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Application to P. Deligne contest.

1. Description of the problem

My research project is devoted to the classification of two classes of group actions:

1. Reductive group actions on normal algebraic varieties.
2. Hamiltonian actions of compact groups on compact symplectic C^∞-manifolds.

Below in this section G denotes a connected reductive group, K a connected compact Lie group, X a normal irreducible algebraic variety, M a connected compact symplectic C^∞-manifold. All algebraic groups and varieties are defined over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Of particular interest are *multiplicity free* actions. The definitions are as follows.

Definition 1.1. An action $G : X$ is called multiplicity free (shortly, MF) if a Borel subgroup B of G has a dense orbit on X. In this case X is called a *spherical* G-variety.

Definition 1.2. An action $K : M$ is said to be MF if the Poisson algebra $C^\infty(M)^K$ is commutative, equivalently, a general K-orbit of M is a coisotropic submanifold.

I am going to consider two general problems.

Problem 1.3. *Classify MF actions $G : X, K : M$ in combinatorial terms.*

Problem 1.4. *To establish certain uniqueness properties for general actions $G : X, K : M$.*

The algebraic part of the second problem was solved in the MF case, see Section 2. Precise statements of conjectures and problems will be given in Section 3.

2. Known results

Our ideology of solving the algebraic parts of Problems 1.4,1.3 goes back to the paper [LV] of Luna and Vust. Essentially, they dealt with the problem of classifying normal irreducible G-varieties in a given class of birational equivalence. To perform this classification one needs to know the following invariants of the G-filed $\mathbb{K}(X)$:

1. The set $\mathbb{K}(X)^{(B)}$ of all rational B-semiinvariants. This set is equipped with the multiplication and partially defined addition, both induced from $\mathbb{K}(X)$.
2. The set $\overline{\mathcal{V}}_{G,X}$ consisting of all geometric $\mathbb{Q}$-valued discrete G-invariant valuations of $\mathbb{K}(X)$. This set is equipped with the map $\overline{\mathcal{V}}_{G,X} \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Q})$ given by $v \mapsto \varphi_v, \varphi_v(f) := v(f)$. This map is known to be injective, see [Kn2], so we consider $\mathcal{V}_{G,X}$ as a subset in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Q})$. Here and below $\mathbb{K}(X)^{(B)\times} := \mathbb{K}(X)^{(B)} \setminus \{0\}$.
3. The set $\mathcal{D}^0_{G,X}$ consisting of all prime Weil divisors on X that are B-stable but not G-stable. Again, there is a natural map $\mathcal{D}^0_{G,X} \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Z})$, $D \mapsto \varphi_D, \varphi_D(f) := \operatorname{ord}_D(f)$.

It also makes sense to consider the set of all B-stable divisors on X, which we denote by $\mathcal{D}_{G,X}$ and which is not a birational invariant of X. Additionally, to any $D \in \mathcal{D}_{G,X}$ we assign its stabilizer G_D in G, which is a parabolic subgroup containing B. We call the triple $(\mathbb{K}(X)^{(B)\times}, \mathcal{V}_{G,X}, \mathcal{D}^0_{G,X})$ (resp., $(\mathbb{K}(X)^{(B)\times}, \mathcal{V}_{G,X}, \mathcal{D}_{G,X})$) the Luna-Vust (shortly, LV) system of X (resp., the *regular* LV system of X)

and denote it by $\mathcal{LV}_{G,X}$ (resp., $\mathcal{LV}^{reg}_{G,X}$). By an *isomorphism* of two LV systems $(F_1,\mathcal{V}_1,\mathcal{D}_1),(F_2,\mathcal{V}_2,\mathcal{D}_2)$ we mean a pair (ψ,ι), where $\psi: F_1\to F_2, \iota:\mathcal{D}_1\to\mathcal{D}_2$ are isomorphisms satisfying the natural compatibility relations. For example, a maximal torus T of B acts by automorphisms of a LV system $\mathcal{LV}=(F,\mathcal{V},\mathcal{D})$ (the action on $\mathcal{D}$ is trivial). By $\mathrm{Aut}^{ess}(\mathcal{LV})$ we denote the quotient of the whole group $\mathrm{Aut}(\mathcal{LV})$ by the image of T.

