REPORT ON THE CONTEST "YOUNG RUSSIAN MATHEMATICS" (2017)

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Results 2017

(1) The framed little n-disks operads fD_n are operads of embeddings of "small" n-dimensional disks in the n-dimensional unit disk. These operads are of fundamental importance in algebraic topology and homological algebra. Surprisingly, the rational homotopy type of the operads fD_n is currently not understood very well. This is in sharp contrast to the rational homotopy type of the non-framed sub-operads $\mathsf{D}_n \subset \mathsf{fD}_n$, which is well understood due to work of Kontsevich and Tamarkin. In particular, the formality of the little 2-discs operad is known to be equivalent to the Kontsevich formality theorem.

We described the real homotopy type of the the action of the orthogonal groups on the operads D_n, from which the real homotopy type of the topological operads fD_n may be deduced. Generally, we showed that the real homotopy type of the $O(n)$-action on D_n is described by a certain Maurer-Cartan element in the Kontsevich graph complex (dg Lie algebra) with coefficients in the cohomology $H(B\mathsf{SO}(n))$

$$m \in \left(\mathsf{GC}_n \hat{\otimes} H(B\mathsf{SO}(n))\right)^{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ should be thought of as $\pi_0(O(n))$. We derive explicit integral formulas for the element m, and provide a model for the $O(n)$-framed little disks operads depending (only) on m. By a version of equivariant localization we can compute the gauge equivalence type of m, and hence produce explicit combinatorial models for the framed little disks operads. The following results can be read off from the models.

Theorem The $O(n)$-framed and $\mathsf{SO}(n)$-framed little n-disks operads are
- formal over $\mathbb{R}$ if $n \geq 2$ is even, in the sense that the homotopy dg Hopf cooperads of differential forms on these operads can be connected to their cohomologies by zigzags of quasi-isomorphisms.
- not formal over $\mathbb{R}$ if $n \geq 3$ is odd.

(2) The category of representations over a quantum group $U_q(g)$ form a braided tensor category that produces an action of the (pure) braid groups on tensor products by permuting factors. Respectively, the category of crystals (which is a limit for q tends to zero) form a coboundary category together with an action of (pure) cacti group on tensor products (that is a corresponding limit of the pure braid group). The little discs operad is an operad whose space of n-ary operations is the Eilenberg-Maclane space of the pure braid groups with n braids. Correspondingly, the real loci of the moduli spaces of stable rational curves with marked points $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ assemble an operad of the Eilenberg-Maclane spaces of pure cacti groups. We give a universal description of the algebraic model of the latter operad as a homotopy quotient of an operad of associative algebras. We use this model to show that this operad is not

formal, present different combinatorial models of the cooperad of differential forms on $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and prove the conjectural description of the lower central series filtration of the pure Cacti group. In particular, we prove that the spaces $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are rational $K(\pi, 1)$ spaces.

Papers 2017

(1) "Differentials on graph complexes" with T. Willwacher and M. Živković
    published in Advances in Mathematics, Volume 307 (2017), pp. 1184–1214

(2) "Differentials on graph complexes II - hairy graphs" (with T. Willwacher and M. Živković)
    Letters in Mathematical Physics October 2017, Volume 107, Issue 10, pp. 17811797

(3) "Real models for the framed little n-disks operads" (with T. Willwacher)
    preprint available at math.arXiv:1705.08108

(4) "Implementing the operad of real moduli space of rational curves" (with T. Willwacher)
    in preparation

Scientific conferences and seminar talks (2017)

(1) Conference "Hot Topics: Galois Theory of Periods and Applications".
    MSRI, Berkeley, March 27–31, 2017. The link of the conference:
    https://www.msri.org/workshops/826
    Title of the talk: "The operad structure of $\overline{M_{0,n+1}}(\mathbb{R})$"
(2) Workshop Graph complexes, props and homotopy algebras.
    University of Luxembourg, 8-10 November 2017. The link of the conference:
    https://wwwen.uni.lu/content/download/104168/1242061/file/Workshop.pdf Title
    of the talk: "On god-given relations in graded Hopf operads."

Teaching (2017)

(1) Operads in Algebra and Topology, NRU Higher School of Economics, III-IV year
    bachelor students and magisrants, fall 2017, 2 hours lectures per week.
    Program include
    - Definitions, examples and simplest computations for Nonsymmetric and Symmetric operads;
    - Resolutions of algebras over an operad and resolutions of operads;
    - A_∞-algebras and the operad of Stasheff polytopes;
    - Homotopy transfer theorems for complexes of vector spaces and for algebras over an operad;
    - Rational homotopy type, Loop spaces and modules over Associahedron operad;
    - Closed model categories: definitions and examples of CMC on the category of dg-algebras over an operad;
    - Koszul duality for algebras and for operads.

(2) Basic algebra for II year bachelor students, NRU Higher School of Economics, fall
    2017, 3 hours lectures per week + 3 hours exercises class per week.
    This course was separated into two parts: basics of Galois theory and basics of the representation theory of finite groups. Parts of the Galois theory discussed in the course include:
    - Field extensions, algebraic and trancendental numbers;
    - Existence and uniqueness of splitting fields;
    - Normal, separable and Galois extensions;

- Galois theorem and Galois correspondence;
- Abel's theorem, solvable field extensions.

Parts of the representation theory discussed in the course include:
- irreducible and indecomposable representations, Mashke's theorem and Schur's Lemma;
- regular, quasiregular and induced representations, Frobenius reciprocity;
- Characters of representations, orthogonality and character tables for standard groups.

(3) Basic algebra for I year bachelor students, NRU Higher School of Economics, spring 2017, 3 hours exercises class per week.

(4) I am currently a scientific supervisor at NRU HSE of
- 5 bachelor students of 2-nd year,
- 1 bachelor student of 4-th year,
- 1 master student,
- 1 bachelor diploma thesis was written this year under my supervision.