordan’s Theorem, and hence does not involve the corresponding difficulties.

1.5. Planarity of Disks with Ribbons

Consider a word of length $2n$ in which each of n letters occurs exactly twice. Take a convex polygon in the plane. Choose an orientation of the closed polygonal line that bounds it. Take $2n$ disjoint segments on this polygonal line corresponding to the letters of the word in the order they occur in it. For each letter, join (not necessarily in the plane) the two corresponding segments by a ribbon (i.e., a ‘stretched’ and ‘creased’ rectangle) so that different ribbons do not intersect each other. The **disk with ribbons** corresponding to the given word is the union of the constructed (two-dimensional) convex polygon and the ribbons[6].

A ribbon is said to be **twisted** if the arrows on the boundary of the polygon have the same direction ‘when translated’ along the ribbon, and **untwisted** if they have opposite directions (Fig. 1.5.1).

For example, the annulus and the cylinder (Fig. 2.1.2 and the text before it) are disks with one untwisted ribbon, while the disk with

[5]Here are two examples. In [Pr14′, proof of Theorem 1.6], it is not explained why “deleting one boundary edge decreases the number of faces by 1”; this fact is not simpler than Jordan’s Theorem 1.4.3 (b), whose proof [Pr14′, p. 19—20] is nontrivial for a beginner and contains the gap described at the end of Remark 1.4.8. The proof of Euler’s Formula in [Om18, Chapter 7, § 2] also includes neither explanations of a similar fact, no references to Jordan’s Theorem (though the nontriviality of this theorem is discussed earlier).

[6]More precisely, a disk with ribbons is *any* shape obtained by this construction; cf. the remark before Problem 2.2.2. Still more precisely, it is the pair consisting of this union and the union of loops corresponding to the ribbons. This terminological distinction is not relevant for the realizability we study here, but it is important for calculating the number of disks with ribbons, see § 1.7 and [Sk, ‘Orientability and classification of thickenings’].
This informal definition can be formalized using the notions of homeomorphism and gluing (§ 2.7 and Example 5.1.1.c); cf. § 1.7.

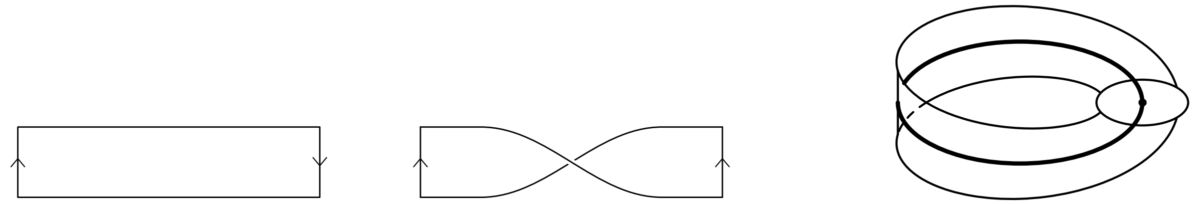


Figure 1.5.1. Left: arrows that have opposite directions ‘when translated’ along the ribbon. Right: a disk with a twisted ribbon (the Möbius strip)

n holes (Fig. 3.9.2) is a disk with n untwisted ribbons. For other examples of disks with untwisted ribbons, see Figs. 1.5.2 and 1.5.3.

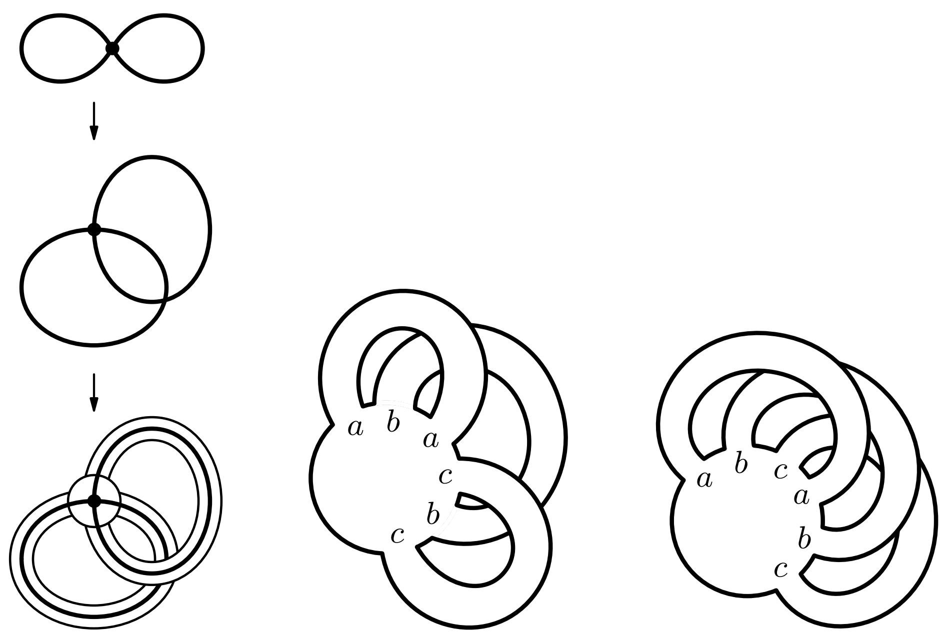


Figure 1.5.2. Left: the top picture shows a multigraph with one vertex and two loops, the middle one is a drawing of this multigraph in the plane, and the bottom one is the corresponding disk with untwisted ribbons; it corresponds to the word $(abab)$. Middle and right: the disks with three untwisted ribbons corresponding to the words $(abacbc)$ and $(abcabc)$.

Ribbons a and b in a disk with untwisted ribbons are said to **interlace** if the segments along which they are glued to the polygon alternate along its boundary, i.e., occur in the cyclic order $(abab)$, and not $(aabb)$.

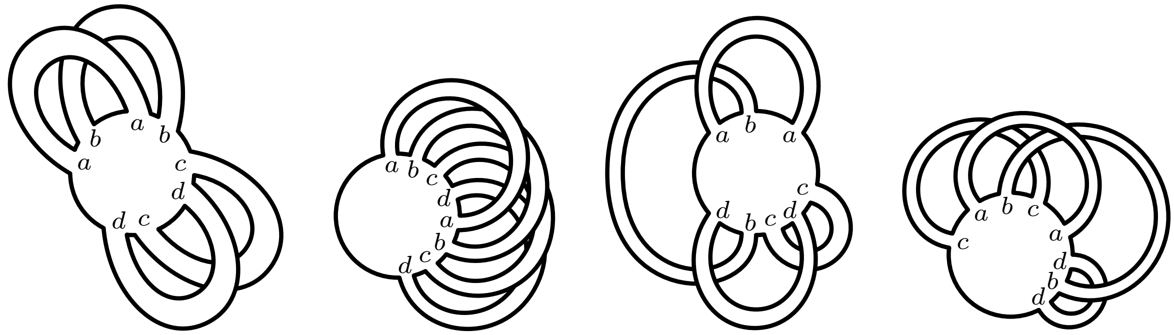


Figure 1.5.3. Disks with four untwisted ribbons (which cannot be realized on the torus)

Lemma 1.5.1. *A disk with untwisted ribbons can be cut out of the plane if and only if it has no interlacing ribbons.*

A **boundary circle** of a disk with ribbons is a connected part of the set of its points that it approaches 'from one side'. This informal definition is formalized in § 5.5. In Fig. 1.5.2 (middle and right), the boundary circles are shown in bold. For example, the disks with untwisted ribbons in Fig. 1.5.2 have one, two, and two boundary circles, respectively.

1.5.2. (a) How many boundary circles can a disk with two untwisted ribbons have (more precisely, find all F for which there exists a disk with two untwisted ribbons that has F boundary circles)?

(b) How many boundary circles do the disks with untwisted ribbons in Fig. 1.5.3 have?

(c) How many boundary circles can a disk with five untwisted ribbons have?

(d) Adding a non-twisted ribbon changes the number of boundary circles by ± 1.

1.5.3. (a) The number of boundary circles of a disk with n untwisted ribbons does not exceed $n+1$.

(a') The number of boundary circles of a disk with n ribbons, of which at least one is twisted, does not exceed n.

(b) *Lemma.* For a disk with n untwisted ribbons, each of the assumptions of Lemma 1.5.1 is equivalent to the number of boundary circles being equal to $n+1$.

(c) Given a word of length $2n$ in which each of n letters occurs exactly twice, construct a graph with the number of connected components equal to the number of boundary circles of the disk with untwisted ribbons corresponding to this word. (Thus, this number can be found by computer without drawing a figure.)

1.6. Planarity of Thickenings

Given a graph with n vertices, consider the union of n pairwise disjoint convex polygons in the plane. On each of the closed polygonal lines bounding the polygons take disjoint segments corresponding to the edges incident to the corresponding vertex. For each edge of the graph, join (not necessarily in the plane) the corresponding two segments by a ribbon so that the ribbons do not intersect each other (Fig. 1.6.1). A **thickening** of the graph is the union of the constructed convex polygons and ribbons. The graph is called the *spine*, or the *thinning*, of this union. A remark similar to that in footnote 6 at the beginning of § 1.5 applies to this case as well.

Figure 1.6.1. Joining disks with a ribbon

A thickening is said to be **orientable** if the boundary circles of the polygons can be endowed with orientations so that every ribbon becomes untwisted, i.e., the arrows on the boundaries of the polygons have the *opposite direction* 'when translated' along the ribbon (Fig. 1.5.1, left). Note that each of the pictures in Fig. 1.6.1 can correspond to such a way of joining disks with ribbons. A thickening is said to be **non-orientable** if there are no such orientations.

For example, orientable thickenings of the graphs $K_{3,2}$ and $K_{3,3}$ are shown in Fig. 1.6.2.

A disk with ribbons (§ 1.5) is a thickening of a multigraph consisting of one vertex with several loops.

The **regular neighborhood** of a graph drawn in the plane (or on a surface, see § 2.1) without edges crossing is the union of caps and ribbons constructed as shown in Fig. 1.6.3 (left). For a rigorous

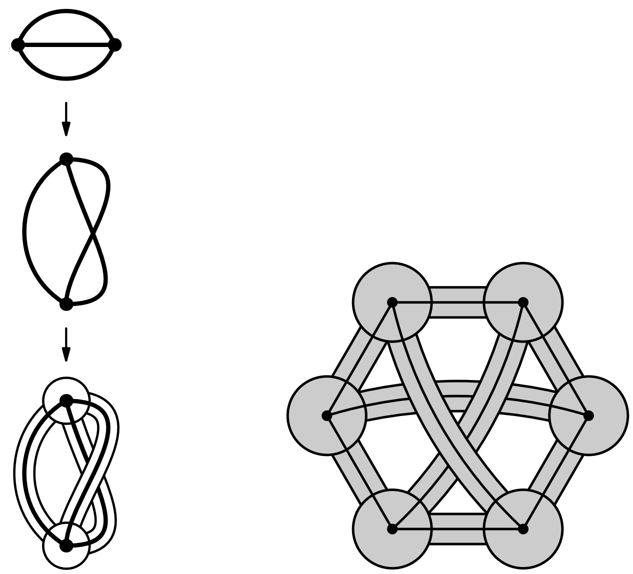


Figure 1.6.2. Left: the top picture shows the graph $K_{3,2}$, the middle one is a drawing of this graph in the plane, and the bottom one is the corresponding thickening.
Right: an oriented thickening of the graph $K_{3,3}$

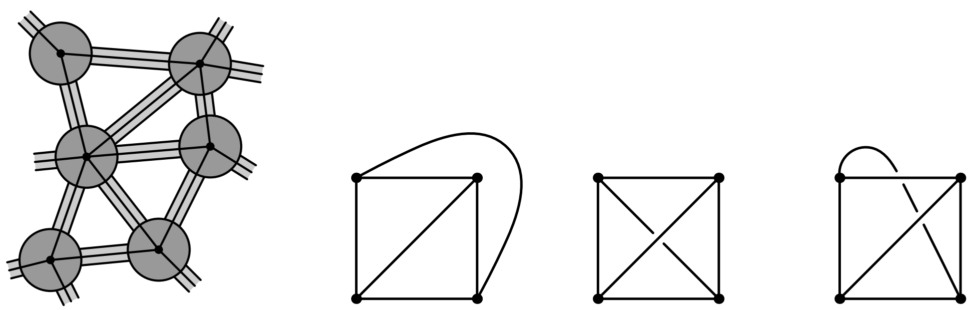


Figure 1.6.3. Left: the caps and ribbons (called clusters and pipes in [MT01]) form the regular neighborhood (thickening) of a graph on a surface.
Right: drawings of the graph K_4 in the plane

definition, see § 5.9. The regular neighborhood of a graph G is an oriented thickening of G (Fig. 1.6.3 (left)). More generally, if we have a *general position* map of a graph G to the plane (or to a surface, see § 2.1), then we can construct an oriented thickening of G 'corresponding' to this map (Figs. 1.5.2 and 1.6.2 (left), Fig. 1.6.3 (right)).

An oriented thickening is said to be **planar** if it can be cut out of the plane.

1.6.1. (a) Every thickening of a tree is planar.

(b) Every orientable thickening of a cycle is planar.

(c) Every orientable thickening of a unicyclic graph is planar. (A graph is said to be *unicyclic* if it becomes a tree after deleting an edge.)

(d) Is the orientable thickening of the graph $K_{3,2}$ shown in Fig. 1.6.2 (left) planar?

(e) Which of the orientable thickenings of the graph K_4 (Fig. 1.6.3 (right)) are planar?

(f) A graph is planar if and only if it has a planar orientable thickening.

(g) A *rotation system* of a graph is an assignment to each vertex of an oriented cyclic order on the edges incident to this vertex. Every graph has finitely many rotation systems (moreover, there is an algorithm searching through those rotation systems).

Deciding the planarity of graphs reduces to deciding the planarity of orientable thickenings, see Assertion 1.6.1 (f, g).

1.6.2. (a) Define the operation of *contracting an edge* of a thickening so that it would give the operation of contracting an edge of a graph and preserve planarity.

(b) Draw the thickenings obtained from the thickenings of the graph K_4 (Fig. 1.6.3 (right)) by contracting the 'top horizontal' edge.

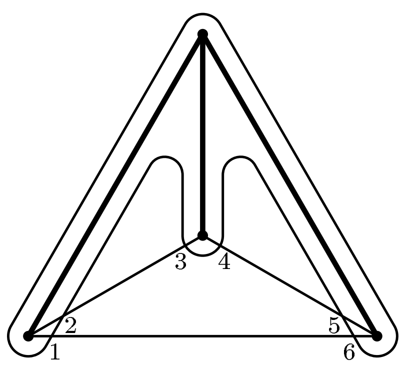


Figure 1.6.4. Walking around a spanning tree

Theorem 1.6.3. *(a) There is an algorithm for deciding the planarity of thickenings.*

(b) Each of the following conditions on an orientable thickening of a connected graph G is equivalent to the planarity of this thickening.

(I) *For every spanning tree T, going along the boundary of the thickening of T (Fig.* 1.6.4*) we obtain a cyclic sequence of edges not from T, in which every edge occurs twice; then any two edges in this sequence do not alternate, i.e., occur in the cyclic order $(aabb)$, and not $(abab)$.*

(E) *The number of boundary circles of the thickening is $E - V + 2$, where V and E are the numbers of vertices and edges.*

(Boundary circles of a thickening are defined analogously to boundary circles of a disk with ribbons.)

(S) *The thickening 'does not contain' the 'figure eight' and 'letter theta' subthickenings shown in Figs.* 1.5.2 *and* 1.6.2 *(left). (More precisely, the graph does not contain a subgraph* homeomorphic *to one of the graphs shown in the top pictures of these figures such that the* restriction *of the thickening to this subgraph is* homeomorphic *to one of the thickenings shown in the bottom pictures of these figures.)*

1.6.4. Every orientable thickening

(a) of a tree has one boundary circle;

(b) of a cycle has two boundary circles.

(c) of a connected graph with V vertices and E edges has at most $E - V + 2$ boundary circles.

1.6.5. Every non-orientable thickening of a connected graph with V vertices and E edges has at most $E - V + 1$ boundary circles.

Hint: Assertions 1.6.4.c and 1.6.5 follow from Assertions 1.5.3.a,a'.

1.7. Hieroglyphs and Orientable Thickenings*

In this subsection we give an interpretation of the constructions from §§ 1.5 and 1.6. A *representation* of a hieroglyph is a word of length $2n$ in which each of n letters occurs exactly twice. A *hieroglyph* is an equivalence class of such words up to renaming of letters and cyclic shift. Other names: chord diagram, one-vertex multigraph with rotations.

Hieroglyphs are drawn as shown in Figs. 1.5.2 (left) and 1.7.1, i.e., as families of loops in the plane with a common vertex. A cyclic order is determined by enumerating the segments incident to the vertex in the counterclockwise direction.