In general, a LV system is an extremely complicated invariant. However, everything becomes much easier when X is spherical. In this case, $\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times$ is just a subgroup in the group $\mathfrak{X}(B)$ of characters of B called the weight lattice of X and denoted by $\mathfrak{X}_{G,X}$. The subset $\mathcal{V}_{G,X}\subset\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X},\mathbb{Q})$ is known to be a finitely generated convex cone, [BriPo]. Moreover, w.r.t. the scalar product on $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X},\mathbb{Q})$ induced from an invariant nondegenerate symmetric form on $\mathfrak{g}$ the cone $\mathcal{V}_{G,X}$ becomes a Weyl chamber of a unique linear group $W_{G,X}$ generated by reflections, see [Bri1], called the Weyl group of X. Finally, the set $\mathcal{D}_{G,X}$ is finite. So the $\mathcal{LV}_{G,X}$ is a purely combinatorial invariant of X.

The birational classification of spherical varieties reduces to the classification of spherical homogeneous spaces. The basic result here is the following theorem.

Theorem 2.1 ([Lo5], Theorem 1). *Let X_1,X_2 be spherical homogeneous G-spaces, whose LV systems are isomorphic. Then X_1,X_2 are isomorphic.*

This theorem was conjectured by Luna, [Lu1] and proved by him, [Lu2], for G of type A and then by Bravi-Pezzini, [BraPe] for G of type $A-D$.

Another important result obtained in [Lo5] is the description of the automorphism group $\mathrm{Aut}^G(X)$ for a spherical homogeneous space X in terms of $\mathcal{LV}_{G,X}$ (Theorem 2). In particular, this theorem implies that the natural map $\mathrm{Aut}^G(X)\to\mathrm{Aut}^{ess}(\mathcal{LV}_{G,X})$ is surjective.

In fact, if X is affine, then there is a combinatorial invariant of X, which is more basic then the (regular) LV system: the *weight monoid* $\mathfrak{X}^+_{G,X}$ of X consisting of all weights of B in the algebra $\mathbb{K}[X]$ of regular functions on X. Using Theorem 2.1 the author proved the following uniqueness result for affine spherical G-varieties.

Theorem 2.2 ([Lo4], Theorems 1.2,1.3). *Let X_1,X_2 be affine spherical G-varieties such that $\mathfrak{X}^+_{G,X_1}=\mathfrak{X}^+_{G,X_2}$. Suppose at least one of the following conditions holds:*

(1) *Both X_1,X_2 are smooth.*

(2) $\mathcal{V}_{G,X_1}=\mathcal{V}_{G,X_2}$.

Then X_1,X_2 are G-equivariantly isomorphic.

The theorem in the case of smooth X_1,X_2 was conjectured by Knop in connection with MF Hamiltonian actions. It was proved by Camus, [Ca], for G of type A.

The question of existence of a spherical homogeneous space with given LV system is much more complicated. Luna in [Lu1] obtained some explicit combinatorial conditions on $\mathcal{LV}_{G,X}$. He conjectured that any LV system satisfying these conditions is the LV system of some spherical homogeneous space. So far this conjecture is proved in types A, [Lu2], and $A-D$, [BraPe]. Besides Camus in [Ca] classified all regular LV systems of smooth affine spherical varieties in type A.

To complete the discussion of spherical varieties and their LV systems let us mention the results of the author, [Lo1],[Lo2],[Lo3],[Lo6][1]. These papers deal with

[1] [Lo2] is an immense preprint in Russian, its results concerning the computation of Weyl groups and weight lattices are available in [Lo3],[Lo6]

not necessarily spherical varieties. The definition of weight lattices is generalized to this case directly, while the existence of Weyl groups was proved by Knop in [Kn2].

Proceed to MF Hamiltonian actions $K : M$. To such an action one assigns its moment polytope and a certain subgroup in K called the *principal isotropy subgroup*. Delzant conjectured in [D2] that these invariants determine M uniquely. When K is a torus this conjecture becomes the celebrated Delzant theorem, [D1]. In some special case the Delzant conjecture was proved by Woodward, [W]. Finally, Knop reduced the Delzant conjecture to the first assertion of Theorem 2.2 (unpublished). He also reduced the existence question to the description of all possible weight monoids of smooth affine spherical varieties.

