Frobenius endomorphisms of linear spaces

Alexander E. Guterman

During the last century many efforts were devoted to classifying Frobenius endomorphisms of matrices, i.e., transformations preserving certain matrix invariants. The first result in this area was obtained by Frobenius in 1896, who characterized all bijective linear transformations on the space of complex matrices $M_n(\mathbb{C})$ that preserve the determinant function. In 1925 Schur, in 1949 Dieudonné, and in 1950 Dynkin continued the work in this direction. The detailed and self-contained information on Frobenius endomorphisms can be found in the special volumes of the journal Linear and Multilinear Algebra, volumes 33 and 48, completely devoted to the survey of results in this area.

Main results of my previous work are:

1) Kaplansky-Watkins problem, which is posed in 1976, is solved, namely we obtained a classification of non-linear surjective transformations preserving zeros of matrix polynomials. In order to do this special technique, called the method of elementary operators, is developed which reduces a non-linear problem to the group of linear problems.

2) Method of matrix deformations and chain method are developed in order to classify monotone linear and additive transformations on matrices. We investigate monotone transformations with respect to the following matrix partial orders: Drazin star order, left and right star orders, diamond order, singular value orders.

3) We characterized Frobenius endomorphisms for Dieudonné determinant.

4) Negative solution of Polya problem over finite fields: it is proved that there is no bijective converters of permanent to determinant for the matrix space over finite fields of sufficiently large cardinality.

The following further investigations are planned:

1) To find Gibson barriers for the number of ones in sign-convertible $(0,1)$-matrices and symmetric $(0,1)$-matrices, namely, to determine the maximal integer ω_n such that for any $(0,1)$ matrix A with the number of ones $v(A) < \omega_n$ it holds that A is convertible; to determine the minimal integer Ω_n such that for any $(0,1)$ matrix A with $v(A) > \Omega_n$ it holds that A is non-convertible; to characterize convertible matrices with $v(A) = \omega_n$ or $v(A) = \Omega_n$, and to prove that for any r, $\omega_n \leq \Omega_n$ there exist both a convertible matrix A and a non-convertible matrix B with $v(A) = r = v(B)$. Here convertibility means that permanent of A is equal to the determinant of a certain matrix obtained from A by changing some of its $+1$ entries into -1.

2) To prove that a tropical polyhedral cone can be reduced to the zero vector if and only if a corresponding mean payoff game has at least one winning initial state.

3) To characterize non-linear and even non-additive monotone matrix transformations.

4) To characterize Frobenius endomorphisms for C-determinantal range and radii.

Teaching experience and plans:

Since 2001 I work at the Department of Mathematics and Mechanics of MSU on the full time position. My affiliation is Associate professor. In particular I deliver one year lecture courses in matrix theory and ring theory for advanced students. I was a visiting professor in many different universities, in particular, Sung Kyun Kwan University, Korea, 2003, the University of Dortmund, 2004, Birmingham University, 2005, INRIA and Ecole Polytechnique, Paris, 2006, 2008, Institute of Mathematics, Physics, and Mechanics, Slovenia, 2008, 2009, University of University of Trás-os-Montes and Alto Douro, Portugal, and delivered courses in Matrix theory. 8 M.Sc. thesis and 1 Ph.D. dissertation were defended under my supervision. Currently 7 students, including 2 Ph.D. students, are working under my supervision in MSU. I am preparing a textbook and a computer course in my research area.