§ 2. Intuitive Problems About Surfaces

Wissen war ein bisschen Schaum, der über eine Woge tanzt. Jeder Wind konnte ihn wegblasen, aber die Woge blieb.

E. M. Remarque. *Die Nacht von Lissabon*[7]

In § 2.1 we recall the definitions of basic surfaces. The reader may omit this subsection and return to it when necessary. Subsection 2.2 contains intuitive problems about cutting surfaces and cutting out of surfaces. Here we state Riemann's and Betti's Theorems 2.3.5, which are used to prove than a surface cannot be cut out of another surface. Subsection 2.4 contains basic results about graphs and map colorings on surfaces (Theorems 2.4.4, 2.4.5 (b), 2.4.7). They are similar to the results from §§ 1.1 and 1.3 about graphs and map colorings in the plane. The proofs involve an analog of Euler's Formula, namely, Euler's Inequality 2.5.3 (a). This inequality is proved in § 2.5 together with Riemann's Theorem 2.3.5 (a). In § 2.6, an algorithm is constructed for deciding whether a graph can be realized on a given surface (i.e., Theorem 2.4.5 (b) is proved). In § 2.7 we informally introduce and study the notion of topological equivalence of surfaces. In particular, Assertions 2.7.8 (b) and 2.7.9 (b) demonstrate one of the main ideas of the proof of Theorem 5.6.2 on classification of surfaces. Subsection 2.8 contains versions of the previous examples and results for non-orientable surfaces.

2.1. Examples of Surfaces

If you are not familiar with Cartesian coordinates in the space, then at the beginning of the book you may omit coordinate definitions and work with intuitive descriptions and drawings (given after coordinate definitions).

[7]Knowledge was a speck of foam dancing on top of a wave. Every gust of wind could blow it away; but the wave remained. (E. M. Remarque. *The Night in Lisbon*)

The **sphere** S^2 is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 1$:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}.$$

This is the same thing as the set of all points (x, y, z) of the form

$$(\cos\varphi\cos\psi, \sin\varphi\cos\psi, \sin\psi).$$

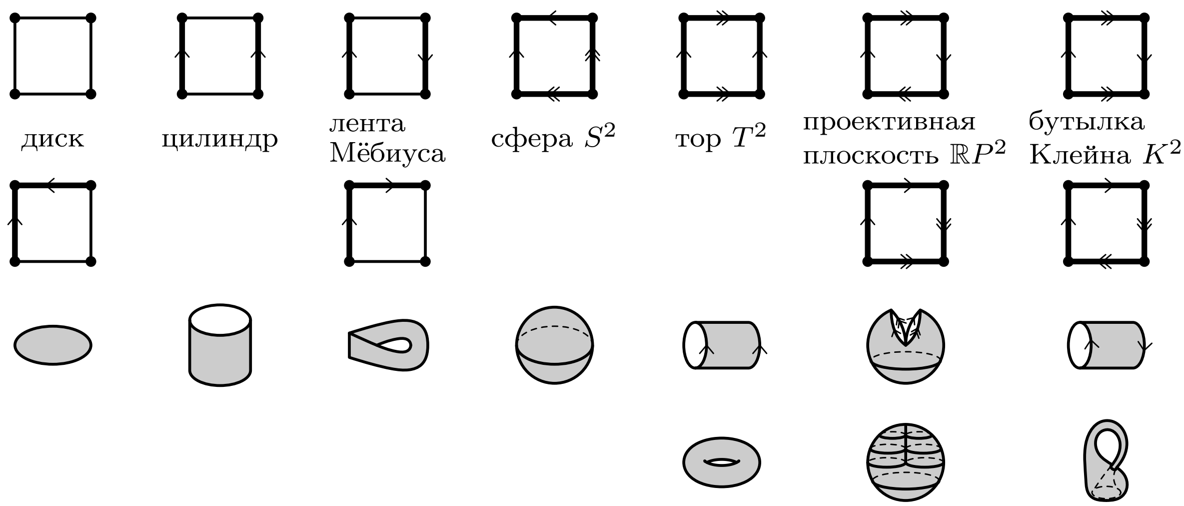


Figure 2.1.1. The surfaces obtained by gluing together sides of a rectangle

In what follows, by a rectangle we mean a two-dimensional part of the plane (and not its boundary), and 'gluing' includes a 'continuous deformation' that drags the points to be glued to each other.

The sphere is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AD}$, $\overrightarrow{CB}$ and $\overrightarrow{CD}$ with the directions indicated in the picture (the fourth column in Fig. 2.1.1).

The **annulus** is the set $\{(x, y) \in \mathbb{R}^2 \ : \ 1 \leqslant x^2 + y^2 \leqslant 2\}$ (Fig. 6.3.1). The **lateral surface of a cylinder** (Fig. 2.1.2 (right)) is the set

$$\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1, 0 \leqslant z \leqslant 1\}.$$

Each of these shapes is obtained from a rectangle $ABCD$ by 'gluing together' the pair of opposite sides $\overrightarrow{AB}$ and $\overrightarrow{DC}$ 'with the same direction' (the second column in Fig. 2.1.1).

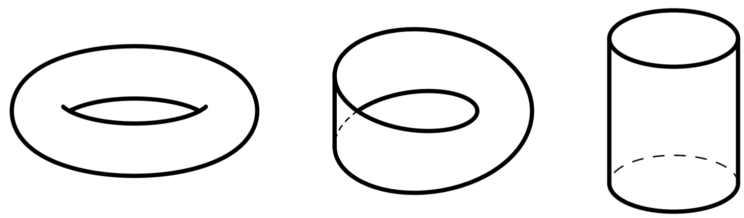


Figure 2.1.2. The torus, Möbius strip, and lateral surface of a cylinder

The **torus** T^2 is the shape obtained by rotating the circle $(x-2)^2+y^2=1$ about the Oy axis (Fig. 2.1.2 (left)).

The torus is the 'surface of a doughnut'. It is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of opposite sides $\overrightarrow{AB}$ and $\overrightarrow{DC}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ 'with the same direction' (the fifth column in Fig. 2.1.1).

The **Möbius strip** is the set of points in $\mathbb{R}^3$ swept by a bar of length 1 rotating uniformly about its center as this center moves uniformly along a circle of radius 9 when the bar makes half a turn (Fig. 2.1.2 (middle)).

The Möbius strip is obtained from a rectangle $ABCD$ by 'gluing together' two opposite sides $\overrightarrow{AB}$ and $\overrightarrow{CD}$ 'with opposite directions' (the third column in Fig. 2.1.1).

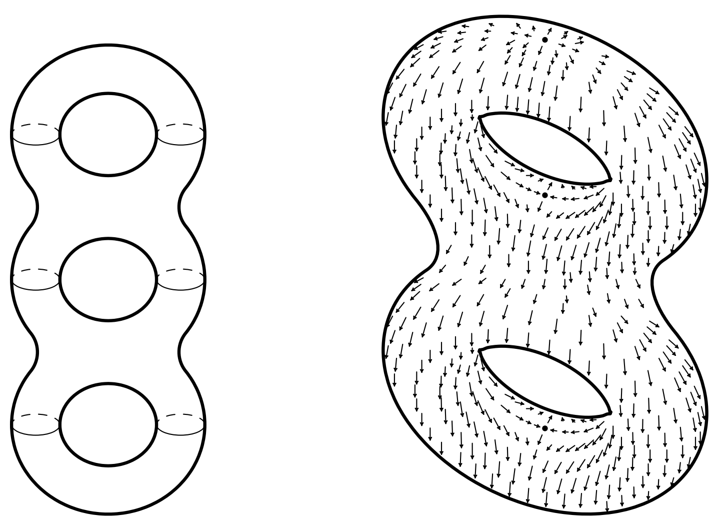


Figure 2.1.3. The spheres with two and three handles

The **sphere with g handles** S_g, where $g \geqslant 1$, is the set of points $(x, y, z) \in \mathbb{R}^3$ such that

$$x^2 + \prod_{k=1}^{g} ((z-4k)^2 + y^2 - 4)^2 = 1.$$

The *sphere with* 0 *handles* is the sphere S^2. The sphere with one handle is the torus. The spheres with two and three handles are shown in Fig. 2.1.3.

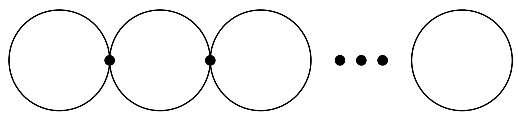


Figure 2.1.4. A ‘chain of circles’ in the plane

The equation $\prod\limits_{k=1}^{g} ((z-4k)^2 + y^2 - 4) = 0$ defines a ‘chain of circles’ in the plane Oyz (Fig. 2.1.4). The sphere with g handles is the boundary of the ‘tubular neighborhood’ of this chain in the space. Hence, the sphere with g handles is obtained from the sphere by ‘cutting out’ $2g$ disks and then attaching g curvilinear lateral surfaces of cylinders to g pairs of boundary circles of these disks (Fig. 2.1.5).

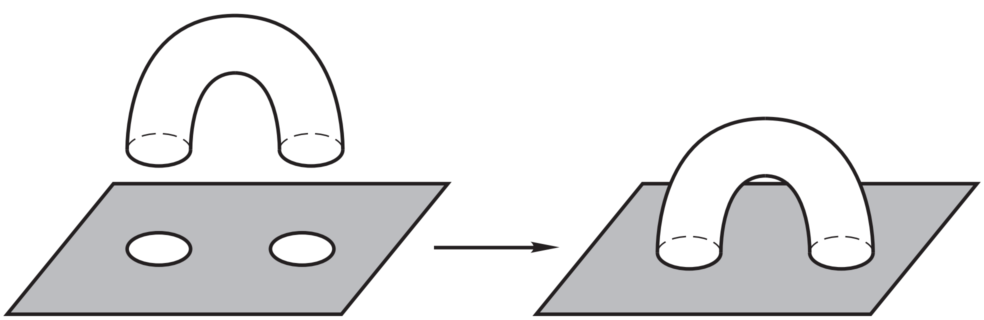


Figure 2.1.5. Attaching a handle

The **sphere with g handles and a hole** $S_{g,0}$ is the part of the sphere with g handles that lies below or on the plane situated slightly below the tangent plane at the top point (i.e., the part of S_g that lies in the domain $z \leqslant 4g + 2$). This shape is obtained from the sphere with handles by ‘cutting out a hole’.

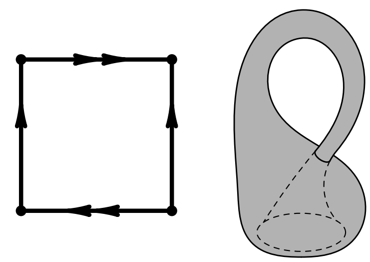


Figure 2.1.6. The Klein bottle: (a) gluing; (b) a drawing in $\mathbb{R}^3$

Informally, the *Klein bottle* is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of opposite sides, the pair $\overrightarrow{AB}$, $\overrightarrow{DC}$ 'with the same direction', and the other pair $\overrightarrow{BC}$, $\overrightarrow{DA}$ 'with opposite directions' (Fig. 2.1.6 (a)).

Consider in $\mathbb{R}^4$ the circle $x^2 + y^2 = 1$, $z = t = 0$ and the family of three-dimensional normal planes to this circle. Strictly speaking, the **Klein bottle** K is the set of points in $\mathbb{R}^4$ swept by a circle ω as its center moves uniformly along the circle under consideration, while the circle ω itself rotates uniformly by angle π (in the moving three-dimensional normal plane, about its own diameter moving together with this plane).

The projection of the Klein bottle to $\mathbb{R}^3$ is shown in Fig. 2.1.6 (b).

In what follows, 'surface' is a collective term for the shapes defined above, and not a mathematical term (cf. the definition of a 2-manifold in § 4.5).

2.2. Cutting Surfaces and Cutting out of Surfaces

In the problems of this subsection, you are asked to give not rigorous proofs, but large, comprehensible, and preferably beautiful pictures.

2.2.1. (a) For every n there exist n points in $\mathbb{R}^3$ such that the segments between them have no common interior points (i.e., every graph can be drawn in $\mathbb{R}^3$ without edges crossing).

(b) Every graph can be drawn without edges crossing on a book with a certain number of sheets (Fig. 2.2.1; the definition is given after the figure) depending on the graph. More precisely, for every n there exists an integer k, as well as n points and $n(n-1)/2$ non-self-intersecting polygonal lines on a book with k sheets such that every pair of points is joined by a polygonal line and no polygonal line intersects the interior of another polygonal line.

(c) The same as in part (b) with 3 sheets instead of k.

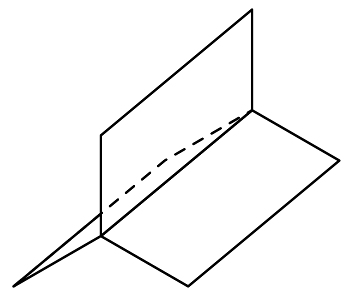


Figure 2.2.1. A book with three sheets

In $\mathbb{R}^3$ consider n rectangles XYB_kA_k, $k = 1, 2, \ldots, n$, any two of which have only the segment XY in common. The *book with n sheets* is the union of these rectangles; see Fig. 2.2.1 for the case $n = 3$.

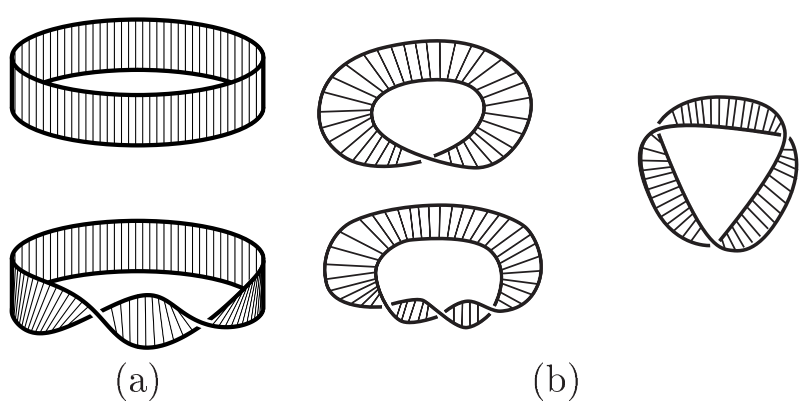


Figure 2.2.2. Nonstandard (a) annuli; (b) Möbius strips

A *nonstandard annulus* is any shape obtained from a rectangle by gluing a pair of opposite sides 'with the same direction' (Fig. 2.2.2 (a)). This informal definition can be formalized using the notions of homeomorphism and gluing (§ 2.7 and Example 5.1.1.c). In a similar way one defines a nonstandard Möbius strip (Fig. 2.2.2 (b)), torus with a hole, Klein bottle with a hole, etc. They will be used only in this subsection (one cuts nonstandard shapes out of standard ones); the word 'nonstandard' will be omitted.

2.2.2. Cut the Möbius strip so as to obtain

(a) an annulus; (b) an annulus and a Möbius strip.

2.2.3. Cut the Klein bottle (Fig. 2.1.6) so as to obtain
(a) two Möbius strips; (b) one Möbius strip.

2.2.4. Cut out the following shapes from the book with three sheets (Fig. 2.2.1):
(a) a Möbius strip; (b) a torus with a hole;
(c) a sphere with two handles and a hole;
(d) a Klein bottle with a hole.

2.2.5. Let A, B, C, D be points on the boundary circle of a torus with a hole (in this order along the circle). A rectangle $A'B'D'C'$ is attached to the torus with a hole by gluing AB to $A'B'$ and CD to $C'D'$. From the resulting shape (i.e., from a torus with a hole and a Möbius strip), cut out three pairwise disjoint Möbius strips.

2.3. Impossibility of Cutting and Separating Curves

2.3.1. (a) A torus with a hole cannot be cut out of the plane.

(b) For $k < n$, a sphere with n handles and a hole cannot be cut out of the sphere with k handles.

(c) Two disjoint Möbius strips cannot be cut out of the Möbius strip.

(d) Find all g, m, g', m' for which g' tori with a hole and m' Möbius strips (all $g' + m'$ shapes pairwise disjoint) can be cut out of a disk with g handles and m Möbius strips (see the definitions before Figs. 2.1.5 and 2.8.1).

Proof of (a). Part (a) follows from the Intersection Lemma 1.4.4 or from the (essentially equivalent) nonplanarity of the graph K_5 (Assertion 1.3.2 (a)), because the analogues of these results for the torus are false (cf. Assertion 2.4.1 (a)).

Alternatively, assume to the contrary that a torus with a hole is cut out of the plane. Take a closed non-self-intersecting curve γ on this torus with a hole such that γ does not separate it (Assertion 2.3.2.a). In the next paragraph we prove that γ does not separate the sphere, contradicting Jordan's Theorem 1.4.3 (b) (the details are necessary because e.g. the boundary circle of the disk does not separate the disk, but does separate the plane containing the disk).