3. Plans

Consider the problem of classifying multiplicity free actions.

Let X be a spherical homogeneous space. Luna in [Lu1],[Lu2] found explicit combinatorial restrictions on $\mathfrak{X}_{G,X}, \mathcal{D}_{G,X}, \mathcal{V}_{G,X}$. He developed some reductions of LV systems. They should enjoy the following property:

(*) if a reduced system arises from a spherical homogeneous space, then an initial one does.

Roughly speaking, the Luna program looks as follows:

1. Prove that the reductions do enjoy (*).
2. Classify all LV systems that cannot be reduced.
3. For any such system construct a spherical homogeneous space corresponding to this system.

Luna's approach is "algebro-geometric", his constructions use the geometry of wonderful varieties. On the other hand, I plan to use a "group-theoretic" approach applying the structure theory of algebraic subgroups in a reductive group.

Luna proposed three reductions and proved that the first two ("parabolic induction" and "fiber product") always enjoy (*), while the third one ("projective fibration") does so only under some restriction on the LV system. I am going to prove that this reduction always satisfies (*). In particular, this would complete the classification in the case when all simple normal subgroups of G are of types A, C started by Pezzini in his PhD thesis.

Step 3 does not seem to be really difficult (for one LV system). Algorithms from [Lo2],[Lo3],[Lo6] should be of help here. On the contrary, it seems that one cannot perform step 2 without restrictions on G, for there are too many irreducible LV systems. However, it should be easy to solve problem (2) when G is simple (which does not imply the classification of all spherical homogeneous spaces for simple G). Some cases (A, C, D, E, the latter unpublished) are already done by Luna, Bravi and Pezzini.

To overcome the difficulty in the general case I am going to reduce it to the case when G is simple. The basic idea here is to use Knop's results, [Kn1], that describe all spherical subgroups containing a given one in terms of LV systems: these subgroups are in one to one correspondence with so called *colored subspaces*. I am going to study combinatorial properties of these objects. Then the idea is to assume that minimal colored subspaces from a certain class correspond to some subgroups $\widetilde{H}_1, \ldots, \widetilde{H}_k \subset G$ and try to recover H from $\widetilde{H}_1, \ldots, \widetilde{H}_k$. The reconstruction seems to be possible provided $k \geqslant 2$. Likely, the last condition holds in most cases when G is not simple.

Classification of all spherical homogeneous spaces together with the results of [Lo4] and [Bri2] will make possible the description of all *compatible* Kähler structures on a given multiplicity free compact Hamiltonian manifold (at least, in the case when the moment polytope is rational). While calling a Kähler structure compatible we mean that it is invariant and satisfies some other minor restrictions.

Proceed to MF Hamiltonian actions of compact groups. Knop's unpublished results reduce the classification to the question whether a given monoid is the weight monoid of a smooth affine spherical G-varieties. There are algorithms to determine the regular LV system of a smooth affine spherical variety from its weight monoid. They are implicitly contained in [Lo4]. So to classify MF Hamiltonian actions one needs to classify smooth affine spherical varieties in terms of their regular LV systems. However, unlike in the homogeneous case, it seems that it is not possible to give an explicit axiomatic description of all such LV systems. Again, we are going to solve a weaker problem.

Problem 3.1. *Find a combinatorial algorithm to determine whether a given LV system is a regular LV system of a smooth affine spherical G-variety.*

We are not going to use the classification of spherical homogeneous spaces in the solution of this problem. Instead, we plan to use partial classification of smooth affine spherical varieties due to Knop and Van Steirteghem, [KVS].

Proceed to Problem 1.4. The following conjecture generalizes of Theorem 2.1.

Conjecture 3.2. *Let X_1, X_2 be G-varieties with isomorphic LV systems. Then X_1, X_2 are birationally equivalent.*

This would immediately imply

Corollary 3.3. *Let X_1, X_2 be normal* affine *G-varieties with isomorphic* regular *LV systems. Then X_1, X_2 are isomorphic.*

The crucial role in the proof should be played by the following result.

