

Summary of A. Skopenkov's proposal

This proposal concerns the classical Knotting Problem in topology: *classify embeddings of a given space into another given space up to isotopy*. These problems have played an outstanding role in the development of topology. Various methods for the investigation of the Knotting Problems were created by classical figures.

The Knotting Problem is known to be hard. E.g. for the best known specific case of codimension 2 embeddings a complete classification is neither known nor expected.

The Knotting Problem is most interesting for manifolds of dimension at most 4, because embeddings of such manifolds often appear in other branches of mathematics and its applications.

I work in the smooth category unless PL (piecewise linear) category is explicitly mentioned.

Classical results of Wu, Haefliger, Hirsch (1960-s) on embeddings of n -dimensional manifolds into \mathbb{R}^m have the *metastable dimension restriction*

$$2m > 3n + 3.$$

In particular, in low dimensions Haefliger and Hirsch classified embeddings of 3-dimensional manifolds into \mathbb{R}^m for $m \geq 7$, and of 4-dimensional manifolds into \mathbb{R}^m for $m \geq 8$.

The main intention of this research proposal is to classify embeddings for

$$2m \leq 3n + 3$$

and for *closed connected* manifolds. For N not a homology sphere until 2005 no classification was known, in spite of the existence of interesting partial results, results in the PL category and approaches of Browder-Wall and Goodwillie-Weiss.

Embeddings $S^n \rightarrow S^m$ for $m > n + 2$ were classified by A. Haefliger in 1960s. There is the '*connected sum*' action $\#$ of the group of embeddings $S^n \rightarrow S^m$ on the set of embeddings $N \rightarrow S^m$ for a closed connected orientable n -manifold N .

The quotient set of this action was known for some cases including the case $m = 6 = 2n$ and the case N simply-connected, $m = 7 = 2n - 1$; there remained to find the orbits of $\#$. For N not a homology sphere until 2005 no description of the orbits was known.

A description of orbits for these cases appeared in papers by Crowley and myself in 2008-2010. They yielded *a classification of embeddings for 3-dimensional manifolds in \mathbb{R}^6 and for simply-connected 4-dimensional manifolds in \mathbb{R}^7* .

D. Crowley and I plan to obtain a classification of embeddings of non-simply-connected 4-manifolds in \mathbb{R}^7 . This is a significant new step as it is explained in the proposal. We also plan to obtain a piecewise linear analogue of this result. My contribution to our joint plan would be a description of the quotient set of $\#$ and the 'lower estimation' in the description of the orbits of $\#$.

I also plan to study the following Compression Problem for $n = 4$ and $m = 6, 7$: characterize embeddings of n -manifolds into \mathbb{R}^{m+1} that are isotopic to embeddings into \mathbb{R}^m .

Many interesting examples of embeddings are embeddings $S^p \times S^q \rightarrow S^m$. A classification of such embeddings is a natural next step (after the link theory and the classification of embeddings of highly-connected manifolds) towards classification of embeddings of arbitrary manifolds.

Assume that $1 \leq p \leq q$. A classification for $m \geq 2q + p + 1$ and $2m \geq 3q + 3p + 4$ was obtained by Haefliger-Hirsch in 1963, and for $m \geq q + 2p + 3$ and $2m \geq 3q + 2p + 3$ by myself recently.

I plan to obtain a classification of embeddings $S^p \times S^q \rightarrow S^m$ for $1 \leq p \leq q$ and $m \geq q + 2p + 3$. The classification would be in terms of homotopy groups of spheres and embeddings $D^{p+1} \times S^q \rightarrow S^m$ and $S^{p+q} \rightarrow S^m$ which are easier to describe.

I plan to define an action of the homology group of a manifold N on the set of embeddings $N \rightarrow S^m$. This would require a partial solution of the following problem of E. Rees: describe the action of self-diffeomorphisms of $S^p \times S^q$ on isotopy classes of embeddings $S^p \times S^q \rightarrow S^m$.