Pick any two points in the plane that do not lie on γ. Join them with a polygonal line α 'in general position' with respect to γ. This polygonal line meets γ in a finite number of points. For each such point A, take a small segment α_A of α that contains A in its interior. The endpoints of α_A lie on the torus with a hole. Hence, they can be joined by a polygonal line α'_A that does not intersect γ. Replace each segment α_A with α'_A. We obtain a polygonal line that joins the given points and does not intersect γ. □

Comments on the proof of (b,c,d). Part (b) follows from Theorem 2.3.5 (c) and Assertion 2.3.3.c. Part (b) can also be deduced from Assertion 2.4.4 (c), or from Theorem 2.3.5 (a) and Assertion 2.3.3.a (observe that both Assertion 2.4.4 (c) and Theorem 2.3.5 (a) use Euler's Inequality 2.5.3 (a)). The details of deduction from Theorems 2.3.5 (c) or 2.3.5 (a) have to be checked, cf. (a).

Analogously, parts (c) can be deduced from either of Assertions 2.8.2 (a), 2.8.2 (c) or 2.8.3 (b).

To solve part (d), it is helpful to use Assertion 2.8.5 (c), see also Assertion 2.6.6 and Problem 6.7.7. □

2.3.2. (a) Draw a closed curve on the torus such that cutting along this curve does not separate the torus.

(b) The same for the Möbius strip.

(c) Draw two closed curves on the torus such that cutting along their union does not separate the torus.

(d) Draw two closed disjoint curves on the Klein bottle such that cutting along their union does not separate the Klein bottle.

Curves and graphs on the torus can be easily defined by regarding the torus as obtained from a rectangle by gluing. A (*piecewise linear*) *curve on the torus* is then a family of polygonal lines in the rectangle satisfying certain conditions (work out these conditions!). In a similar way, other surfaces can be obtained from plane polygons by gluing (for spheres with handles, see Problem 2.3.4). This allows one to define curves and graphs on other surfaces. Another formalization is given in § 5, see also § 4.

2.3.3. On the sphere with g handles S_g there are

(a) g closed pairwise disjoint curves, whose union does not separate S_g.

(b) $2g$ closed curves, of which any two intersect by a finite number of points, and whose union does not separate S_g.

(c) a non-separating wedge of $2g$ cycles.

2.3.4. For every $g > 0$, obtain S_g by gluing together sides of a $4g$-gon. (See visualization in `https://www.youtube.com/watch?v=G1yyfPShgqw` and in `https://www.youtube.com/watch?v=U5N5mg3MePM`.)

It turns out that cutting the torus along the union of any two disjoint closed curves inevitably separates the torus. This is a special case of the following generalizations of Jordan's Theorem 1.4.3 (b).

Theorem 2.3.5. (a) (Riemann) *The union of any $g+1$ pairwise disjoint closed curves on S_g separates S_g.*

(b) (Betti) *Suppose that on S_g there are $2g+1$ closed curves, of which any two intersect by a finite number of points. Then the union of the curves separates the sphere with g handles.*

(c) *Any wedge of $2g+1$ cycles drawn without self-intersections on S_g separates S_g.*

Here the curves are allowed to be self-intersecting; however, the case of non-self-intersecting curves is the most interesting, and the general case can be easily reduced to it.)

These results (strictly speaking, for the *piecewise linear case*) follow from Euler's Inequality 2.5.3 (a). For part (c) the deduction is clear, for parts (a,b) see § 2.5.

2.4. Graphs on Surfaces and Map Colorings

The definition and discussion of a drawing of a graph on a surface without edges crossing is analogous to the case of the plane, see § 1.3. The formalization is outlined after Problem 2.3.2 and described in § 5.2, but can be omitted on first acquaintance.

The torus, Möbius strip, and other shapes are assumed to be *transparent*, i.e., a point (or a subset) that 'lies on one side of a surface' 'lies on the other side as well'. In a similar way, in geometry we speak about a triangle in the plane, rather than a triangle on the upper (or lower) side of the plane.

2.4.1. Draw the following graphs on the torus without edges crossing:

(a) K_5; (b) $K_{3,3}$; (c) K_6; (d) K_7; (e)* $K_{4,4}$; (f)* $K_{6,3}$.

The definition of a graph **realizable** on the torus or on a sphere with handles is analogous to that of a planar graph.

Proposition 2.4.2. *Any graph can be realized on a sphere with a certain number* (*depending on the graph*) *of handles.*

2.4.3. (a) The graph K_8; (b) the graph $K_{5,4}$; (c)* the graph $K_5 \sqcup K_5$ are not realizable on the torus.

To prove Assertions 2.4.3 and 2.4.4, we need Euler's Inequality 2.5.3 (a).

Here is a version of Assertion 2.4.3 for spheres with handles.

Proposition 2.4.4. (a) *The graph K_n is not realizable on a sphere with less than $(n-3)(n-4)/12$ handles.*

(b) *The graph $K_{m,n}$ is not realizable on a sphere with less than $(m-2)(n-2)/4$ handles.*

(c)* *The disjoint union of $g+1$ copies of the graph K_5 is not realizable on the sphere with g handles S_g.*

In view of Assertions 2.4.4 (a, c), for every g there is a graph (for example, K_{g+15} or the disjoint union of $g+1$ copies of K_5) that is not realizable on S_g (the second of these graphs is realizable on S_{g+1}). The estimations in Assertion 2.4.4 are sharp [Pr14, 13.1].

Theorem 2.4.5. *For every g there is an algorithm for deciding whether a graph is realizable on S_g.*

This result is deduced from Theorem 2.6.8 (a).

2.4.6. A *map on the torus* is a partition of the torus into (curved) polygons. A coloring of a map on the torus is said to be *proper* if different polygons sharing a common boundary curve have different colors. Is it true that any map on the torus has a proper coloring with

(a) 5 colors; (b) 6 colors?

It turns out that any map on the torus has a proper coloring with 7 colors. This is a special case of the following result. A *map on S_g handles* and a *proper* coloring of such a map are defined analogously to the case of the torus.

Theorem 2.4.7 (Heawood)**.** *If $0 < g < (n-2)(n-3)/12$, then every map on S_g has a proper coloring with n colors.*

The version of this theorem for $g = 0$ is true: this is the Four Color Conjecture. In view of Ringel's results on embeddings of K_n [Pr14, 13.1] $n-1$ colors are not sufficient for $g \geqslant (n-2)(n-3)/12$.

Heawood's Theorem 2.4.7 is implied by the following result, whose proof relies on Euler's Inequality 2.5.3 (a).

2.4.8. (a) Any graph drawn on the torus without edges crossing has a vertex with at most 6 incident edges.

(b) If $0 < g < (k-1)(k-2)/12$, then any graph drawn on S_g without edges crossing has a vertex with at most k incident edges.

2.5. Euler's Inequality for Spheres with Handles

Given a graph drawn on a surface without edges crossing, a **face** is any of the connected parts into which cutting along all edges of the graph divides the surface.

On the torus there are two closed curves such that cutting along them divides the torus into different numbers of parts (Problem 2.3.2 (a)). So, the number of faces depends on the way the graph is drawn on the given surface. However, we still have a version of Euler's Formula for surfaces. These are the following inequalities 2.5.1 (d) and 2.5.3 (a).

2.5.1. (a, b, c, d) The same as in Assertions 1.4.2, with the plane replaced by a sphere with handles and a planar graph replaced by a graph drawn on the sphere with handles without edges crossing.

(d′) In a parliament consisting of n members there are several (pairwise distinct) 3-person commissions. It is known that if two persons x, y belong to a commission, then the set $\{x, y\}$ is contained in exactly two commissions. Such two commissions are said to be *adjacent.* It is also known that for any two persons A, B there is a sequence of commissions such that A is in the first commission, B is in the last commission, and any two consecutive commissions are adjacent. Show that the number of commissions is not less than $2n-4$.

(e) If G is a subgraph of a connected graph H on a sphere with handles, then $V_G - E_G + F_G \geqslant V_H - E_H + F_H$.

Hint. Part (e) follows from part (c). Use the operations of deleting an edge, or deleting a hanging vertex.

Warning. Part (e) is not true for a disconnected graph H, but is true for a disconnected graph H if every connected component contains a vertex of G.

2.5.2. Given a connected graph with V vertices and E edges drawn on the torus without edges crossing, denote by F the number of faces.

(a) If the graph (more exactly, its drawing) contains a parallel, then $F = E - V$.

Hint. Cut the torus along the parallel. The result is a plane graph lying between two its cycles. Apply Euler's Formula to this graph.

(b) $F \geqslant E - V$.

Clarification. Prove the assertion under the following assumption: the graph meets a parallel in a finite number of points, and cutting the graph along the parallel with subsequently unfolding it into the plane results in a union of polygonal lines (a more learned way of saying this is that the given embedding of the graph into the torus is piecewise linear and in general position with respect to the parallel).

Hint. Use part (a) and Assertion 2.5.1 (e).

2.5.3. (a) **Euler's Inequality**[8]. *Given a connected graph with V vertices and E edges drawn on S_g without edges crossing, denote by F the number of faces. Then*

$$V - E + F \geqslant 2 - 2g.$$

(b) Given a graph with V vertices, E edges, and s connected components drawn on S_g without edges crossing, denote by F the number of faces. Then $V - E + F \geqslant 1 + s - 2g$.

Euler's Inequality 2.5.3 (a) can be proved analogously to the case of the torus 2.5.2 (b) using Assertion 2.3.4.

Sketch of proof of Riemann's Theorem 2.3.5 (a). Consider the case of the torus (the general case is proved analogously). Suppose that the union of two disjoint closed curves does not separate the torus. We may assume that the curves are simple. Similarly to the proof of Jordan's Theorem 1.4.3 (b), we use the orientability of the torus to conclude

[8] Usually, instead of Euler's *Inequality*, which is sufficient for solving many interesting problems, one considers the more complicated Euler's *Formula* 5.9.2 (cf. Assertion 2.5.2 (a)), whose statement involves the notion of a *cellular subgraph*.

that there are a 'figure eight' and a circle that are non-self-intersecting, disjoint, and whose union does not separate the torus. Joining the figure eight and the circle by an arc on the torus, we obtain a graph with $V - E = -2$ that does not separate the torus, contradicting Euler's Inequality. $\square$

Betti's Theorem 2.3.5 (b) follows from Euler's Inequality 2.5.3 (b) (or from Euler's Inequality 2.5.3 (a) and Riemann's Theorem 2.3.5 (a); the details are similar to the arguments in [Bi20, bottom of p. 6]).

2.6. Realizability of Hieroglyphs and Orientable Thickenings

Disks with untwisted ribbons are defined in § 1.5. We will call them hieroglyphs, cf. § 1.7. A hieroglyph is said to be **realizable** on a given surface if it can be cut out of this surface.

2.6.1. (a, b, c) The hieroglyphs corresponding to the words ($abab$), ($abcabc$), and ($abacbc$) (Fig. 1.5.2) are realizable on the torus.

A solution of (b, c) is presented in Fig. 2.6.1.

2.6.2. The hieroglyphs shown in Fig. 1.5.3
(a′, b′, c′, d′) are realizable on the sphere with two handles.
(a, b, c, d) are not realizable on the torus.

For a proof of (a′, b′, c′, d′) pick two interlacing ribbons and show that the disk with the two remaining ribbons is realizable on the torus (a proof via attaching ribbons one by one also works, but is more complicated). Parts (a,b,c,d) are proved analogously to Assertion 2.3.1 (b) (in fact, every hieroglyph with 4 ribbons that has one boundary circle cannot be realized on the torus).

Denote by $h(M)$ the number of boundary circles of a hieroglyph or a thickening M.

2.6.3. (a) If a hieroglyph M is cut out of the sphere with g handles S_g, then the number of obtained connected components of $S_g - M$ does not exceed $h(M)$.

(a') If a hieroglyph M with n ribbons is cut out of S_g, then $h(M) \geqslant n + 1 - 2g$.

(b) For every g there exists a hieroglyph not realizable on S_g.

(c) If a hieroglyph M is realizable on S_g and removing any of its ribbons results in a hieroglyph non-realizable on S_g, then M has $2g+2$ ribbons.

Here part (a') follows from part (a) and Euler's Inequality 2.5.3 (a) (cf. Assertion 2.3.1 (b)). Part (b) follows by part (a') (take e.g. hieroglyph $(a_1b_1a_1b_1 \dots a_{g+1}b_{g+1}a_{g+1}b_{g+1})$).

2.6.4. (a) Every hieroglyph with 3 ribbons is realizable on the torus.

(b) Does there exist a hieroglyph with 4 ribbons that has two boundary circles?

(c) Every hieroglyph with 4 ribbons that has three boundary circles is realizable on the torus.

(d) Every hieroglyph with n ribbons that has at least $n-1$ boundary circles is realizable on the torus.

The proof is analogous to that of Assertions 2.6.2(a$'$, b$'$, c$'$, d$'$), cf. Assertions 1.5.3 (a, b).

Theorem 2.6.5. (a) *For every g there is an algorithm for deciding whether a hieroglyph is realizable on S_g.*

(b) *Each of the following conditions on a hieroglyph M with n ribbons is equivalent to its realizability on S_g.*

(E) *The inequality $h(M) \geqslant n+1-2g$ holds.*

(I) *Among any $2g+1$ rows of the interlacement matrix (see the definition below) there are several ($\geqslant 1$) rows whose sum is zero modulo 2. (In other words, the rank of the interlacement matrix over $\mathbb{Z}_2$ does not exceed $2g$.)*

The *interlacement matrix* of a hieroglyph with n ribbons is the $n \times n$ matrix whose $a \times b$ cell contains 1 if $a \neq b$ and the letters a and b do not interlace, and 0 otherwise. Cf. § 6.7.

Here part (a) follows from (b). The condition (E) is necessary for the realizability by Assertion 2.6.3.a'. The sufficiency of (E) is proved analogously to Assertion 2.6.4, cf. Assertion 2.7.8 (b) and its proof. Criterion (I) can be proved analogously to Assertion 2.7.8 (c).

The *rank* $\operatorname{rk} M$ of a hieroglyph M is the rank of its interlacement matrix over $\mathbb{Z}_2$. The rank measures the 'complexity of intersections' on the hieroglyph.

2.6.6. A hieroglyph M can be cut out of a hieroglyph M' if and only if $\operatorname{rk} M \leqslant \operatorname{rk} M'$.

Orientable thickenings are defined in §§ 1.6 and 1.7. A thickening is said to be **realizable** on a given surface if it can be cut out of this surface.

2.6.7. Does there exist an orientable thickening of
(a) the graph K_4; (b) the graph K_5
that is not realizable on the torus?

Theorem 2.6.8. (a) *For every g there is an algorithm for deciding whether a thickening is realizable on S_g.*

(b) *Each of the following conditions on an orientable thickening M of a connected graph is equivalent to its realizability on S_g.*

(E) *The inequality $2g \geqslant 2 - V + E - h(M)$ holds, where V and E are the numbers of vertices and edges of the graph.*

(I) =2.6.5.b(I).

Given an orientable thickening of a connected graph G and a spanning tree, we construct a hieroglyph corresponding to the edges not in the tree (Fig. 1.6.4). The *interlacement matrix*, corresponding to the tree, of the orientable thickening is the interlacement matrix of the resulting hieroglyph. The *rank* of an orientable thickening is the rank of its interlacement matrix (corresponding to an arbitrary tree) over $\mathbb{Z}_2$.

Theorem 2.6.8 is reduced to Theorem 2.6.5 by contracting an edge or considering a spanning tree.

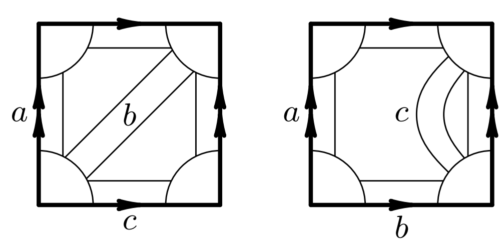


Figure 2.6.1. The disks with ribbons corresponding to the words ($abcabc$) and ($abacbc$) on the torus

2.7. Topological Equivalence (Homeomorphism)

2.7.1. Can the graph K_5 be drawn without edges crossing

(a) on the sphere; (b) on the lateral surface of a cylinder (Fig. 2.1.2)?

In this section, we do not give a rigorous definition of the notion of *homeomorphism* (topological equivalence); for a rigorous definition, see § 5.2. To 'prove' that shapes are homeomorphic, in this section you must draw a chain of pictures similar to Fig. 2.7.1.

