PIERRE DELIGNE SCHOLARSHIP REPORT. DMITRY CHELKAK, 2010.

1. Results

In 2010 I made some progress in several research projects concerning a sharp behavior of the critical Ising model as the mesh size tends to zero (see the list in Sect. 4 below), and the most important one was done in collaboration with *K. Izyurov* (Geneva-St. Petersburg).

As a starting point, *we constructed a spinor* generalization of Smirnov's *fermionic observable* in the critical Ising model *for non-simply connected domains* (or Riemann surfaces). Remaining quite simple, this generalization allows one to attack several questions which were out of reach having only the original Smirnov construction. The most important application is that, interpreting a single inner face as a microscopic hole, one obtains a way to identify (conformally covariant) limits of various *spin-correlations* in the critical spin-Ising model (this is an ongoing project of *C.Hongler* (New York), K.Izyurov and myself) via discrete complex analysis tools.

In order to describe this generalization in more details, let us remind that the original fermionic observable is defined for any discrete domain Ω^δ with a source point a on the boundary and its value $F_a^\delta(z)$ is given by the partition function of all configurations consisting of a (discrete) curve γ running from a to z and a number of loops, counted with a complex (fermionic) weight $\exp[-\frac{i}{2}\text{winding}(a \to z)]$ depending on the winding number of γ. It was shown by S.Smirnov and myself that this observable has a conformally covariant scaling limit when the mesh size δ tends to zero. In particular, this implies

> convergence of spin-Ising interfaces in a simply-connected domain Ω^δ with two marked boundary points a, b (and Dobrushin $+/-$ boundary conditions) to Schramm's SLE(3) curves running from a to b.

Indeed, winding$(a \to b)$ is defined uniquely by topological considerations, so by defintion $F_a^\delta(b)$ is a fair partition function for the Ising model with Dobrushin $+/-$ boundary conditions, and for any z inside Ω^δ the ratio $F_a^\delta(z)/F_a^\delta(b)$ is a martingale with respect to the growing curve γ. Then, the identification of (random) limit interfaces as SLE curves follows from the so-called "conformal martingale principle".

Further, it was shown by C.Hongler and S.Smirnov that, taking the source point inside Ω^δ, the similar observable allows one to compute (properly renormalized)

> conformally covariant limit of an energy density s_z in the critical Ising model at $z \in \Omega$ and, more generally, to obtain limits of all (properly renormalized) correlations $\mathbb{E}[s_{z_1} s_{z_2} ..]$ of the energy field.

For non-simply connected domains one can define the same F_a^δ, but it is not sufficient: e.g., this observable allows one to describe an interface running from a to the boundary point lying on the same component of $\partial\Omega^\delta$ (since winding$(a \to b) \mod 4\pi$ is defined uniquely) but doesn't allow to describe an "interface" running from one boundary component to another since in this setup winding$(a \to b)$ is defined uniquely only up to 2π term and so $F^\delta(b)$ is not a fair partition functions anymore (one has different signs for different terms).

Our *"twisted" observable* lives on a double cover $\widetilde{\Omega}^\delta$ of Ω^δ and changes its sign between sheets. Its value $\widetilde{F}_a^\delta(z)$ at some $z \in \Omega^\delta$ is defined as a similar sum over all configurations in Ω^δ consisting of a curve from a to z and a number of loops but with two additional signs: first depending on the sheet where γ "arrives at z" and second depending on a number of topologically nontrivial loops in the configuration. It's easy to see that, say, for doubly connected domains the values of $\widetilde{F}_a^\delta$ on the other component of the boundary are fair partition functions. Thus, e.g., this modification allows one to describe the limiting law (variant of SLE(3)) of an "interface" running from one boundary component to another.

Even more interesting is to compare values of $\widetilde{F}_a^\delta$ with the original observable F_a^δ. Again, for simplicity, let Ω^δ be *doubly connected*, and let both $a, b \in \partial\Omega^\delta_{\mathrm{out}}$ lie on the outer boundary. Let us consider the critical Ising model in this domain with $+/-$ boundary conditions on $\partial\Omega^\delta_{\mathrm{out}}$ and "unknown monochromatic" b.c. on the inner part $\partial\Omega^\delta_{\mathrm{int}}$. It directly follows from combinatorial definitions of "normal" and "twisted" observables that

$$\frac{\widetilde{F}_a^\delta(b)F_a^\delta(a)}{F_a^\delta(b)\widetilde{F}_a^\delta(a)} = \frac{\mathbb{E}_{+/-}[\sigma(\partial\Omega^\delta_{\mathrm{int}})]}{\mathbb{E}_{+}[\sigma(\partial\Omega^\delta_{\mathrm{int}})]}$$

where $\sigma(\partial\Omega^\delta_{\mathrm{int}})$ denotes the (unknown) spin of $\partial\Omega^\delta_{\mathrm{int}}$, $\mathbb{E}_{+/-}$ means $+/-$ b.c. on $\partial\Omega^\delta_{\mathrm{out}}$ while $\mathbb{E}_+$ stands for all $+'s$. One can identify the scaling limit of the l.h.s. as $\delta \to 0$ via solutions of some Riemann-Hilbert-type boundary value problems. In particular, when the inner boundary is shrunk to a single face $z \in \Omega$ and Ω is *simply connected*, one obtains the following result:

