Project Summary
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ASYMPTOTIC PROPERTIES OF ZETA FUNCTIONS

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The starting point of our research project is the classical Brauer–Siegel theorem which is one the milestones of number theory of the past century. It describes the behaviour of the important invariants of number fields: the class number and the regulator.

M. Tsfasman and S. Vlăduţ generalized this theorem to the case of families of almost normal global fields where the degree growths fast enough compared to the discriminant. Such families are called asymptotically good and they stand at the heart of our project. The techniques of M. Tsfasman and S. Vlăduţ led to the new fundamental concepts of an infinite global field and a limit zeta function which are the important objects for further investigation.

One knows that the Brauer–Siegel theorem is equivalent to a statement about the asymptotic behaviour of the residue of the Dedekind zeta functions at $s = 1$. The concept of a limit zeta function permitted us to extend the Brauer–Siegel theorem yet further describing the asymptotic behaviour of the Dedekind zeta function for $\operatorname{Re} s > \frac{1}{2}$ in families of global fields (assuming the Generalized Riemann Hypothesis (GRH) in the number field case).

One should mention a d-dimensional variant of the Brauer–Siegel theorem proven by the author and the conjectures by M. Hindry, A. Pacheco, B. Kunyavskii and M. Tsfasman for families of elliptic curves over global fields which are close in spirit to it. We managed to generalize these conjectures to the case of arbitrary zeta functions over finite fields and proved several results in the direction of these conjectures. We also established theorems connected to point counting on varieties over finite fields as well as to zero distribution of L-functions over finite fields.

In this project we plan develop further the asymptotic theory of zeta and L-functions, especially in the case of characteristic zero. We will pay particular attention to finding arithmetic and geometric applications of the theory as well as to studying new concrete examples. Let us describe individual problems we plan to address.

1. Asymptotic theory in characteristic zero: analytic aspects. In particular, we want to investigate the following three issues: basic inequalities, Brauer–Siegel type results and zero distribution problems.

2. Applications of asymptotic results to the study of arithmetic properties of varieties over global fields. We are going to study in more detail elliptic curves and algebraic tori over global fields having in mind the goal of obtaining new information on the group of rational points and on their various arithmetic invariants.

3. Asymptotic properties of Selberg zeta function. Here we hope to get new results concerning the properties of discrete subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ and of the quotient of the upper half plane $\Gamma\backslash\mathbb{H}$. For instance, we aim at the theorems describing the distribution of eigenvalues of the Laplace operator acting on $\Gamma\backslash\mathbb{H}$.

4. Extension of Ihara's work on M-functions to L-functions of modular forms. In this part we plan to obtain finer asymptotic results on the values of L-functions of modular forms "on average". This will possibly shed some light on important problems related to moments of L-functions.

5. Asymptotic properties of families of fields coming from modular forms. As a final step we would like to investigate the possibility of constructing asymptotically good families of number fields from modular Galois representations with values in $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$.