Here it is allowed to temporarily *cut* a shape, and then *glue together* the 'edges' of the cut. For example,

• the sphere with a point removed is homeomorphic to the plane, and the lateral surface of a cylinder is homeomorphic to the annulus on the plane (here a chain of pictures can be obtained from the solution of Problem 2.7.1);

• the sphere with one handle (Fig. 2.1.5) is homeomorphic to the torus (Fig. 2.1.2);

• the disk with two ribbons (Fig. 2.7.1 (right)) is homeomorphic to the torus with a hole (Fig. 2.7.1 (left));

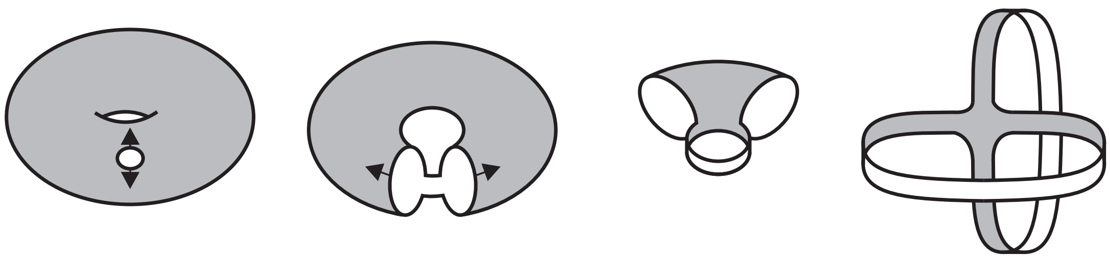


Figure 2.7.1. The torus with a hole is homeomorphic to the disk with two ribbons

• the three ribbons in Fig. 2.2.2 (b) are homeomorphic (here we can no longer do without cutting);

• the two ribbons in Fig. 2.2.2 (a) are homeomorphic (here again we cannot do without cutting).

The ribbons in Fig. 2.2.2 (a) and in Fig. 2.2.2 (b) are not homeomorphic. We will deal with *nonhomeomorphic* shapes in § 5, after introducing a rigorous definition and other notions, which allow one to turn the informal arguments of this section into rigorous proofs.

One should not confuse the notion of homeomorphism with that of *isotopy*, see Problem 6.6.1 (b) and § 15.5.

2.7.2. (a, b) The shapes in Fig. 1.5.2 (middle and right) are homeomorphic to the torus with two holes.

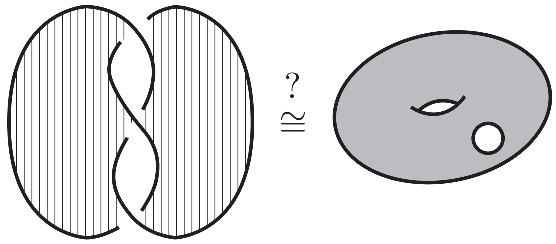


Figure 2.7.2. Are these shapes homeomorphic?

(c) The shape in Fig. 2.7.2 (left) is homeomorphic to the torus with a hole.

(d) Is the shape in Fig. 1.6.2 (right) homeomorphic to a sphere with handles and holes? If yes, with how many handles and holes?

2.7.3. (a, b, c, d) The shapes in Fig. 1.5.3 are homeomorphic to the sphere with two handles and a hole.

2.7.4. Cutting the torus

(a) along any non-separating cycle results in a shape homeomorphic to the annulus;

(b) along any non-separating 'figure eight' results in a shape homeomorphic to the disk (i.e., to a convex polygon).

2.7.5. The regular neighborhoods of different drawings of a graph in the plane without edges crossing (i.e., of isomorphic plane graphs, see Fig. 1.3.1) are homeomorphic.

Concerning hieroglyphs and thickenings, see §§ 2.6 and 1.5–1.7.

2.7.6. (a) Every hieroglyph with two ribbons is homeomorphic either to the disk with two holes or to the disk with one hole.

(b) (Riddle) To what surfaces can an orientable thickening of the graph K_4 be homeomorphic?

Proposition 2.7.7. *(a) Any thickening of a tree is homeomorphic to the disk D^2.*

(b) Any disk with non-twisted ribbons, for which no two ribbons interlace, is homeomorphic to the disk with holes.

(c) Let M be a thickening of a connected graph with V vertices and E edges. If $V - E + h(M) = 2$, then M is homeomorphic to the sphere with $h(M)$ holes.

Proposition 2.7.7.c is proved using Proposition 2.7.7.ab (together with Assertions 1.5.3.a,b and 1.6.4.c; cf. Euler formulas 2.7.9 (b) and 2.8.11 (b)).

Proposition 2.7.8. (a) *Two hieroglyphs with the same number of ribbons are homeomorphic if and only if they have the same number of boundary circles.*

(b) Euler's Formula. *Let M be a hieroglyph with n ribbons. Then $h(M) - n$ is odd, $h(M) \leqslant n + 1$, and M is homeomorphic to the sphere with $(n + 1 - h(M))/2$ handles and $h(M)$ holes.*

(c)* Mohar's Formula. *Let M be a hieroglyph of rank r with n ribbons. Then r is even and M is homeomorphic to the sphere with $r/2$ handles and $n + 1 - r$ holes.*

The names 'Euler's Formula' and 'Mohar's Formula' for Assertions 2.7.8, 2.7.9, and 2.8.8 (see below) are not widely used. Cf. Problems 5.9.2 and 6.7.5 (f, g).

Proposition 2.7.9. (a) *Two orientable thickenings of a connected graph are homeomorphic if and only if they have the same number of boundary circles.*

(b) Euler's Formula. *Assume that M is an orientable thickening of a connected graph with V vertices and E edges. Then $V - E + h(M)$ is even, $V - E + h(M) \leqslant 2$, and M is homeomorphic to the sphere with $(2 - V + E - h(M))/2$ handles and F holes.*

(c)* Mohar's Formula. *Assume that M is an orientable thickening of rank r of a connected graph with V vertices and E edges. Then r is even, $V - E + r \leqslant 1$, and M is homeomorphic to the sphere with $r/2$ handles and $2 - V + E - r$ holes.*

2.8. Non-Orientable Surfaces*

Graphs and Map Colorings on a Disk with Möbius strips

2.8.1. Draw the following graphs on the Möbius strip without edges crossing:

(a) $K_{3,3}$; (b) $K_{3,4}$; (c) K_5; (d) K_6.

2.8.2. (a) *Euler's Inequality.* Assume that a connected graph with V vertices and E edges is drawn on the Möbius strip without edges crossing so that it does not intersect the boundary circle. Denote by F the number of faces. Then $V - E + F \geqslant 1$.

(b) The graph K_7 cannot be realized on the Möbius strip.

(c) The graph $K_5 \sqcup K_5$ cannot be realized on the Möbius strip.

(d) Any map on the Möbius strip has a proper coloring with 6 colors.

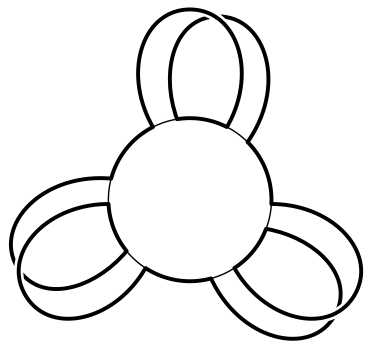


Figure 2.8.1. The disk with Möbius strips

The **disk with m Möbius strips** (Fig. 2.8.1) is the union of the disk and m ribbons such that

• each ribbon is glued along a pair of opposite sides to the boundary circle S of the disk, and the directions on these sides determined by an arbitrary direction on S 'coincide along the ribbon',

• the ribbons are 'separated', i.e., there are m pairwise disjoint arcs on S such that the endpoints of the ith ribbon are glued to two disjoint subarcs contained in the ith arc for every $i = 1, 2, \dots, m$.

2.8.3. (a) Draw m closed non-self-intersecting pairwise disjoint curves on the disk with m Möbius strips such that their union does not separate the disk with m Möbius strips.

(b) The union of any $m + 1$ pairwise disjoint closed curves on the disk with m Möbius strips separates it.

(c) Any graph can be drawn without edges crossing on a disk with a certain number (depending on the graph) of Möbius strips.

(d) For every $m > 0$, obtain the disk with m Möbius strips by gluing from a regular $4m$-gon.

2.8.4. (a) *Euler's Inequality.* Assume that a connected graph with V vertices and E edges is drawn without edges crossing on the disk with

m Möbius strips, so that the graph does not intersect the boundary circle. Denote by F the number of faces. Then $V - E + F \geqslant 2 - m$.

(b) State and prove versions of Theorem 2.4.4 for the disk with m Möbius strips, where $m \neq 2$.

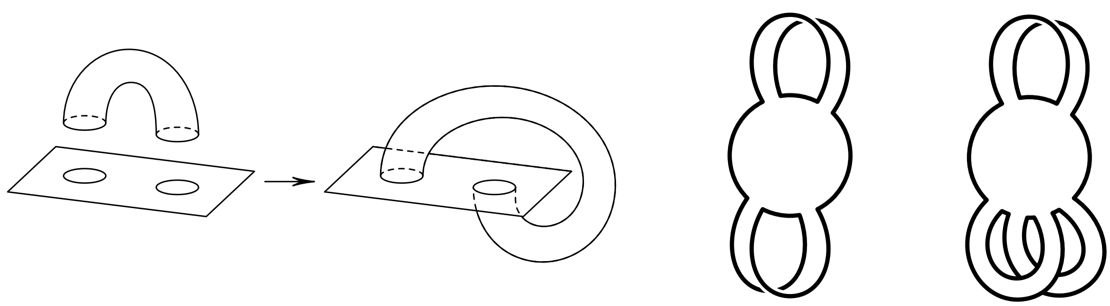
(c) State a prove a version of Heawood's Theorem 2.4.7 for the disk with m Möbius strips, where $m \neq 2$.

It turns out that the graph K_7 cannot be realized on the Klein bottle (i.e., on the disk with 2 Möbius strips), and that any map on the Klein bottle has a proper coloring with 6 colors [Fr34, SK86].

Homeomorphic Non-Orientable Surfaces

2.8.5. (a) The Möbius strip with a handle is homeomorphic to the Möbius strip with an inverted handle, see Fig. 2.1.5, 2.8.2 (a).

(b) The shape in Fig. 2.8.2 (b) (i.e., the disk with two 'twisted' 'separated' ribbons) is homeomorphic to the Klein bottle with a hole (Fig. 2.1.6).



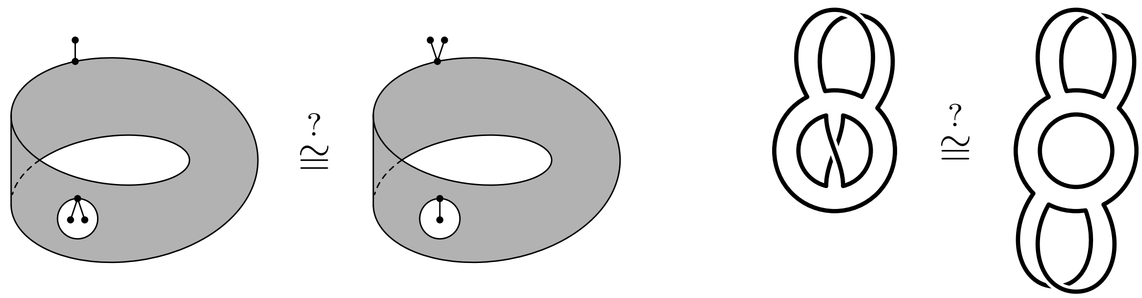
(a) (b) (c)

Figure 2.8.2. (a) Attaching an inverted handle (cf. Fig. 2.1.5). (b) The disk with two 'twisted' 'separated' ribbons (c) The disk with ribbons corresponding to the word $(aabcbc)$ with $w(a) = 1$ and $w(b) = w(c) = 0$.

(c) The shape in Fig. 2.8.2 (c) is homeomorphic to the disk with three Möbius strips.

(d) The shapes in Fig. 2.8.3 (a) are homeomorphic.

(e) The shapes in Fig. 2.8.3 (b) (i.e., an annulus with two 'twisted' 'separated' ribbons glued to the same boundary circle and an annulus



(a) (b)

Figure 2.8.3. (a) Are the boundary circles of the Möbius strip with a hole equivalent? (b) Are these annuli with two Möbius strips homeomorphic?

with two 'twisted' ribbons glued to different boundary circles) are homeomorphic.

Beautiful examples from Problems 2.8.5 (d, e) are of importance since they show that dissimilar shapes can still be homeomorphic.

Disks with Twisted Ribbons

Given a disk with ribbons and a ribbon k in it, set $w(k) = 1$ if the ribbon is twisted, and $w(k) = 0$ otherwise.

Figures 2.8.2 (b, c) and 1.5.1 (right), 2.8.1 show, respectively,

- the disk with ribbons corresponding to the word $(aabb)$ for which $w(a) = w(b) = 1$;
- the disk with ribbons corresponding to the word $(aabcbc)$ for which $w(a) = 1$ and $w(b) = w(c) = 0$;
- the disk with n Möbius strips, i.e., the disk with ribbons corresponding to the word $(1122 \ldots nn)$ for which $w(1) = w(2) = \ldots = w(n) = 1$.

2.8.6. (a) How many boundary circles can a disk with two ribbons have?

(b) To what surfaces can a disk with two ribbons be homeomorphic?

(c) To one of the boundary circles of the disk with n Möbius strips and $k > 0$ holes, a *twisted* (with respect to this boundary circle) ribbon is attached. The resulting shape is homeomorphic to the disk with $n + 1$ Möbius strips and k holes.

2.8.7. State and prove versions of Theorems 2.6.5 (a, b) for the realizability of disks with ribbons on the disk with m Möbius strips.

Proposition 2.8.8. (a) *Two disks with the same number of ribbons are homeomorphic if and only if they have the same number of boundary circles and either both have a twisted ribbon or neither has one.*

(b) Euler's Formula. *Assume that M is a disk with n ribbons among which there is a twisted one, and M has h boundary circles. Then $h \leqslant n$, and M is homeomorphic to the disk with $n+1-h$ Möbius strips and $h-1$ holes.*

(c)* Mohar's Formula. *The* interlacement matrix *of a hieroglyph with ribbons $1, 2, \dots, n$ and nonzero map $w\colon \{1, 2, \dots, n\} \to \{0, 1\}$ is defined analogously to the interlacement matrix of a hieroglyph, with the difference that the diagonal cell $a \times a$ contains the number $w(a)$. Denote by r the rank of the interlacement matrix over $\mathbb{Z}_2$. Then the corresponding disk with ribbons is homeomorphic to the disk with r Möbius strips and $n-r$ holes.*

Thickenings of Graphs

2.8.9. (a) The thickening in Fig. 2.8.4 cannot be realized on the Möbius strip.

(b) Every thickening of a unicyclic graph can be realized on the Möbius strip.

(c) Which thickenings of the graph K_4 can be realized on the Möbius strip?

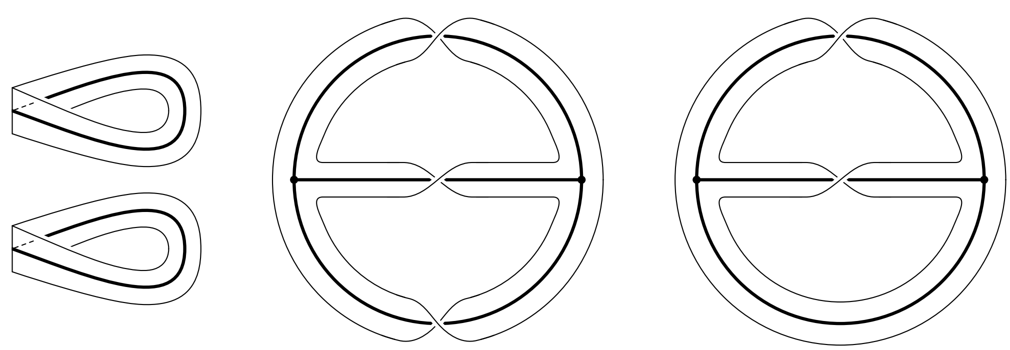


Figure 2.8.4. Thickenings that cannot be realized on the Möbius strip

§ 5. Two-dimensional manifolds

> I should say it meant something simple and obvious, but then I am no philosopher!
>
> I. Murdoch. The Sea, the Sea.

5.1. Hypergraphs and their geometric realizations

Let us give a combinatorial definition of two-dimensional surfaces (and somewhat more general objects). This definition is convenient for theoretical purposes as well as for storing in computer memory; cf. §1.2.

A **two-dimensional hypergraph**[13] (or **2-hypergraph**, for short) (V, F) is a collection F of three-element subsets of a finite set V. The elements of V and F are called **vertices** and **faces** (or *hyperedges*) of the 2-hypergraph. An **edge** of a 2-hypergraph is a two-element subset of the vertex set that is contained in a face.

