Lecture 1. Main Principles

The prerequisites for this course are a solid knowledge of mathematics (in Russian) and a sound knowledge of mathematical terminology in English. No knowledge of the English language is required.

The goal of the course are to teach you to correctly write and speak mathematical English.

To do this, you must understand, first of all, that English, unlike Russian, is not a grammatical language – correct English cannot be obtained by following a finite set of grammatical rules,

GOOD MATHEMATICAL ENGLISH COMES FROM USAGE

i.e., from employing only those *standard constructions* that native English speaking mathematicians ordinarily use.

Are there many such constructions? Actually very few are needed to produce a good text – this is a wonderful property of *mathematical* English, a linguistic fact that makes our approach to teaching that language simple and efficient.

You will never master mathematical English if you don't follow the *main rule*:

NEVER TRANSLATE FROM RUSSIAN!

In order to produce a good mathematical text in English, *never* begin by expressing your mathematical thoughts in Russian, first clarify to yourself what it is you want to say *mathematically*, and then express it using the English constructions that you are familiar with. Even when you are in fact performing a translation of a Russian mathematical text, **don't translate** – first understand mathematically what the author wants to say, then express it in your own words, by means of those English constructions that you are familiar with. To do that, you need to have a collection of standard expressions in your memory. I call these expressions clichés.

By a *cliché* I mean a fixed text with variable entries (i.e., blank spaces to be filled in by words, expressions or formulas of the appropriate type). Here is an example:

FOR ANY [ ] THERE EXISTS A [ ] SUCH THAT [ ].

B filling in the empty spaces (specifying the variable entries), we can obtain the following sentences:

- For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$.
- For any braid $b \in B_n$ there exists a braid $b' \in B_n$ such that $bb' = \mathrm{id}.$,

as well as many other popular mathematical statements.

In order to obtain a syntactically correct sentence from a cliché, we need to fill in the empty spaces by the appropriate *type of variable*. In our study of mathematical English, we will consider five types of variables, which we call *parts of speech* (although they are completely different from the parts of speech of traditional grammar); we simply list them with their abbreviations (without giving any formal linguistic definition) followed by typical examples.

• Objects (*obj*): Banach space, $f(x)$, $\varepsilon > 0$, the Abelian group G, a second order differential equation solved w.r.t. the highest derivative,...

• Openers (*open*): Then, Therefore, Without loss of generality we can assume that, Suppose that, This means that, ...

• References (*ref*): Theorem 1, the previous lemma, Gauss' proof, the Poincaré conjecture, Hilbert's 10th Problem, Definition 3.1, ...

• Links (*link*): and, or, if, such that, whenever, when, which, ...

• Properties (*prop*): continuity, smoothness, Lebesgue integrability, Jordan measurability, infinite differentiability, ...

In some clichés, there are empty spaces that must be filled not by parts of speech, but by mathematical statements, e.g. formulas or sentences constructed from other clichés, so that our constructions can be, in a sense, recursive. Empty spaces for formulas or statements (which are not parts of speech!) will be denoted by the word *claim*. In particular, claims often appear in empty spaces on both sides of links in clichés such as

[*claim*] AND [*claim*] or [*claim*] SUCH THAT [*claim*].

To see how much mathematics can be correctly expressed by using a very small number of clichés, let us now carefully read the following mathematical text (an introduction to the theory of smooth manifolds).

Example of a simple mathematical text

Definition. A *manifold* is a pair $(M, \mathcal{A})$, where M is a topological space and $\mathcal{A}$ is an atlas; here the *atlas* $\mathcal{A}$ is a set $\mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ such that

(i) $U_\alpha \subset M$ is an open set;

(ii) $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism;

(iii) $\cup_{\alpha \in J} U_\alpha = M$.

Examples. 1) M is $\mathbb{R}^n$ and $\mathcal{A} = \{\text{ id} : \mathbb{R}^n \to \mathbb{R}^n\}$.

2) M is the sphere $\mathbb{S}^n$ and $\mathcal{A} = \{p_i : \mathbb{S}^n \setminus n_i \to \mathbb{R}^n, i = 1, 2\}$; here p_1 and p_2 are the stereographic projections.

Definitions. Suppose that $(M, \mathcal{A})$ is a manifold and $\alpha, \beta \in J$; then $\phi_\alpha \circ \phi_\beta^{-1} =: t_{\alpha,\beta}$ is a *transition function*.

Further, $(M, \mathcal{A})$ is a *smooth manifold* if $t_{\alpha,\beta} \in C^\infty(\mathbb{R}^n)\ \forall \alpha, \beta \in J$, where

$$C^\infty(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R}^n | f \text{ is an infinitely differentiable map}\}.$$

Suppose $(M, \mathcal{A})$ is a smooth manifold; a *smooth embedding* is a map $h : M \to \mathbb{R}^n$ such that $h \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to h(U_\alpha)$ is a diffeomorphism $\forall \alpha$.

Theorem 1. [Whitney, 1921] *Suppose that M is a smooth manifold and* $\dim M = n$. *Then there is a smooth embedding $h : M \to \mathbb{R}^{2n}$.*

Analyzing this text, we see that – amazingly – all its sentences are obtained from three clichés (basically, from only one cliché), namely:

[*obj*] IS [*obj*]; [*objs*] ARE [*objs*]; [*ref*].

with the help of only four links (*where, and, such that, if*) and three openers (*here, suppose that, then*)!

To conclude this lecture, let us list eight more basic clichés, which we will constantly use in what follows.

- CONSIDER [*obj*]
- LET [*obj*] BE A [*obj*]
- [*ref* or *prop*] IMPLIES [*ref* or *prop*]
- THERE EXISTS A [*obj*] SUCH THAT [*claim*]
- THERE EXISTS A UNIQUE [*obj*] SUCH THAT [*claim*]
- [*obj*] IS CALLED [*mod* or *obj*] IF [*claim*]
- DENOTE BY [*symbol*] THE [*obj*]
- THE SET OF ALL [*objs*] IS A [*obj*] W.R.T. [*obj*]

Homework assignement. Write a one-page introduction to group theory using only constructions based on the clichés that we have indicated in this lecture.