KOSZUL DUALITY AND SEMI-INFINITE COHOMOLOGY

RESEACH STATEMENT

LEONID POSITSELSKI

Extension of the Koszul duality to ungraded modules over ungraded algebras requires elaboration of the distinction between derived categories of the first and the second kind. Certain mixtures of the two kinds of derived categories, called semiderived categories, play an important role in the theory of semi-infinite cohomology of associative algebras. The aim of this project is to extend Koszul duality to the relative situations with a base ring, a base coalgebra, or even a base coring; prove comparison theorems connecting the semi-infinite cohomology of Lie and associative algebras; study the relations between relative Koszul duality and semi-infinite cohomology; and look for generalizations of the existing definitions of semi-infinite cohomology.

1. Past Research

1.1. **Nonhomogeneous quadratic duality.** A graded algebra $A = \bigoplus_i A_i$ over a field k such that $A_i = 0$ for $i < 0$ and $A_0 = k$ is called *Koszul* if $\mathrm{Tor}^A_{i,j}(k,k) = 0$ for all $i \neq j$. A graded coalgebra $\mathcal{C} = \bigoplus_i \mathcal{C}_i$ over k such that $\mathcal{C}_i = 0$ for $i < 0$ and $\mathcal{C}_0 = k$ is called *Koszul* if $\mathrm{Ext}^{i,j}_{\mathcal{C}}(k,k) = 0$ for all $i \neq j$. The functors $A \longmapsto \mathcal{C} = \mathrm{Tor}^A(k,k)$ and $\mathcal{C} \longmapsto A = \mathrm{Ext}_{\mathcal{C}}(k,k)$ are mutually inverse equivalences between the categories of Koszul algebras and Koszul coalgebras. The algebra A and the coalgebra $\mathcal{C}$ are said to be quadratic dual to each other.

This equivalence can be extended to ungraded algebras as follows. A *nonhomogeneous Koszul algebra* is an associative algebra A over k endowed with an increasing filtration $F_iA \subset A$ such that the associated graded algebra $\mathrm{gr}_F A = \bigoplus_i F_iA/F_{i-1}A$ is Koszul. The nonhomogeneous quadratic duality assigns to a nonhomogeneous Koszul algebra a Koszul CDG-coalgebra (curved differential graded coalgebra).

A *CDG-algebra* $B^\bullet$ over a field k is a graded algebra endowed with an odd derivation d of degree $+1$ and an element $h \in B^2$ such that $d^2(x) = hx - xh$ for any $x \in B$ and $d(h) = 0$. A *morphism of CDG-algebras* $B'^\bullet \longrightarrow B''^\bullet$ is a pair (g, a) consisting of a morphism of graded algebras $g\colon B' \longrightarrow B''$ and an element $a \in B''^1$ such that $g(d'x) = d''g(x) + [a, x]$ for any $x \in B'$ and $g(h') = h'' + d''a + a^2$.

A thematic example of a CDG-algebra is the algebra $\Omega^\bullet(M, \mathrm{End}(E))$ of differential forms with values in the bundle of endomorphisms of a vector bundle E over a (smooth, analytic, or algebraic) variety M such that E is endowed with global connection ∇_E. The derivation d is the de Rham differential corresponding to the induced connection $\nabla_{\mathrm{End}(E)}$, while the element h is the curvature of ∇_E. CDG-algebras corresponding to different connections are naturally isomorphic.

A *CDG-coalgebra* $\mathcal{C}_\bullet$ over k is a graded coalgebra endowed with an odd coderivation d of degree -1 and a linear function $h\colon \mathcal{C}_2 \longrightarrow k$ satisfying the dual equations. A CDG-coalgebra is said to be Koszul if the underlying graded coalgebra is Koszul.

There is a natural equivalence between the categories of nonhomogeneous Koszul algebras and Koszul CDG-coalgebras such that the associated graded algebra of a nonhomogeneous Koszul algebra is quadratic dual to the underlying graded coalgebra of the corresponding CDG-coalgebra (essentially [P1], see also [6] and [PP]). This result is a generalization of the Poincare–Birkhoff–Witt theorem to algebras with nonhomogeneous quadratic relations of Koszul type.