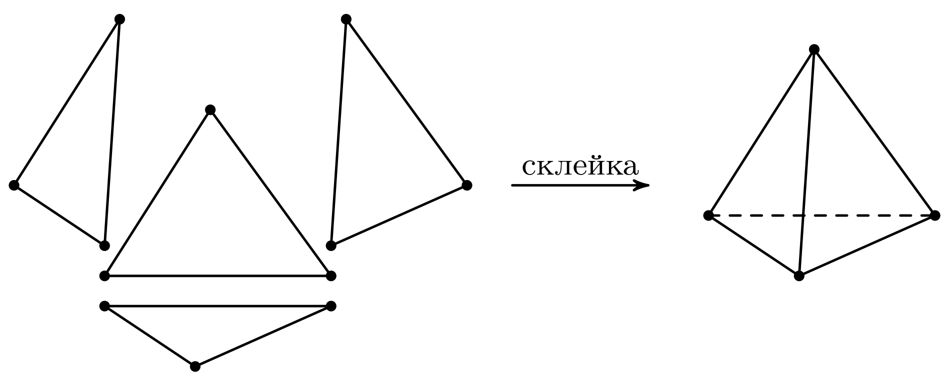


Figure 5.1.1. Building (the geometric realization of) a complete 2-hypergraph with 4 vertices

Example 5.1.1. (a) A *complete 2-hypergraph with n vertices* (or the *two-dimensional skeleton of an $(n-1)$-dimensional simplex*) is the collection of all three-element subsets of an n-element set. See Figure 5.1.1 for $n = 4$ and Figure 5.1.2 for $n = 5$. In this section the complete 2-hypergraph on 4 vertices is called the **sphere** S^2.

[13]Sometimes called a 3-*uniform* hypergraph, or a *dimensionally homogeneous* (*pure*) two-dimensional simplicial complex, see [Sk, § 5]

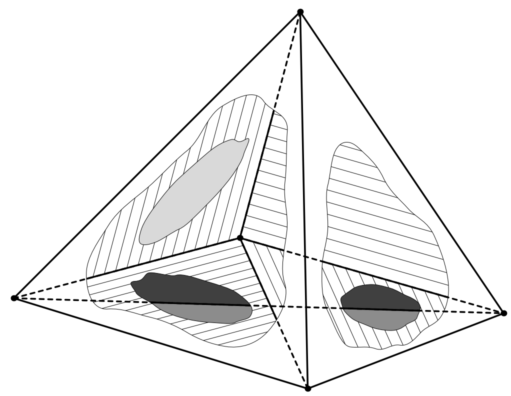


Figure 5.1.2. A complete 2-hypergraph with 5 vertices

(b) The *book with n pages* is the 2-hypergraph with vertices $a, b, 1, 2, \dots, n$ and faces $\{a, b, j\}$, $j = 1, 2, \dots, n$. See Figure 2.2.1 for $n = 3$.

(c) Suppose one has a 2-hypergraph, and a gluing diagram showing which pairs of edges should be identified, so that no two vertices of intersecting faces get identified. Then one can obtain a new 2-hypergraph by gluing the edges according to the diagram. For instance, Figure 2.1.1 shows the 2-hypergraphs obtained by gluing the sides of a square (triangulations are not shown there; see §5.9 and §6.2 for the formalization).

(d) A triangulation of 2-manifold (see §4.6) can be naturally viewed as a 2-hypergraph, which is also called a triangulation.

For $1 \leqslant i \leqslant n$, denote by $e_{n,i} \in \mathbb{R}^n$ the point whose i-th coordinate is 1 whereas the others are 0. The convex hull Δ_n of the points $e_{n+1,1}, \dots, e_{n+1,n+1} \in \mathbb{R}^{n+1}$ is called[14] the *n-dimensional simplex*. It is a convex polyhedron with $n+1$ vertices; the union of its edges 'forms' the complete graph K_{n+1}. The **geometric realization** (or *body*) of a 2-hypergraph (V, F) is the union of those two-dimensional faces of the simplex with vertex set V that correspond to the faces from F.

Main results stated in this section (but not used later) are Theorems 5.2.4, 5.3.1, 5.3.3, and 5.6.2.

[14] One could define the n-dimensional simplex as the convex hull of $(0, \dots, 0), e_{n,1}, \dots, e_{n,n} \in \mathbb{R}^n$. This might be more visually intuitive but this is less convenient for us.

Remark 5.1.2 (on geometric realization of hypergraphs)**.** Similarly to the case of graphs, one builds a geometric shape from a 2-hypergraph, and calls it the *geometric realization* (cf. the above rigorous definition). Informally speaking, the shape is obtained by gluing several triangles corresponding to the faces. The gluing procedure does not have to happen in three-dimensional space; the procedure is either done in higher dimensions, or even abstractly, without any reference to an ambient space.

For example, Figure 5.1.1 shows how to build the geometric realization of the complete 2-hypergraph with 4 vertices. The geometric realization of the 2-hypergraph that is obtained as a surface triangulation is homeomorphic to that surface. More generally, 2-hypergraphs, just like graphs, can be specified by *geometric shapes*, including 'smooth' or self-intersecting ones. See the last two rows of Figure 2.1.1. One shape specifies multiple 2-hypergraphs.

Usually all these 2-hypergraphs are homeomorphic (see §5.2, Theorem 5.2.4 and the example before Problem 10.3.3). Then a 2-hypergraph bears the name of the shape. In this case non-isomorphic but homeomorphic 2-hypergraphs have the same name.

Despite having a geometric realization, a 2-hypergraph is a combinatorial object. It is impossible, say, to take a point on its face. However, 'taking a point on a face of the geometric realization of a 2-hypergraph' can be formalized as 'taking the newly added vertex of the new 2-hypergraph obtained by the subdivision of that face'; see Figure 5.2.2 on the right. We will not follow such a level of formality.

The definition of a 2-hypergraph isomorphism is analogous to the one for graphs. 2-Hypergraphs (V,F) and (V',F') are called **isomorphic** if there is a 1–1 correspondence $f\colon V\to V'$ satisfying the following property: *vertices $A,B,C\in V$ lie in the same face if and only if the vertices $f(A),f(B),f(C)\in V'$ lie in the same face.*

5.2. Homeomorphic 2-hypergraphs

Remark 5.2.1 (homeomorphism of graphs)**.** (a) The operation of *edge subdivision* is shown in Figure 5.2.1. Two graphs are called *homeomorphic* if one of them can be obtained from the other using edge subdivisions and the inverse operations. Equivalently, two graphs are homeomorphic if there is a graph that can be obtained from either of the two using edge subdivisions.

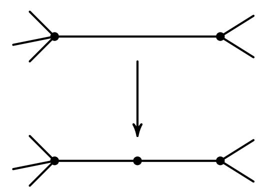


Figure 5.2.1. Edge subdivision

(b) The definition of a homeomorphism for subsets of Euclidean space is given in §3.1. It turns out that *graphs* G_1 *and* G_2 *are homeomorphic if and only if the realizations* $|G_1|$ *and* $|G_2|$ *are homeomorphic.* This criterion *motivates* the definition of a graph homeomorphism, which allows us to study certain shapes using combinatorial language.

(c) A *one-dimensional polyhedron* is a homeomorphism class of graphs. A topologist is usually interested in polyhedra even if calling them graphs. On the other hand, graphs and their realizations are convenient tools for studying polyhedra and storing them in computer memory. A combinatorialist or discrete geometer are mostly interested in graphs, though they might find polyhedra useful as well.

The definition of homeomorphic (combinatorial topology equivalent) 2-hypergraphs is analogous to the one for graphs.

The operation of an **edge subdivision** of a 2-hypergraph is shown in Figure 5.2.2, on the left.

5.2.2. The operation of a *face subdivision* in Figure 5.2.2, on the right, can be expressed using edge subdivision and its inverse.

Two 2-hypergraphs are said to be **homeomorphic**, if one of them can be obtained from the other (to be precise, from a 2-hypergraph isomorphic to the latter, see the end of §5.1) using the operations of edge subdivision and its inverse.

5.2.3. (a) The 2-hypergraph with vertices $0, 1, \dots, n$ and faces $\{0, 1, 2\}, \{0, 2, 3\}, \dots, \{0, n-1, n\}$ is homeomorphic to complete 2-hypergraph with three vertices.

(b) The same for the set of faces $\{0, 1, 2\}, \{0, 2, 3\}, \dots, \{0, n-1, n\}, \{0, n, 1\}$.

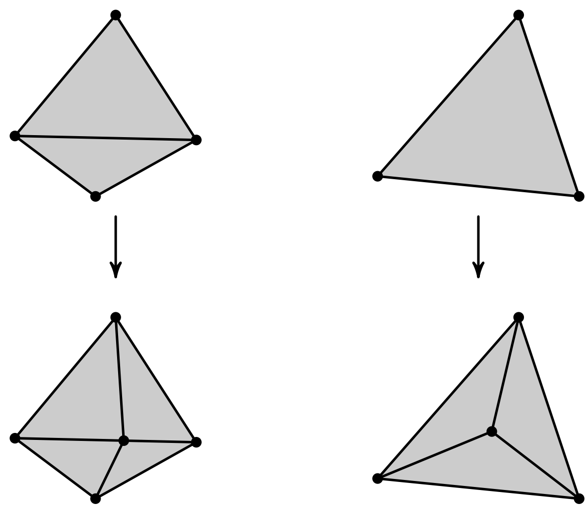


Figure 5.2.2. Subdivision of an edge and a face

(c) The 2-hypergraphs in each separate column of Figure 2.1.1 are homeomorphic to each other (for some triangulation of square), while the 2-hypergraphs from different columns are not.

Hint: the material of the following sections can be used in order to prove that certain 2-hypergraphs are not homeomorphic.

(d)* Any two triangulations of triangle are homeomorphic.

Theorem 5.2.4. *(a) Two-dimensional hypergraphs are homeomorphic if and only if their geometric realizations are homeomorphic.*

(b) The 2-hypergraphs corresponding to different triangulations of the same 2-manifold in $\mathbb{R}^m$ (see §4.5) are homeomorphic.

This is an important statement ('Hauptvermutung'). It illustrates the connection between the notions of 'combinatorial' homeomorphism of 2-hypergraphs and 'topological' homeomorphism of their geometric realizations.

Theorem 5.2.4 is neither proved nor used in this book. This result is nontrivial even when one of the 2-hypergraphs is a triangle (Assertion 5.2.3 (d)) or a sphere with handles (§2.1).[15]

[15] Be careful: visually intuitive explanations of this and analogous results might not be proofs! For example, in [Pr14, proof of Theorem 11.5] the following things are not defined: 'surface edges', 'piecewise linear graph on the surface', and 'transverse intersection of edges'. To overcome this, one needs a version of Triangulation Theorem 4.6.4. An easier way is to prove the equality of the Euler characteristics not for arbitrary closed two-dimensional surfaces, but for the examples in question,

A *two-dimensional polyhedron* is a homeomorphism class of 2-hypergraphs. An analogue of Remark 5.2.1.c is valid for 2-hypergraphs.

A graph is said to be **embeddable** (or *realizable*) in a 2-hypergraph if a certain 2-hypergraph homeomorphic to the given one contains a graph homeomorphic to the given one.

5.3. Recognition of 2-hypergraphs being homeomorphic

Theorem 5.3.1. *There exists an algorithm deciding whether*

(a) *a 2-hypergraph is homeomorphic to the sphere S^2;*

(b) *two arbitrary 2-hypergraphs are homeomorphic.*

Theorem 5.3.1 is neither proved nor used in this book. Theorem 5.3.1 (a) follows from Theorem 5.3.3 on sphere recognition. The latter and Theorem 5.6.2 on classification of surfaces can be regarded as important special cases of Theorem 5.3.1 (b), which suggest how to prove the general case (see Problem 5.5.2 (b) and the notion of *attaching word* before Problem 10.5.4). Let us introduce the notions required to state these special cases.

A 2-hypergraph is called **connected**, if any two vertices can be joined by a path along the edges.

A 2-hypergraph is called **locally Euclidean**, if for every its vertex v, the faces containing v form a chain

$$\{v, a_1, a_2\},\ \{v, a_2, a_3\},\ \dots,\ \{v, a_{n-1}, a_n\} \quad \text{or}$$
$$\{v, a_1, a_2\},\ \{v, a_2, a_3\},\ \dots,\ \{v, a_{n-1}, a_n\},\ \{v, a_n, a_1\}$$

for some pairwise distinct vertices $a_1, \dots, a_n$.

E.g. 2-hypergraphs that are triangulations of surfaces in Figure 2.1.1, or of a disk with ribbons (§ 1.5), are locally Euclidean.

5.3.2. (a) For which n is the complete 2-hypergraph on n vertices locally Euclidean?

(b) There is a 2-hypergraph that is not locally Euclidean but with each edge incident to two faces.

(c) A 2-hypergraph homeomorphic to a locally Euclidean one is locally Euclidean itself.

---

and take in place of G_2 the specific triangulation that we constructed (this suffices for Theorem 11.5). Even after this, the phrase 'Graph G_1 can be modified in order to...' in not obvious; it seems that this fact is as difficult as Theorem 5.2.4.b.

The **Euler characteristic** of a 2-hypergraph K with V vertices, E edges and F faces is the number

$$\chi(K) := V - E + F.$$

The methods for computing the Euler characteristics are presented in §5.4.

Theorem 5.3.3 (Sphere recognition). *A 2-hypergraph is homeomorphic to the sphere S^2 if and only if it is connected, locally Euclidean, and its Euler characteristic equals* 2.

A sketch of the proof is presented in §5.5. For higher dimensional analogues see §10.1.

5.4. Euler characteristic of a 2-hypergraph

5.4.1. (a—i) Find the Euler characteristic of the triangulations constructed in your solution of Problem 4.6.3.

The Euler characteristic can be computed easier (for example, in Problem 5.4.1) not by definition but using its properties. They are presented below.

5.4.2. (a) (Riddle) Guess and prove the formula for the Euler characteristic of a union.

(b) Cutting a hole decreases the Euler characteristic by 1.

5.4.3. (a) The Euler characteristics of homeomorphic 2-hypergraphs are equal.

(b) The triangulations of spheres with distinct numbers of handles, which you constructed in Problem 4.6.3 (e), are not homeomorphic. (This fact is not obvious since seemingly different shapes might happen to be homeomorphic, see §2.7 and especially §2.8.)

5.4.4. Find the Euler characteristic of

(a) the disk with m Möbius bands (see Figure 2.8.1 and definition thereafter);

(b) the Klein bottle with g handles;

(c) the projective plane with g handles;

(d) the sphere with m Möbius bands attached;

(e) the sphere with m Möbius bands attached, and h holes cut.

5.4.5. Which 2-hypergraphs from Problem 5.4.4 (b, c, d) are homeomorphic?

5.4.6. Denote by K a triangulation of 2-manifold.

(a) *The Riemann Theorem.* Suppose $g+m$ pairwise disjoint loops are chosen in K so that cutting along any of the first g of them gives two boundary circles, and cutting along any of the last m of them gives one boundary circle. If $2g+m>2-\chi(K)$ then the union of these loops splits the triangulation.

(b) *The Euler inequality.* A connected subgraph G of K with V vertices and E edges splits the triangulation into at least $E-V+\chi(K)$ parts. In other words, $\chi(G)\geqslant\chi(K)$.

(c)* What is the minimum number of parts in a splitting of K by a subgraph with V vertices, E edges and s connected components?

The Riemann Theorem 5.4.6 (b) generalizes the Riemann Theorem 2.3.5 (a) and is implied by the following assertion (cf. [Pr14, § 11.4]).

5.4.7. Cut a triangulation of 2-manifold along a non-splitting curve that is built from some edges of the triangulation. The resulting triangulation of 2-manifold has the same Euler characteristic as the original one.

Answers to 5.4.1. (a, b) 0; (c, h) 2; (d, i) 1; (e, f, g) $2g$.

5.5. Proof of Sphere Recognition Theorem 5.3.3

Theorem 5.3.3 is reduced to its version for thickenings (Proposition 2.7.7.c) using Assertion 5.5.1.d.

The **boundary** ∂N of a locally Euclidean 2-hypergraph N is the union of all its edges each of which is contained in a single face.

5.5.1. (a) The boundary is a disjoint union of cycles, i.e., graphs homeomorphic to a triangle.

(b) The number of boundary circles is the same for homeomorphic locally Euclidean 2-hypergraphs.

(c) 2-Hypergraphs ‘representing’ annulus and Möbius band are not homeomorphic.

(d) Let $K\cong L$ be triangulations of 2-manifolds. Let S_K and S_L be connected components of ∂K and ∂L, respectively. Then $K\cup_{S_K}\text{con}\,S_K\cong L\cup_{S_L}\text{con}\,S_L$.

