4. COMPLEXES AND CELL HOMOLOGY

Problem 1 (5-lemma)**.** Consider a commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\ p\downarrow & & q\downarrow & & r\downarrow & & s\downarrow & & t\downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E' \end{array}$$

The lines of the diagram are exact, q and s are isomorphisms, p is an epimorphism, and t is a monomorphism. Prove that r is an isomorphism.

Problem 2 (Bockstein's construction)**.** Let $0 \to A \xrightarrow{p} B \xrightarrow{q} C \to 0$ is an exact sequence of complexes with the differentials ∂_A, ∂_B and ∂_C, respectively. Let $c \in C_i$ be such that $\partial_C c = 0$. The map $q : B_i \to C_i$ is onto, so take $b \in B_i$ such that $q(b) = c$. Denote $\beta = \partial_B b \in B_{i-1}$; then $q(\beta) = p(\partial_C c) = 0$; hence, there exists $\alpha \in A_{i-1}$ such that $p(\alpha) = \beta$. (a) Prove that $\partial_A \alpha = 0$. (b) Prove that the homology class $[\alpha] \in H_{i-1}(A)$ is well-defined, that is, does not depend on the choice of b (which is not unique). (c) Prove that $[\alpha]$ depends only on $[c] \in H_i(C)$; this defines a map $\delta_i : H_i(C) \to H_{i-1}(A)$. (d) Prove that the sequence $\dots \to H_i(A) \xrightarrow{p_*} H_i(B) \xrightarrow{q_*} H_i(C) \xrightarrow{\delta} H_{i-1}(A) \to \dots$ is exact.

Problem 3. Let $A \xrightarrow{p} B \xrightarrow{q} C \to 0$ be an exact sequence of Abelian groups. Prove that for any Abelian group G the sequence $A \otimes G \xrightarrow{p\otimes \mathrm{id}} B \otimes G \xrightarrow{q\otimes \mathrm{id}} C \otimes G \to 0$ is exact. Show that the similar statement for an exact sequence $0 \to A \xrightarrow{p} B \xrightarrow{q} C$ may be not true.

Problem 4. Let $\dots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$ be a chain complex of free Abelian groups. Consider a sequence $\dots \xleftarrow{\partial_{i+1}^*} C_i^* \xleftarrow{\partial_i^*} C_{i-1}^* \xleftarrow{\partial_{i-1}^*} \dots \xleftarrow{\partial_1^*} C_0^* \longleftarrow 0$ where $C_i^* = \mathrm{Hom}(C_i, \mathbb{Z})$. (a) Prove that this is a cochain complex. (b) Prove that if $H_i(C) = \mathbb{Z}^{\beta_i} \oplus G_i$ where G_i is finite, then $H^i(C^*) = \mathbb{Z}^{\beta_i} \oplus G_i'$ where G_i' is finite, too. Show that G_i and G_i' may be not isomorphic.

Problem 5. For the following spaces find CW-complexes homeomorphic to them and compute their homology with the coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. If two spaces X and Y and a map $f : X \to Y$ are given then do this for both spaces and compute the homomorphism $f_* : H_*(X) \to H_*(Y)$. (a) X is a sphere with g handles; (b) X is the Klein bottle with g handles; (c) X is $\mathbb{R}P^2$ with g handles; (d) X is the sphere with g handles and n holes; Y is the sphere with g handles, $f : X \to Y$ is the natural embedding. (e) $X = S^n$, $Y = \mathbb{R}P^n$, $f : X \to Y$ is the universal cover. (f) $X = S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^n \mid |z_0|^2 + \dots + |z_n|^2 = 1\}$, $Y = \mathbb{C}P^n$, $f : S^{2n+1} \to \mathbb{C}P^n$ is $f(z_0, \dots, z_n) = [z_0 : \dots : z_n]$ (the generalized Hopf bundle). (g) $X = S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$; the group $\mathbb{Z}/p\mathbb{Z} = \{\zeta^m \stackrel{\text{def}}{=} e^{2\pi i m/p} \mid m = 0, \dots, p-1\}$ is acting on X by the maps $\zeta^m(z, w) = (\zeta^m z, \zeta^{qm} w)$ where p and q are coprime; Y is the orbit space for this action (called the (p, q)-lens, $L(p, q)$), and $f : X \to Y$ is the map sending every point to its orbit. (h) $X = S^\infty$ is the set of sequences $(x_1, \dots, x_n, \dots)$, such that in every sequence there are finitely many nonzero elements, and the sum of their squares is 1; $Y = \mathbb{C}P^\infty$ (what is it?), $f : S^\infty \to \mathbb{C}P^\infty$ is the inifinity-dimensional analog of the Hopf bundle. (i) $Y = G(2, 4, \mathbb{R})$ (the set of 2-dimensional subspaces in $\mathbb{R}^4$), $X = G_+(2, 4, \mathbb{R})$ (the set of oriented 2-dimensional subspaces in $\mathbb{R}^4$), $f : X \to Y$ is forgetting the orientation.