1.2. **Koszul duality and exotic derived categories.** A thematic example of nonhomogeneous Koszul duality is the functor assigning to a complex of modules $M^\bullet$ over a Lie algebra $\mathfrak{g}$ its standard homological complex $C_\bullet(\mathfrak{g}, M^\bullet)$ considered as a DG-comodule over the DG-coalgebra $C_\bullet(\mathfrak{g})$. Since the standard complexes of nontrivial irreducible modules over semisimple Lie algebras are acyclic, one has to consider exotic derived categories in order to make this functor an equivalence of categories.

A *left CDG-module* over a CDG-algebra $B^\bullet$ is a graded left B-module $M^\bullet$ endowed with a differential d_M of degree $+1$ compatible with the differential d in B as an odd derivation and satisfying the equation $d_M^2(m) = hm$ for all $m \in M^\bullet$. A *left CDG-comodule* over a CDG-coalgebra $\mathcal{C}_\bullet$ is a graded left $\mathcal{C}$-comodule $\mathcal{M}_\bullet$ endowed with an odd coderivation $d_\mathcal{M}$ of degree -1 satisfying the dual equation.

Left CDG-comodules over a given CDG-coalgebra $\mathcal{C}_\bullet$ form a DG-category: for any CDG-comodules $\mathcal{L}_\bullet$ and $\mathcal{M}_\bullet$ there is a complex $\mathrm{Hom}_\mathcal{C}(\mathcal{L}_\bullet, \mathcal{M}_\bullet)$. Hence one can consider the homotopy category of CDG-comodules. A CDG-comodule $\mathcal{M}_\bullet$ over a CDG-coalgebra $\mathcal{C}_\bullet$ is called *coacyclic* if it is homotopy equivalent to a CDG-comodule obtained from total CDG-comodules of exact triples of CDG-comodules using the operations of cone and infinite direct sum. The *coderived category* of left CDG-comodules over $\mathcal{C}_\bullet$ is defined as the quotient category of the homotopy category of CDG-comodules over $\mathcal{C}_\bullet$ by the thick subcategory of coacyclic CDG-comodules. Notice that one cannot even speak of CDG-comodules acyclic in the conventional sense, as CDG-comodules have no cohomology.

To an augmented DG-algebra one can assign its bar construction, which is a conilpotent DG-coalgebra. Extending this construction, one can assign to a nonaugmented DG-algebra $A^\bullet$ a conilpotent CDG-coalgebra $\mathrm{Bar}(A^\bullet)$. Conversely, to a coaugmented CDG-coalgebra $\mathcal{C}_\bullet$ one can assign its cobar construction $\mathrm{Cob}(\mathcal{C}_\bullet)$, which is a DG-algebra. *Then the derived category of DG-modules over $A^\bullet$ is equivalent to the coderived category of CDG-comodules over* $\mathrm{Bar}(A^\bullet)$. *The coderived category of CDG-comodules over $\mathcal{C}_\bullet$ is equivalent to the coderived category of DG-modules over* $\mathrm{Cob}(\mathcal{C}_\bullet)$. *When the coalgebra $\mathcal{C}$ is conilpotent, the coderived category of CDG-comodules over $\mathcal{C}_\bullet$ is equivalent to the derived category of DG-modules over* $\mathrm{Cob}(\mathcal{C}_\bullet)$. For a nonhomogeneous Koszul algebra A and the Koszul CDG-coalgebra $\mathcal{C}_\bullet$ dual to it, the derived category of A-modules is equivalent to the coderived category of CDG-comodules over $\mathcal{C}_\bullet$, since A is quasi-isomorphic to $\mathrm{Cob}(\mathcal{C}_\bullet)$.

These results were presented in an informal series of talks in IAS, Princeton in the Spring of 1999; later I spoke about them at various seminars in Moscow, MPIM-Bonn, etc. They remain unpublished. In 2003 similar results appeared in the dissertation of K. Lefèvre-Hasegawa [9, 8], who also introduced the "coderived categories" terminology. His definition of coacyclic DG-comodules in terms of the cobar construction is different from, though equivalent to, the above.