Conjecture 3.4. *The natural map $\mathrm{Aut}^G(\mathbb{K}(X)) \to \mathrm{Aut}^{ess}(\mathcal{L}\mathcal{V}_{G,X})$ is surjective.*

The strategy I plan to use to prove these conjectures is the same as in [Lo5]:

Step 1. Reduce Conjectures 3.2,3.4 to the case of homogeneous spaces (this step does not occur in the spherical case).

Step 2. Prove Conjectures 3.2, 3.4 for affine homogeneous spaces.

Step 3. Develop the theory of inclusions of subgroups of G on the language of LV systems. Such a theory was developed by Knop, [Kn1] in the spherical case.

Step 4. Prove Conjecture 3.4 generalizing the proof of [Lo5], Theorem 2.

Step 5. Prove Conjecture 3.2 generalizing the proof of [Lo5], Theorem 1.

Now proceed to a generalization of Theorem 2.2. According to [Lo4], we make the following two conjectures.

Conjecture 3.5. *Let X_1, X_2 be normal affine G-varieties such that $\mathbb{K}[X_1]^U \cong \mathbb{K}[X_2]^U$ (an isomorphism of $T = B/U$-algebras); here U denotes the unipotent radical of B. Suppose that one of the following conditions is satisfied:*

(1) *Both X_1, X_2 are smooth.*
(2) *$\mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}$ (note that an isomorphism $\mathbb{K}[X_1]^U \cong \mathbb{K}[X_2]^U$ gives rise to the isomorphism $\mathbb{K}(X_1)^{(B)} \cong \mathbb{K}(X_2)^{(B)}$).*

Then there is an isomorphism $\mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$ compatible with that of $\mathbb{K}(X_1)^{(B)}, \mathbb{K}(X_2)^{(B)}$.

Conjecture 3.6. *Suppose that X_1, X_2 are smooth G-varieties such that there are compatible isomorphisms $\psi : \mathbb{K}[X_1]^U \to \mathbb{K}[X_2]^U, \iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$. Then $\psi^*(\mathcal{V}_{G,X_2}) = \mathcal{V}_{G,X_1}$.*

These two conjectures together with Corollary 3.3 would immediately imply

Corollary 3.7. *Let X_1, X_2 be such as in Conjecture 3.5. Then $X_1 \cong^G X_2$.*

Again, to prove Conjectures 3.5,3.6 we plan to follow the strategy of [Lo4].

Let us discuss a uniqueness property for Hamiltonian actions $K : M$. Fix a maximal torus $T \subset K$. In [GJS] Guillemin, Jeffrey and Sjamaar constructed the so called *imploded cross-section* M_{impl} of M, which is a stratified symplectic space acted on by T such that the restriction of the action to each symplectic stratum is Hamiltonian and the moment map is continuous. Moreover, the topological quotients M/K and M_{impl}/T are naturally identified.

Problem 3.8. *To construct a category $\mathcal{C}$ of singular compact Hamiltonian T-spaces enjoying the following properties.*

1. *$\mathcal{C}$ contains all imploded cross-sections.*
2. *A MF object in $\mathcal{C}$ is determined uniquely (up to an isomorphism) by the inefficiency kernel of the action and the image of the moment map.*
3. *If M_1, M_2 are compact Hamiltonian K-manifolds such that their imploded cross-sections are isomorphic in $\mathcal{C}$, then M_1, M_2 are isomorphic.*

4. Teaching experience and plans

1) Fall 2005, Independent University of Moscow: A course "Moment map", lecturer. The program and lecture notes (in Russian) are available at:
www.ium.mccme.ru/f05/momentum.html

2) Fall and Spring 2006, Independent University of Moscow: Algebra seminars for first year students, teaching assistant.

3) Fall 2006, Independent University of Moscow: A course "Lie groups and Lie algebras", lecturer. The program and exercise sheets (in Russian) are available at:
www.ium.mccme.ru/f06/lie.html

4) Fall 2006, Independent University of Moscow: Algebra seminars for first year students, teaching assistant.

5) Fall, 2007, Independent University of Moscow: Algebra seminars for first year students, teaching assistant.

After defending a PhD thesis (October, 12, 2007) I am going to return to Minsk and take a position at the Belarusian state university. There I will work personally with several strong students using lecture notes and exercise sheets of IUM.

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