Proof of Theorem 5.3.3. The 'only if' part follows from Assertion 5.3.2 (c) and Assertion 5.4.3 (a), along with the result of Problem 5.4.1 (a).

The 'if' part is harder. (Being closed and orientable, see §§5.6, 5.7, is also needed for this part, but is implied by the other hypothesis in Theorem 5.3.3.) Denote by K the given triangulation of 2-manifold. Denote by V, E, F, n the number of its vertices, edges, faces, and boundary circles.

Take the union M of *caps* and *ribbons* corresponding to vertices and edges of the triangulation. (See an informal explanation near Fig. 1.6.3 (left) and a rigorous definition below.) By Assertions 5.2.3.a,b any patch, any ribbon, and any cap is homeomorphic to D^2. Hence M is a thickening of the union of edges. Clearly, M has $F+n$ boundary circles. Since $V-E+F=2$, by and connectivity and Assertions 1.6.4.c, 1.6.5 we have $n=0$. Then by Proposition 2.7.7.c M is homeomorphic to the disk with $h-1=F-1$ holes. The thickening M is K with F holes. Hence by Assertion 5.5.1.d $K \cong S^2$. □

The *barycentric subdivision G' of a graph G* is obtained by subdividing all its edges. The *barycentric subdivision of a face of a 2-hypergraph* is the result of the replacement of the face by six new faces that are obtained by drawing the 'medians' in the triangle representing the face (Figure 5.5.1). The **barycentric subdivision K' of a 2-hypergraph** K is the result of the barycentric subdivision of all its faces.

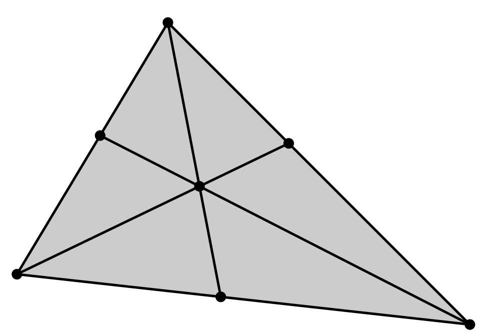


Figure 5.5.1. Barycentric subdivision

Since the barycentric subdivision can be obtained via edge subdivisions, $K' \cong K$.

Denote by K'' the 2-hypergraph obtained from a 2-hypergraph K by barycentrically subdividing it twice. We will use the following notation (see Figure 1.6.3 on the left, where a triangulation of 2-manifold K is shown):

• a **cap** is the union of the faces of the triangulation K'' that contain a certain vertex of the triangulation K;

• a **ribbon** is the union of the faces of the triangulation K'' that intersect a certain edge of the triangulation K but avoid the vertices of the triangulation K;

• a **patch** is a connected component of the union of the remaining faces of the triangulation K'', i.e., the union of all faces of K'' belonging neither to caps nor to ribbons.

5.5.2. (a) There exists an algorithm that takes a 2-hypergraph homeomorphic to S^2 and outputs a sequence of edge subdivisions and inverse operations that transform the 2-hypergraph to S^2.

(b) There exists an algorithm recognizing whether a 2-hypergraph is homeomorphic to the book with 3 pages.

5.6. Classification of surfaces

Lemma 5.6.1 (homogeneity)**.** *Let p and q be any two faces of a locally Euclidean 2-hypergraph K. If both p and q are disjoint from ∂K, then $K-p$ and $K-q$ are homeomorphic.*

From a locally Euclidean 2-hypergraph one can obtain other locally Euclidean 2-hypergraphs by

• *cutting a hole* (more precisely, removing a face disjoint from the boundary; this is well-defined by Homogeneity Lemma 5.6.1),

• *attaching a handle*, see Figure 2.1.5 (more precisely, cutting two holes and attaching to their boundary a certain triangulation of the lateral surface of a cylinder), or

• *attaching a Möbius band*, or a *cross-cap*, see Figure 5.6.1.

Before we prove in Assertions 5.8.1, 5.8.2 that attaching a handle, and a Möbius band operations are well-defined, we do not assume that.

Theorem 5.6.2 (Classification of surfaces)**.** *Every connected locally Euclidean 2-hypergraph is homeomorphic to a triangulation of either a sphere with handles and holes, or a sphere with Möbius bands (attached to the sphere) and holes. These triangulations are not homeomorphic*

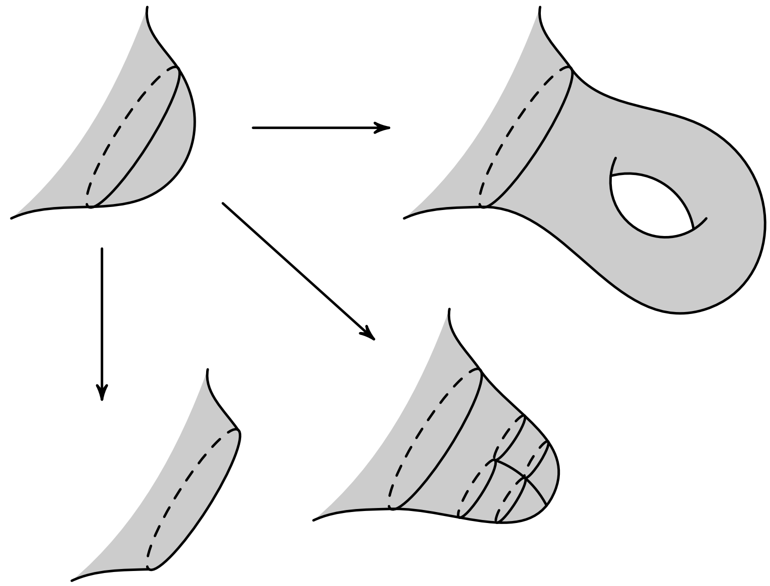


Figure 5.6.1. Attaching a handle and a Möbius band; cutting a hole

for different triples (ε, g, h)*, set to* $(0, g, h)$ *for a sphere with* g *handles and* h *holes, and to* $(1, g, h)$ *for a sphere with* g *Möbius bands and* h *holes.*

A proof is sketched in 5.7. It gives an algorithm detecting homeomorphism between a 2-hypergraph and the aforementioned classes (ε, g, h) of 2-hypergraphs, as well as an algorithm detecting homeomorphism between locally Euclidean 2-hypergraphs. Compare to Theorem 6.7.6.

A *piecewise linear (PL) two-dimensional manifold* is a homeomorphism class of locally Euclidean 2-hypergraphs. If there is no ambiguity with the notion of 2-manifolds from §4.5, we say '**2-manifold**' as a shorthand for 'PL two-dimensional manifold'.

From now on, instead of the term 'locally Euclidean 2-hypergraph' we use a common term '**triangulation of 2-manifold**'. Earlier it would not be convenient for a beginner, since in the study of 2-manifolds from the piecewise linear viewpoint, the primary object is a 2-hypergraph, and not a 2-manifold.

A locally Euclidean 2-hypergraph is called **closed**, if each its edge belongs to two faces (as opposed to one; that is, for each vertex the second option from the definition of being locally Euclidean takes place). For instance, in Figure 2.1.1 only the four last 'hypergraphs' are closed.

By 'sealing' (capping with a disk) each boundary circle of a disk with ribbons one obtains a closed locally Euclidean 2-hypergraph.

5.7. Orientable triangulations of 2-manifolds

An *orientation* of a two-dimensional triangle is an ordering of its vertices up to an even permutation. An orientation is conveniently pictured by a closed curve with an arrow inside the triangle (or by an ordered pair of non-collinear vectors).

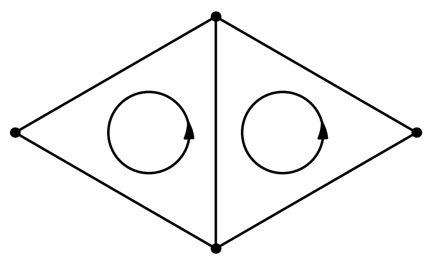


Figure 5.7.1. Agreeing orientations

An *orientation* of a triangulation of 2-manifold is a choice of face orientations *agreeing* with one another on each edge contained in two faces, in the sense that the orientations of adjacent faces induce the *opposite* directions on their common edge (Figure 5.7.1).

A triangulation of 2-manifold is called *orientable* if there exists an orientation of it[16].

It is not difficult to see that *a smooth* 2-*manifold is orientable in the sense of* §4.10 *if and only if it has an orientable triangulation.*

5.7.1. (a) Homeomorphic triangulations of 2-manifold are simultaneously orientable or non-orientable.

(b) The sphere, the torus, a sphere with handles are orientable.

(c) The Möbius band, the Klein bottle, the projective plane (Figure 2.1.1) are non-orientable.

(d) The torus is not homeomorphic to the Klein bottle.

5.7.2. (a) The orientability is preserved when cutting a hole.

[16]The notion of orientability is 'impossible' to introduce for arbitrary 2-hypergraphs (think why), but is could be introduced for 2-hypergraphs each of whose edges is contained in at most two faces.

(b) A disk with ribbons (see §1.5) is orientable if and only if no ribbon is twisted.

5.7.3. (a) A triangulation of 2-manifold is orientable if and only if no homeomorphic triangulation contains a triangulation of Möbius band.

(b)* Does there exist a non-orientable triangulation of 2-manifold that does not contain a triangulation of Möbius band?

(c) A closed triangulation of 2-manifold is orientable if and only if there exists a collection of faces of its barycentric subdivision such that every edge of the subdivision is incident to exactly one face of the collection.

The criterion from part (a) does not give an *algorithm* recognizing orientability. Such an algorithm is obtained from the following strengthening of the criterion: one needs to replace the words 'no homeomorphic triangulation contains' with the words 'its second barycentric subdivision does not contain'. However, the corresponding algorithm is slow (has 'exponential complexity'). A polynomial algorithm is presented in §6.1 (or can be obtained from part (c)).

Sketch of the proof of Surface Classification Theorem 5.6.2. The lack of homeomorphism (i.e., the second assertion of the theorem) is proved using orientability, the number of connected boundary components, and the Euler characteristic; that is, the lack of homeomorphism follows from Assertions 5.7.1 (a), 5.5.1 (b), 5.4.3 (a) and the results of Problems 5.4.4 (e), 5.4.1 (g).

The proof of homeomorphism (i.e., the first assertion of the theorem) is analogous to that of Theorem 5.3.3; that is, the proof of homeomorphism follows from Assertions 2.7.9 (b), 2.8.11 (b), and Assertions 5.7.2 (a, b). □

In Theorem 5.6.2, the number g of handles is called the *orientable genus* of a triangulation of 2-manifold. It can be found from the equation $2-2g-h=\chi$. The number m of Möbius bands is called the *non-orientable genus* and can be found from the equation $2-m-h=\chi$. See Problems 5.4.1 (g) and 5.4.4 (a).

By Theorem 5.6.2 (or by Assertion 6.7.3 (b)) the Euler characteristic of a closed orientable triangulation of 2-manifold is even.

5.8. Attaching a handle or a Möbius band is well-defined

The 2-hypergraphs obtained from a given locally Euclidean one by attaching a handle or a Möbius band, are unique up to a homeomorphism. The fact that the result of attaching a handle or a Möbius band does not depend on the disks to which the handle is attached, also follows from Homogeneity Lemma 5.6.1. However, the independence from the attaching map is a priori not obvious (though it is usually not discussed in textbooks). Indeed, the result of gluing two quadrilaterals $ABCD$ and $A'B'C'D'$ to one another along the edges AB and $A'B'$, CD and $C'D'$, depends on the choice of attaching map (i.e., on the choice of directions along the edges used for gluing). Moreover, in the following paragraph we define a analogous operation of 'attaching a candle', which is not well-defined up to a homeomorphism.

A *candle* is the union of a quadrilateral $ABCD$ with segments CC_1, DD_1, DD_2. Given a surface M and an arc XY in its boundary, *attaching a candle* is taking the union of M and the candle, and identifying the arcs AB and XY. This can be done in two ways: identify A with X, and B with Y, or vice versa. The two thus obtained shapes are homeomorphic when M is a disk, but any homeomorphism between them reverses the orientation on the disk. The two thus obtained shapes are not homeomorphic when M is a disk with candle.

For higher-dimensional manifolds, the result of the attaching an analogue of a handle may depend on the choice of gluing (a remark for experts: $\mathbb{C}P^2\#\mathbb{C}P^2$ and $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ are not homeomorphic).

In order to have the independence of the way of gluing one needs the object being attached to be 'symmetric'. For attaching a handle, the independence follows from Assertion 5.8.1 (b) (or 5.8.1 (c, d)), while for attaching a Möbius band this follows from Assertion 5.8.2 (b).

5.8.1. (a) The quadrilateral whose antipodal sides are endowed with 'agreeing' directions is homeomorphic to the quadrilateral whose antipodal sides are endowed with the opposite 'agreeing' directions. Formally, there exists a refinement K of the 2-hypergraph with vertices 1, 2, 3, 4 and faces $\{1, 2, 3\}, \{1, 3, 4\}$, and an isomorphism $K \to K$, sending 1, 2, 3, 4 to 2, 1, 4, 3, respectively.

(b) The annulus whose boundary circles are endowed with 'agreeing' directions is homeomorphic to the annulus whose boundary circles are endowed with the opposite 'agreeing' directions.

(c) The torus with a hole and with a choice of direction along the boundary circle is homeomorphic to the torus with a hole and with the opposite choice of direction along the boundary circle.

(d) The result of attaching a handle is homeomorphic to the result of the operation in Figure 5.6.1, at the top, and is homeomorphic to the result of cutting out square $ABCD$ and gluing directed edges AB and DC, AD and BC.

In order to prove Assertion 5.8.1 (d), as well as the following claim, you can exhibit a sequence of pictures, as in §2.7.

5.8.2. (a) The projective plane (see Example 4.5.3) with a hole is homeomorphic to the Möbius band.

(b) The Möbius band with a hole and with a choice of direction along its boundary circle is homeomorphic to the Möbius band with a hole and with the opposite choice of direction along the boundary circle.

(c) The result of attaching a Möbius band is homeomorphic to the result of cutting a hole and identifying the antipodal points of its boundary circle.

(d) The Klein bottle is homeomorphic to the sphere with two Möbius bands attached.

(e) The torus with a Möbius band attached is homeomorphic to the Klein bottle with a Möbius band attached.

5.9. Regular neighborhoods and cellular subgraphs

The notion of a regular neighborhood is informally explained near Fig. 1.6.3 (left). An example of a regular neighborhood of a subgraph in a hypergraph one can take the union U of caps and ribbons corresponding to the vertices and the edges of the subgraph; that is, the union of those faces of the second barycentric subdivision that intersect the subgraph. Let us give the general definition.

A hypergraph L is obtained from a complex K by an *elementary collapse* if $K = L \cup \sigma$ and $L \cap \sigma = \partial\sigma - \operatorname{Int}\tau$ for some faces σ, τ of K such that $\tau \subset \partial\sigma$. A hypergraph K *collapses to* L (notation: $K \searrow L$) if there exists a sequence of elementary collapses $K = K_0 \searrow K_1 \searrow \ldots \searrow K_n = L$. A hypergraph K is *collapsible* if it collapses to a point.

A **regular neighborhood** of a subhypergraph A in a hypergraph K is a subhypergraph of some subdivision of K which contains A and collapses to A.

5.9.1. (a) The cone of any graph is collapsible.

(b) Construct three hypergraphs none of which collapses to a hypergraph homeomorphic to any other.

(c) The Euler characteristic is preserved under collapses.

(d) The Euler characteristic of a subgraph and of its regular neighborhood in a 2-hypergraph are equal.

(e) The union U is indeed a regular neighborhood.

The *complement $G - H$ in a graph G to a vertex set H* is formed by the vertices of the graph G that do not lie in H, and the edges of the graph G without endpoints in H.

Let G be a subgraph of a hypergraph K (i.e., a subgraph of the graph formed by the vertices and the edges of the hypergraph K). The *complement* $K - G$ is formed by the faces of the hypergraph K that do not intersect G.

The following definition formalize the construction of gluing a hypergraph out of a square (Figure 2.1.1) or a polygon.

Denote by $|K|$ the geometric realization of a graph K or a hypergraph K.

A vertex set A in a graph K is called (topologically) *cellular* if each connected component of $|K| - |A|$ is homeomorphic (topologically) to the open interval. We will be using the following (equivalent) combinatorial definition. A vertex set H in a graph G is called **cellular** if each connected component of the complement $G'' - H$ is homeomorphic to a segment each of whose endpoints belongs to an edge of the graph G'' incident to a vertex from H.