$$\mathbb{E}_{+/-}[\sigma_z]/\mathbb{E}_{+}[\sigma_z] \to \cos[\pi\omega(z,(ba),\Omega)] \quad \text{as } \delta \to 0,$$

where ω denotes the harmonic measure of the arc (ba) from z. More advanced considerations (when the source point is taken inside Ω^δ) allows one to compute scaling limits of (properly renormalized) quantities like

$$\mathbb{E}_{+}[\sigma_z s_w]/\mathbb{E}_{+}[\sigma_z]$$

for any $z, w \in \Omega$. Further, if z_1 and $w = z_2$ are neighbors, then the last ratio is nothing but

$$\mathbb{E}_{+}[\sigma_{z_2}]/\mathbb{E}_{+}[\sigma_{z_1}],$$

i.e. the logarithmic derivative of the magnetization w.r.t. to the coordinate variable. Despite the fact that several technical details are still missing, now one can hope to make all limiting passages rigorous and to obtain the limit magnetization (and, hopefully, all spin-spin correlations) using discrete complex analysis tools. In case of success, one would be able to claim that *both energy and spin correlations* defined in the critical Ising model in *arbitrary planar domain* converge to conformally invariant limits known from the CFT theory.

2. Publications

Two papers by S. Smirnov and myself were accepted for publication and now are on the proof-reading stage:

- D. Chelkak, S. Smirnov: Discrete complex analysis on isoradial graphs, 34pp. (to appear in *Advances in Mathematics*);
- D. Chelkak, S. Smirnov: Universality in the 2D Ising model and conformal invariance of fermionic observables, 50pp. (to appear in *Inventiones Mathematicae*).

We are working on the third paper in this series, were the convergence of interfaces of the critical spin-Ising model will be established in the topology of curves themselves.

3. Conferences, seminar talks etc.

- Conferences (invited talks):
  "Conformal Structures and Dynamics (CODY-2010)", Seillac (France), May 2–8;
  "73rd Annual Meeting of the Institute of Mathematical Statistics (IMS-2010)", invited session "Scaling limits and conformal invariance", Gothenburg (Sweden), August 9–14.
- Seminar talks, colloquia etc.:
  Helsinki, Paris XIII; St. Petersburg: several talks on various research seminars.
- Workshops, schools:
  "Conformal maps from probability to physics", Ascona (Switzerland), May 23–28;
  "Probability and Statistical Physics in Two and more Dimensions", Clay Mathematical Institute Summer School 2010, Buzios (Brazil), July 16 – August 7.

4. Participation in international research projects

- Research visit to IHES (Bures-sur-Yvette): September 21 – November 4;
- I'm a member of an international team working on a rigorous approach to the conformal invariance in critical two-dimensional lattice models (particularly, spin and random cluster representations of the critical Ising model) via discrete complex analysis tools. I have several ongoing projects (which are on different stages of realization) with:

  S.Smirnov (Geneva-St.Petersburg): improving of the topology of convergence of the spin-Ising interfaces (from driving forces to curves). During 2010, we essentially simplified the core argument (uniform estimate of the crossing probability for $+/-/+/-$ boundary conditions), but a paper is still under preparation.

  C.Hongler (New York) and K.Izyurov (Geneva-St.Petersburg): spin-spin correlations convergence to conformally covariant limits via spinor modification of S.Smirnov's fermionic observables (see Sect. 1 above);

  A.Kemppainen and K.Kytölä (Helsinki): computing the limit of crossing probabilities in the critical FK-Ising model with $2N$ marked points on the boundary;

  H.Duminil-Copin (Geneva) and C.Hongler (New York): developing parts of the Russo-Seymour-Welsh theory for the critical FK-Ising model.

5. Pedagogical activity

I taught the following courses at St.Petersburg State University (Math. & Mech. Faculty):

- Spring 2010: special course "Measure Theory" ("PDMI group" 2nd year students): geometric measure theory, Hausdorff measures, deterministic and random fractals;
- Fall 2010: students seminar "Entire functions and conformal invariants" (3rd year students specialized in mathematical analysis);
- Standard calculus courses (lectures and seminars) for 2nd-3rd years students.

Also, I taught the following course (8 lectures) at "Fizmatclub" PDMI RAS:

- Spring 2010: "Introduction to conformal invariance in two-dimensional lattice models".

I was an advisor of diploma paper "Discrete Extremal Length" by Alexey Vorotov who is now a 1st year graduate student at the St. Petersburg University under the supervision of S.V.Vallander (Probability and Statistics Dept.). Also, I am a co-advisor of 2nd year graduate student Sergey Matveenko (PhD theme: sharp spectral theory of Schrodinger-type operators with matrix potentials) and several undergraduate students.