A subgraph A in a hypergraph K is called (topologically) *cellular* if each connected component of $|K| - |A|$ is homeomorphic (topologically) to the open disk. We will be using the following (equivalent) combinatorial definition. A subgraph G in a hypergraph K is called **cellular** if each connected component C of the complement $K'' - G''$ is homeomorphic to a disk[17] each of whose boundary edges lies in a face of the hypergraph K'' intersecting G. For example,

- a point in the sphere is cellular whereas a point in the torus is not;
- the union of the edges of a hypergraph is cellular.

5.9.2. *The Euler formula.* If K is a 2-hypergraph, and $G \subset K$ is a connected cellular subgraph with V vertices and E edges, then $V - E + F = \chi(K)$, where F is the number of connected components of the complement $K' - G'$.

Hint. The formula follows from the inclusion-exclusion principle (Problem 5.4.2), since $\chi(D^2) = 1$.

5.9.3. (a) If a connected graph can be embedded to the sphere with g handles, then it is homeomorphic to a cellular subgraph of a sphere with at most g handles.

(b) The same for spheres with Möbius bands attached.

[17] In many applications of the notion 'cellular', the condition 'homeomorphic to a disk' could be replaced by a weaker condition $\chi(C) = 1$, which is easier to verify. If the component C is locally Euclidean, then the cellularity condition is equivalent to this weaker condition as well as to the following one: the component C is split by any polygonal line with the endpoints on the boundary of C.

§ 6. Homology of two-dimensional manifolds

> And the leap is not — is not what I think you sometimes see it as — as breaking, as acting. It's something much more like a quiet transition after a lot of patience and — tension of thought, yes — but with that [enlightenment] as its discipline, its orientation, its truth. Not confusion and chaos and immolation and pulling the house down, not something experienced as a great significant moment.
>
> I. Murdoch, The Message to the Planet.

6.1. Orientability criterion

The definitions of a piecewise linear (PL) 2-manifold and its triangulation are presented in §5.6. The definitions of a smooth 2-manifold and its triangulation are presented in §4.5. Either of these two approaches can be used for this section. However, a careful treatment is only presented in the PL language in some places.

The definition of orientability of a triangulation is given in §5.7. There is a nice and simple criterion of orientability: 'does not contain a Möbius band' (a precise formulation is given in Problem 5.7.3 (a)). There is a simple algorithm recognizing orientability as follows. It suffices to check the orientability of each connected component. First, orient a face of the component arbitrarily. Then at each step orient a face adjacent to any of the faces already oriented, until all faces are oriented, or two adjacent faces with disagreeing orientations are found.

In this section we will give an algebraic criterion of orientability, which, basically, is merely a reformulation of the definition of orientability in algebraic language. However, this criterion is important not on its own but rather as an illustration of obstruction theory. Moreover, similar considerations lead to Assertion 6.1.2 (b), and are applied in the *classification of thickenings* [Sk]. Cf. §6.8, §4.11.

Theorem 6.1.1 (Orientability)**.** *A 2-manifold N is orientable if and only if its first Stiefel–Whitney class $w_1(N) \in H_1(N, \partial)$ is zero.*

The group $H_1(N, \partial)$ and the class $w_1(N)$ are defined later. They arise naturally and can be defined rigorously in the process of *inventing* the Orientability Theorem, which we will start in a moment. The computation of the group $H_1(N)$ is given in §6.4.

In this section the word ‘group’ can be regarded synonymous with the word ‘set’ (with the exception of Problems 6.2.5, 6.5.2, and §6.7). The constructions will remain interesting.

6.1.2. (a) Draw a closed non-self-intersecting curve on the disk with three Möbius bands, so that the complement to the curve is orientable.

(b) Any closed 2-manifold admits a closed non-self-intersecting curve whose complement is orientable. (More formally: for any closed triangulation of 2-manifold there is a subgraph of a homeomorphic triangulation T, such that the subgraph is homeomorphic to the circle, and the complement to the image of this subgraph in the second barycentric subdivision of T, see §5.9, is orientable.)

6.2. Cycles

The notion of a cellular decomposition of a hypergraph formalizes the examples ‘glued of polygons’ from Example 5.1.1.c. A **cellular decomposition** of a hypergraph K is a pair $K_0 \subset K_1 \subset K$ of its subhypergraphs in which K_1 is a cellular subgraph in K and K_0 is a cellular set of vertices in K_1 (see §5.9 for definitions). The graph K_1 is called the *one-dimensional skeleton* of the cellular decomposition. *Edges* and *faces* of a cellular decomposition $K_0 \subset K_1 \subset K$ are the connected components of the complement $K_1'' - K_0$ and connected components of the complement $K'' - K_1''$, respectively.

Many constructions are done more conveniently for cellular decompositions rather than for hypergraphs, since many ‘interesting’ hypergraphs have ‘many’ faces, but admit ‘economical’ cellular decompositions. For computations, it is more convenient to draw cellular decompositions rather than more cumbersome polygonal decompositions. Triangulations are special cases of cellular decompositions. Other examples are shown in Figure 2.1.1. In the following considerations, except the examples, the reader may substitute cellular decompositions with triangulations.

In this section T is a cellular decomposition of a 2-manifold N, while o is a choice of orientations on the faces of T.

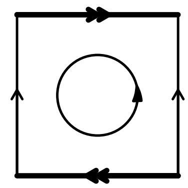


Figure 6.2.1. Collection o of orientations, and the obstruction cycle $\omega(o)$

Color an edge of a cellular decomposition T in red if the orientations of the incident faces *do not agree* along this edge, i.e., induce the same direction on the edge. The collection of the red edges is called the **obstruction cycle** $\omega(o)$.

For instance, in Figure 6.2.1 the Klein bottle is represented as a square with glued sides, i.e., it is decomposed into a single polygon. The faces incident to the horizontal edge from the two sides, coincide. But their (or rather its) orientations do not agree along the edge. Besides, the orientation of the only face agrees with itself along the vertical edge. Hence, in Figure 6.2.1 the obstruction cycle consists of a single horizontal edge (shown in bold).

So, if a decomposition is not a triangulation, then the orientation of a face incident to an edge from two sides does not have to agree with itself along this edge. Moreover, a pair of faces (coinciding or not) might have orientations that agree along one edge but disagree along another edge.

6.2.1. (a) For each edge of the single-face cellular decomposition of the Möbius band (i.e., of the representation of the Möbius band as a square with glued sides, see the third column in Figure 2.1.1), find out if the orientation of the only face agrees with itself along this edge.

(b) The same question for the projective plane (Figure 2.1.1).

6.2.2. (a) Draw the obstruction cycle for the single-face cellular decomposition of the Möbius band.

(b) The same for the projective plane.

Many of the following facts (for example, Problems 6.2.3 (a, b)) can be first proved for triangulations and then for cellular decompositions.

6.2.3. (a) A collection o of face orientations determines an orientation of a cellular decomposition if and only if $\omega(o) = \varnothing$.

(b) If a 2-manifold is closed, then each vertex has an even number of incident edges of the obstruction cycle (by convention, a loop counts with multiplicity two).

(c) The complement to the obstruction cycle $\omega(o)$ (formally, the union of the faces of the second barycentric subdivision that do not intersect $\omega(o)$) is orientable.

(d) For any closed triangulation of 2-manifold, it is possible to orient the (two-dimensional) faces of its barycentric subdivision so that the orientations of any two adjacent faces do not agree.

A **cycle** (homological, one-dimensional, mod 2) in a graph (or in a hypergraph) is an unordered collection of its edges such that any vertex has an even number of incident edges from the collection. The words 'homological', 'one-dimensional' and 'mod 2' will be omitted. Cycles in the sense of graph theory will be called 'closed curves'.

For instance, the graphs in Figure 1.2.1 have 2, 8, and 8 cycles, respectively. The union of edges in the single-face cellular decomposition of the Klein bottle (Figure 6.2.1) is the 'figure eight', so this graph has four cycles.

6.2.4. How many cycles are there in a connected graph with V vertices and E edges?

On the set of all cycles in a given graph (or a hypergraph) consider the operation of the (mod 2) **sum** (i.e., the symmetric difference).

6.2.5. The *homology group* $H_1(G)$ of a graph G (one-dimensional, with coefficients mod 2) is the group of all cycles in the graph G.

(a) The sum of cycles is a cycle.

(b) Homeomorphic graphs have isomorphic homology groups.

(c) For a connected graph G with V vertices and E edges, one has $H_1(G) \cong \mathbb{Z}_2^{E-V+1}$.

(d) Non-self-intersecting closed curves in a graph G generate $H_1(G)$.

6.3. Homologous cycles

If $\omega(o) \neq \varnothing$, then o does not determine an orientation of a cellular decomposition T. All is not lost though: one can try to modify o in order to make the obstruction cycle empty. For this, let us find out how $\omega(o)$ depends on o. The answer is formulated conveniently using the mod 2 sum (i.e., the symmetric difference) of edge sets in an arbitrary graph.

The (homological) **boundary** ∂a of a face a in a hypergraph is the set of edges of the geometric boundary of this face.

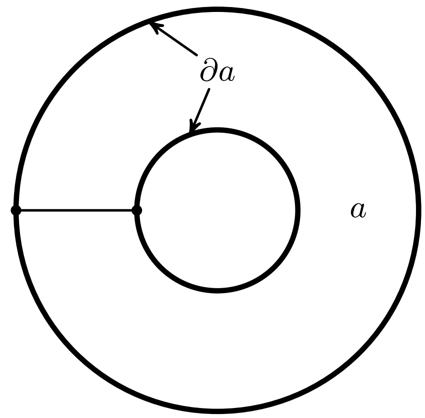


Figure 6.3.1. Homological (algebraic) boundary of a complicated face

For a face of a cellular decomposition, the definition is more involved. The (homological) **boundary** ∂a of a face a is the set of all those edges of the geometric boundary of the face that are adjacent to the face just from one side (Figure 6.3.1).

As for cycles, the word 'homological' will be omitted. For the single-face cellular decomposition of the Klein bottle (Figure 6.2.1) the boundary of the only face is empty.

6.3.1. (a) What is the boundary of the only face in the single-face cellular decomposition of the projective plane (see Figure 2.1.1)?

(b) The boundary of a face is a cycle.

(c) When the orientation of single face a is reverted, the cycle $\omega(o)$ changes to the sum with the boundary of that face: for the resulting collection o' of orientations one has $\omega(o') - \omega(o) = \partial a$.

(d) When the orientations of *several faces* $a_1, \ldots, a_k$ are reverted, the cycle $\omega(o)$ changes to the sum with the boundaries of these faces: for the resulting collection o' of orientations one has

$$\omega(o') - \omega(o) = \partial a_1 + \ldots + \partial a_k.$$

Two cycles are called **homologous** (or *congruent modulo boundaries*), if their difference is the sum of the boundaries of several faces.

6.3.2. (a) When the collection o of orientations is changed, the obstruction cycle $\omega(o)$ is replaced by a homologous cycle.

(b) If $\omega(o)$ is a boundary, then it is possible to change o to o' so that $\omega(o') = \varnothing$.

Proposition 6.3.3. *A closed triangulation of* 2*-manifold is orientable if and only if some* (*or, equivalently, any*) *obstruction cycle is homologous to the empty cycle.*

Sketch of the proof. It is clear that this condition is necessary for orientability. Conversely, suppose that some obstruction cycle is homologous to the empty cycle. Then there exists a collection o of face orientations of which $\omega(o)$ is the boundary. Then by Assertion 6.3.2 (b) it is possible to change o to o' so that $\omega(o') = 0$. Therefore, the triangulation is orientable. □

6.3.4. (a) Any two cycles in the single-face cellular decomposition of the sphere (see Figure 2.1.1) are homologous.

(b) The boundary circles on the torus with two holes are homologous (for any cellular decomposition).

(c) The boundary circle of the Möbius band is homologous to the empty cycle (for any cellular decomposition).

6.3.5. For the single-face cellular decomposition of the torus (Figure 2.1.1)

(a) the 'meridian' cycle is not homologous to the empty cycle;

(b) different cycles are not homologous.

6.3.6. (a) In the single-face cellular decomposition of the projective plane (Figure 2.1.1) different cycles are not homologous.

(b) In the complete hypergraph on 9 vertices any two cycles are homologous.

(c) Any two cycles are homologous in the single-face cellular decomposition of the Zeeman dunce hat.

(The *Zeeman dunce hat* is obtained from a triangle ABC by gluing all three its sides directed so that $\overrightarrow{AB} = \overrightarrow{AC} = \overrightarrow{BC}$.)

6.3.7. (a) Homology is an equivalence relation on the set of cycles.

(b) Any cycle in a connected triangulation T of 2-manifold is homologous to a closed non-self-intersecting polygonal line in some subdivision of T.

(c) Is the same true for an arbitrary connected hypergraph T?

6.3.8. (a) The sum of the boundaries of all faces of a closed triangulation of 2-manifold is empty.

(b) The sum of the boundaries of all faces of a triangulation of 2-manifold equals to the boundary.

(c) The sum of the boundaries of any proper subset of faces of a connected closed triangulation of 2-manifold is non-empty.

6.3.9. (a) Any cycle in a hypergraph is homologous to some cycle in any cellular graph in this hypergraph.

(b) If two cycles in a cellular decomposition of a hypergraph are homologous in the hypergraph, then they are homologous in the cellular decomposition as well.

6.4. Homology and the first Stiefel—Whitney class

Recall the definitions, motivated and introduced in the previous sections. A **cycle** in a hypergraph is an unordered collection of edges such that every vertex is incident to an even number of them. The **boundary** ∂a of a face a in a hypergraph is the collection of all edges of the geometric boundary of this face. Two cycles are called **homologous** if their difference is the sum of several boundaries.

The **homology group** $H_1(K)$ (one-dimensional, with coefficients mod 2) of a hypergraph K is the group of cycles up to homology.

The homology group appears in solutions of specific problems (e.g. in checking orientability, see §6.2-§6.3). It is important that the homology group is defined in a short way regardless of the problems, and for arbitrary hypergraphs.

6.4.1. (a) On the set $H_1(K)$ the sum operation is well-defined by the formula $[\alpha] + [\beta] = [\alpha + \beta]$.

(b) The set $H_1(K)$ with this operation is a group.

(c) The homology groups of homeomorphic hypergraphs are isomorphic. More precisely, if a hypergraph K is obtained from a hypergraph L by edge subdivision, then the naturally defined homomorphism $H_1(L) \to H_1(K)$ is an isomorphism.

The *homology group* $H_1(T)$ (one-dimensional, with coefficients mod 2) of a cellular decomposition T of a hypergraph is defined analogously. By definition, the **boundary** ∂a of a face a of a cellular decomposition of a hypergraph is the collection of those edges of the geometric boundary of a that are adjacent to a from an odd number of sides (Figure 6.3.1).

6.4.2. (a) For the aforementioned single-face cellular decompositions of the sphere, the torus, the projective plane, the Klein bottle (Figures 2.1.1 and 6.2.1) the number of elements in $H_1(T)$ equals 1, 4, 2, 4, respectively.

(b) For a cellular decomposition T of a hypergraph K the following holds: $H_1(T) \cong H_1(K)$.

The *homology group* $H_1(N)$ (one-dimensional, with coefficients mod 2) of a 2-manifold N is the group $H_1(T)$ for any triangulation T of the manifold (or even for any cellular decomposition T of a triangulation). The homology group is well-defined by Assertion 6.4.1 (c) (and 6.4.2 (b)).

The **first Stiefel—Whitney class** of a cellular decomposition T of a closed triangulation of 2-manifold is the homology class of an obstruction cycle:

$$w_1(T) := [\omega(o)] \in H_1(T).$$

This is well-defined by Assertion 6.3.2 (a).

The **first Stiefel—Whitney class** of a closed 2-manifold N is the first Stiefel—Whitney class of any triangulation T of 2-manifold N (or even of any cellular decomposition T of a triangulation): $w_1(N) := w_1(T)$. This is well-defined in the following sense (see also Assertion 6.4.2 (b)).

6.4.3. The map from Assertion 6.4.1 (c) sends $w_1(L)$ to $w_1(K)$.

Orientability Theorem 6.1.1 is a reformulation of Assertion 6.3.3.

6.5. Computations and properties of homology groups

In the arguments involving homology classes of cycles, it is convenient first to work with representing cycles, and then prove that the actual choice of the representatives does not play a role.

6.5.1. Find the homology group and draw the curves forming its basis for your preferred cellular decomposition of

(a) the sphere with g handles;

(b) the sphere with g handles and h holes;

(c) the sphere with m Möbius bands;

(d) the sphere with m Möbius bands and h holes.

6.5.2. If T is a cellular decomposition of a connected closed 2-manifold, then $H_1(T) \cong \mathbb{Z}_2^{2-\chi(T)}$.

6.5.3. (a) If M and N are closed 2-manifolds, then $H_1(M \# N) \cong$ $\cong H_1(M) \oplus H_1(N)$ (the operation $\#$ of connected sum is defined analogously to Figure 5.6.1).

(b) Does that formula hold for non-closed 2-manifolds M and N?

6.5.4. (a) For any hypergraphs K and L sharing at most one point, $H_1(K \cup L) \cong H_1(K) \oplus H_1(L)$.

(b) Does that formula hold if there are two common points?

6.5.5. (a) For any connected graph K one has

$$H_1(K \times I) \cong H_1(K) \quad \text{and} \quad H_1(K \times S^1) \cong H_1(K) \oplus \mathbb{Z}_2.$$

(Come up with your own definitions of the product of a graph with the interval/the circle, or find the definitions in [Sk, Section 5.9.2 "Linear realizability of products"].)

(b) For a regular neighborhood U of a subgraph K in a hypergraph, one has $H_1(U) \cong H_1(K)$.

Let T be a cellular decomposition of a triangulation of 2-manifold N (perhaps, with a non-empty boundary). A *cycle relative to the boundary* (or a *relative cycle*, for brevity) in T is a collection of edges of T such that every non-boundary vertex is incident to an even number of the edges from the collection. Two relative cycles are said to be *homologous relative to the boundary*, if their difference is a sum of the boundaries of several faces and of some boundary edges. The homology groups $H_1(T, \partial)$, $H_1(N, \partial)$ relative to the boundary, and the classes $w_1(T) \in H_1(T, \partial)$, $w_1(N) \in H_1(N, \partial)$ are defined analogously to above.

6.5.6. (a, b) Formulate and solve the analogues of Problems 6.5.1 (b, d) for the homology groups relative to the boundary.

6.6. Intersection form: motivation

The intersection form is among the most important tools and research objects in topology and its applications. See [DZ93]. The intersection form arises naturally, for instance, when proving Assertions 6.6.1 (b) and 6.6.2. See also the Mohar formulas 2.7.8 (c) and 2.8.8 (c).

6.6.1. (a) Regular neighborhoods (see Figure 1.6.3, on the left, and §5.9) of isomorphic graphs in the same surface are not necessarily homeomorphic.

(b) Regular neighborhoods of the images of homotopic embeddings of a given graph into a 2-manifold are homeomorphic. (The definitions of homotopy are analogous to the ones given in §3.2, 3.4, 3.7.)

Two embeddings $f_0, f_1\colon G \to N$ are called *isotopic* if there exists a family $U_t\colon N \to N$ of *homeomorphisms* depending continuously on the parameter $t \in [0, 1]$, such that $U_0 = \mathrm{id}$ and $U_1 \circ f_0 = f_1$. It is clear that regular neighborhoods of the images of homotopic embeddings of a given graph into a surface are homeomorphic. In contrast, Assertion 6.6.1 (b) is not obvious.

6.6.2. On Topologist's planet, shaped as a solid torus, there are rivers Meridian and Parallel. The Little Prince and Topologist traveled around the planet along two different closed routes. The prince crossed the Meridian 9 times and the Parallel 6 times, while Topologist crossed the rivers 8 and 7 times, respectively. Then their routes had to intersect. (When crossing a river a character ends up on the other bank of the river. More rigorously, the intersection of the river and character's path are *transverse*, see the definition below.)

An heuristic argument, leading to the notion of the intersection number. Let N be a 2-manifold and let a, b be closed curves on N. Let us assume that a and b

• are subgraphs of a certain hypergraph representing N;

• are in *general position*; that is, they intersect transversely (Figure 6.6.1) in finitely many points, none of which is a self-intersection point of either a or b.

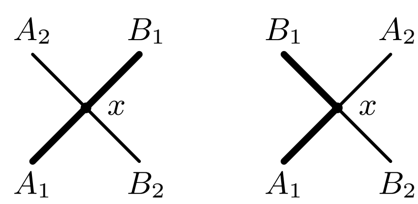


Figure 6.6.1. A transverse intersection and a non-transverse intersection

An intersection point x of two curves on a 2-manifold is called *transverse* if the curves are non-self-intersecting in a neighborhood of the point, and every sufficiently small closed curve S_x winding around x intersects the two curves in two pairs of points that *alternate* along S_x (that is, if A_1, B_1 are the intersection points of the first curve with S_x, and A_2, B_2 are the intersection points of the second curve with S_x, then these points are situated along S_x in the order $A_1A_2B_1B_2$). In other words, in order for the point x to be transverse, two short 'segments' of the first curve that are incident to x need to be on the different sides of the second curve in a small neighborhood of x, see Figure 6.6.1.

In this situation $|a \cap b|$ mod 2 does not change if a and b are replaced by homologous curves satisfying the same condition (the subgraphs, corresponding the curves, are homologous cycles; this is what is meant by 'homologous' curves).

6.7. Intersection form: definition and properties

The argument presented in the preceding section can be reworked in order to give a definition of the intersection form, based on transversality. We will present a different definition. Instead of transversality it will use the following more convenient notion of the dual cellular decomposition. For the definition of a cellular decomposition and its advantage over polygonal decompositions, see the beginning of §6.2.

The definition of the dual cellular decomposition of a cellular decomposition U of a closed 2-manifold N. The definition is obtained from the definition of the dual decomposition into polygons (see §4.8) by requiring an additional condition: the edge a^* intersects the union of edges U_1 of a cellular subgraph U in a single point that belongs to the edge a. The edge a^* is called *dual* to a. The resulting graph U_1^* is cellular for a certain triangulation of 2-manifold N. (This triangulation might be different from the one that participates in the definition of the cellular decomposition U. In the graph U_1^* there might be loops and multiple edges, even if in U_1 there are such edges.) The resulting cellular decomposition U^* is called *dual* to U.

The definition of the intersection of edge collections. Take a cellular decomposition U of a 2-manifold N (to be precise, of a representing hypergraph). Take the dual cellular decomposition U^*. For edge collections X in U, Y in U^*, set $X \cap Y$ to be the number of their intersection points mod 2.

6.7.1. (a) The intersection of edge collections is bilinear: $\alpha \cap (\gamma + \delta) = \alpha \cap \gamma + \alpha \cap \delta$ and $(\alpha + \beta) \cap \gamma = \alpha \cap \gamma + \beta \cap \gamma$.

(b) The intersection of a cycle and a boundary equals zero.

(c) The bilinear multiplication $\cap\colon H_1(U) \times H_1(U^*) \to \mathbb{Z}_2$ is well-defined via the formula $[X] \cap [Y] := X \cap Y$, for a cycle X in a decomposition U and a cycle Y in the decomposition U^*.

(d) Let $T, \overline{T}$ be closed triangulations of 2-manifold N, where $\overline{T}$ is obtained from T by a single edge subdivision. Define 'natural' maps $f\colon H_1(T) \to H_1(\overline{T})$ and $f^*\colon H_1(T^*) \to H_1(\overline{T}^*)$ (cf. Assertion 6.4.1 (c)). Prove that $\alpha \cap \beta = f(\alpha) \cap f^*(\beta)$.

In view of Problems 6.7.1 (a, c, d) one obtains the symmetric bilinear **intersection form**

$$\cap\colon H_1(N) \times H_1(N) \to \mathbb{Z}_2.$$

6.7.2. (a) Find the intersection form of the sphere with g handles (that is, find the matrix of this form in some basis of the homology group).

(b) Find the intersection form of the sphere with m Möbius bands.

(c) The rank of the intersection form of a disk with ribbons is equal to the rank defined in the Mohar formula 2.8.8 (c).

(d) The intersection form is symmetric: $\alpha \cap \beta = \beta \cap \alpha$.

6.7.3. Let N be a closed 2-manifold. The definition of the first Stiefel—Whitney class $w_1(N) \in H_1(N)$ is presented in §6.4.

(a) For any $a \in H_1(N)$, one has $w_1(N) \cap a = a \cap a$.

(b) $w_1(N) \cap w_1(N) = \rho_2 \chi(N)$.

6.7.4. *Poincaré duality.* The intersection form of any closed 2-manifold N is non-degenerate; that is, for any $\alpha \in H_1(N) - \{0\}$ there exists $\beta \in H_1(N)$ such that $\alpha \cap \beta = 1$.

6.7.5. (a—d) Define the intersection form $H_1(N) \times H_1(N) \to \mathbb{Z}_2$ for a 2-manifold N with non-empty boundary. Formulate and prove the analogues of Assertions 6.7.1.

(e) The intersection form can be degenerate.

(f) Find the intersection form and its rank for the sphere with g handles and h holes.

(g) Find the intersection form and its rank for the sphere with m Möbius bands and h holes.

On the path of this book to a reader

Here we give details to 'publishing rights' in p. 2 of this file. As of May, 2022, no public reply from the Editorial Board or from Springer are available. Updates (e.g. a public reply, if available) will be presented here.

A. Skopenkov's letter to the Editorial board of Springer book series 'Moscow Lecture Notes' (Cc M. Peters). Dec 6, 2021.

Dear colleagues,

Hope you are fine and healthy.

Thank you for accepting for publication in 'Moscow Lecture Notes' series of Springer the book Algebraic Topology From a Geometric Standpoint, https://www.mccme.ru/circles/oim/obstructeng.pdf

I'm afraid Springer is disregarding this acceptance decision of the Editorial Board. The Publishing Agreement proposed by Springer in April does not make the Publisher committed to publishing the book. Martin Peters and I found a compromise in May. But our compromise is not realized, and the problem is still unresolved - in spite of my monthly reminders. Natalia Tsilevich did excellent urgent translation work in July, but neither is paid by Springer, nor has a legal document ensuring later payment.

Does Editorial Board have any means to ensure that its acceptance decision is fulfilled by Springer? This information is vital for authors submitting to 'Moscow Lecture Notes' series.

Best wishes, Arkadiy.

PS The translation went fast and was already completed as early as in July (only the introduction and sections 3,4 remained). The translation was stopped for reasons described above.

A. Skopenkov's letter to A. Gorodentsev and V. Bogachev, Editors of Springer book series 'Moscow Lecture Notes' (Cc M. Peters). Dec 15, 2021.

Dear Alexey and Vladimir Igorevich,

Upon request of Vladimir Igorevich I describe how Springer is disregarding the acceptance decision of the Editorial Board of 'Moscow Lecture Notes' series. On compromises, see my letter of 6 Dec.

Could the Editorial Board make minimal efforts supporting its acceptance decision? A possible way is to publicly support the authors' amends to the Agreement proposed by Springer (I am willing to send you the list of amends). The information on whether the acceptance decision of the Editorial Board is final, is vital for authors submitting to the 'Moscow Lecture Notes' series. So the result of your efforts (if you choose to do some) should be widespread throughout the scientific community.

(1) The Agreement proposed by Springer contains the following clause allowing the Publisher to terminate the Agreement without any losses. This makes the publisher not committed to publishing the book, and so makes the acceptance decision of the Editorial Board void.

****

11.2. If the Publisher, acting reasonably, decides that the Work is not suitable for publication in the intended market place and/or community or that there is no substantial market for the Work, or the economic circumstances of publication have substantially changed (in each case other than due to the Work not being of a suitable quality to justify publication) then the Publisher may at any time terminate this Agreement by giving one month's notice to the Author in writing.

****

(2) The Agreement proposed by Springer does not contain a deadline for publication of the book (in terms of months after receipt of the translation). This makes the publisher not committed to publishing the book, and so makes the acceptance decision of the Editorial Board void.

(3) The Agreement proposed by Springer contains the following clause which makes the acceptance decision of the Editorial Board void.

****

13.1. This Agreement, and the documents referred to within it, constitute the entire agreement between the Parties with respect to the subject matter hereof and supersede any previous agreements, warranties, representations, undertakings or understandings. Each Party acknowledges that it is not relying on, and shall have no remedies in respect of, any undertakings, representations, warranties, promises or assurances that are not set forth in this Agreement.

****

(4) The Agreement proposed by Springer does not specify the amount of, and the deadline for, Publisher's payment for translation. For this, the Agreement refers to the Translation Agreement, but gives no guarantee that the terms of that Translation Agreement will be acceptable to the author and other translator. Since the author should not sign such an Agreement, this makes the acceptance decision of the Editorial Board void.

Best Regards, Arkadiy.

A. Skopenkov's letter to V. Bogachev, Editor of Springer book series 'Moscow Lecture Notes' (Cc A. Gorodentsev and M. Peters). Dec 23, 2021.

Dear Vladimir Igorevich,

Thank you for your reply.

Why do you write that my suggestions have been taken into account in a modified contract? This is wrong as I explained in my letter of Dec 15: my suggestions on items (1)-(4) are not taken into account. I forwarded you the last list of my suggestions sent to M. Peters on Nov 17 (analogous suggestions to previous versions of the Publishing Agreement were sent earlier). I received no reply either accepting these suggestions, or stating that Springer would not change the contract, or proposing compromises.

Recall that

(*) Springer is disregarding the acceptance decision of the Editorial Board because the Publishing Agreement proposed by Springer does not make the Publisher committed to publishing the book.

This is justified in my letter of Dec 15 by items (1)-(4). You do not consider those items, so you could not refute the statement (*). You write that the Publishing Agreement proposed by Springer is standard, but again this does not refute the statement (*). If something bad is a standard practice, this does not make it good.

My real experience with Springer is poor. I spent an enormous amount of time correcting errors that appeared during typesetting of my paper in Arnold J. Math. In May M. Peters agreed to take my suggestions into account. As of December, neither this is done, nor he informed me that this would not be done. So publication of the book is unduly postponed for an uncontrolled amount of time. All positive parts of our collaboration with M. Peters are explicitly made void by clause 13.1 of the Agreement:

****

13.1. This Agreement, and the documents referred to within it, constitute the entire agreement between the Parties with respect to the subject matter hereof and supersede any previous agreements, warranties, representations, undertakings or understandings. Each Party acknowledges that it is not relying on, and shall have no remedies in respect of, any undertakings, representations, warranties, promises or assurances that are not set forth in this Agreement.

****

For the moment, I will not comment on the other part of your letter for the following reason. The above (and the rest of your letter) makes me suppose that you confused a responsible business discussion with an irresponsible tea-time talk. If I am wrong, then I am sorry, and I have the following suggestion.

We strongly need this discussion to be responsible. We do not have enough time to discuss premature ideas, whose invalidity becomes clear when their publication (or a mental experiment of publication) is suggested. So I inform you that our correspondence with the Editorial Board on this subject is public.

I will publish all my letters at https://www.mccme.ru/circles/oim/obstructeng.pdf . If you would not send me a public reply to my Dec 15 letter, then the best way is to treat the private reply as non-existent, and inform the community that there is no public reply. If you send me a public reply to my Dec 15 letter (please feel free to edit your private reply), then I will publish it. My reply, your further reply, etc will also be published; presumably the discussion will soon converge by revealing important questions (like Q1, Q2, Q3 below) and the Editors answering them. If I receive a letter not stated to be public, then I will delete it unread (to avoid confusion). If a part of such a public discussion would become obsolete, we could delete that part (only) by our mutual consent.

Such a public discussion would be very useful for potential authors of this book series. In particular, they would be grateful if the Editors could publicly answer the following questions:

(Q1) Is Agreement with the properties (1)-(4) from my Dec 15 letter absolutely standard for this book series?

(Q2) Is Springer not obliged to accept all recommendations of the Editorial Board for this book series?

(Q3) Do Editors advise the authors to sign the Agreement without reading it?

If there is no public answer, a potential author could only assume that the answer is 'yes'.

Such a public discussion would require much effort. So let us find a way to avoid it. E.g., discussion by skype / zoom / phone makes it easier to understand each other and to find compromises.

Best wishes, Arkadiy.

A. Skopenkov's letter to M. Peters, A. Gorodentsev, V. Bogachev, and Yu. S. Ilyashenko. Jan 30, 2022.

Dear Martin, Alexey, Vladimir Igorevich, and Yuliy Sergeevich,

Hope you are fine and healthy.

I am grateful to the Editorial Board of 'Moscow Lecture Notes' of Springer for accepting in January, 2021 for publication the book 'Algebraic Topology From Geometric Standpoint'. (Please see the electronic version of a part at https://www.mccme.ru/circles/oim/obstructeng.pdf.)

The translation was essentially rejected by Springer by sending an unacceptable publishing agreement, promising to make amends suggested by the author in May, 2021, and neither making amends nor informing the author that the amends are not accepted, by January, 2022.

So, however reluctantly, I inform you that this book is no longer submitted to Springer.

We do not have enough time to discuss premature ideas, whose invalidity becomes clear when their publication (or a mental experiment of publication) is suggested. So I inform you that our correspondence on this subject is public. My letters are published at https://www.mccme.ru/circles/oim/obstructeng.pdf. If I receive a letter not stated to be public, then I will delete it unread (to avoid confusion).

I am also open to private discussions by skype / zoom / phone.

Best wishes, Arkadiy.