1.3. **Semi-infinite cohomology.**

1.3.1. A *coring* $\mathcal{C}$ over a noncommutative ring A is a coring object in the tensor category of A-A-bimodules; in other words, it is an A-A-bimodule endowed with a comultiplication map $\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$ and a counit map $\mathcal{C} \longrightarrow A$, which should be A-A-bimodule morphisms satisfying the usual coassociativity and counit equations. A *left comodule* $\mathcal{M}$ over a coring $\mathcal{C}$ is a left A-module endowed with a left A-module morphims of left coaction $\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$ satisfying the usual equations. The category of left $\mathcal{C}$-comodules is abelian whenever $\mathcal{C}$ is a flat right A-module; in the sequel we will presume $\mathcal{C}$ to be a flat left and right A-module.

The *cotensor product* $\mathcal{N} \square_{\mathcal{C}} \mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ is defined as the kernel of the pair of maps $\mathcal{N} \otimes_A \mathcal{M} \rightrightarrows \mathcal{N} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$. This functor is neither left, nor right exact in general.

A right $\mathcal{C}$-comodule is called coflat if the functor of cotensor product with it is exact on the category of left $\mathcal{C}$-comodules. The cotensor product of bicomodules is not always associative, but for, say, bicomodules that are coflat right comodules, it is. A *right coflat semialgebra* $\mathbf{S}$ over a coring $\mathcal{C}$ is a ring object in the tensor category of right coflat $\mathcal{C}$-$\mathcal{C}$-bicomodules; in other words, it is a $\mathcal{C}$-$\mathcal{C}$-bicomodule that is a coflat right $\mathcal{C}$-comodule endowed with a *semimultiplication* map $\mathbf{S} \square_{\mathcal{C}} \mathbf{S} \longrightarrow \mathbf{S}$ and a *semiunit* map $\mathcal{C} \longrightarrow \mathbf{S}$, which should be morphisms of $\mathcal{C}$-$\mathcal{C}$-bicomodules satisfying the usual associativity and unity equations. A *left semimodule* $\mathbf{M}$ over a right coflat semialgebra $\mathbf{S}$ is a left $\mathcal{C}$-comodule endowed with a left $\mathcal{C}$-comodule morphism of *left semiaction* $\mathbf{S} \square_{\mathcal{C}} \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the usual equations. The category of left semimodules over a right coflat semialgebra is abelian; in the sequel we will presume that $\mathbf{S}$ is a coflat left and right $\mathcal{C}$-comodule.

The *semitensor product* $\mathbf{N} \Diamond_{\mathbf{S}} \mathbf{M}$ of a right $\mathbf{S}$-semimodule $\mathbf{N}$ and a left $\mathbf{S}$-semimodule $\mathbf{M}$ is defined as the cokernel of the pair of maps $\mathbf{N} \square_{\mathcal{C}} \mathbf{S} \square_{\mathcal{C}} \mathbf{M} \rightrightarrows \mathbf{N} \square_{\mathcal{C}} \mathbf{M}$. This functor is not everywhere defined, as the triple cotensor product $\mathbf{N} \square_{\mathcal{C}} \mathbf{S} \square_{\mathcal{C}} \mathbf{M}$ is not associative for arbitrary $\mathbf{N}$ and $\mathbf{M}$; but it is defined when one the semimodules $\mathbf{N}$ and $\mathbf{M}$ is a coflat $\mathcal{C}$-comodule, or when one of $\mathbf{N}$ and $\mathbf{M}$ is a flat A-module and the ring A has a finite weak homological dimension.

1.3.2. Assume that the ring A has a finite weak homological dimension. The *semiderived category* of left $\mathbf{S}$-semimodules is defined as the quotient category of the homotopy category of complexes of left $\mathbf{S}$-semimodules by the thick subcategory of complexes that are coacyclic *as complexes of left* $\mathcal{C}$*-comodules.* A complex of right $\mathbf{S}$-semimodules $\mathbf{N}^\bullet$ is called *semiflat* if its semitensor product $\mathbf{N}^\bullet \Diamond_{\mathbf{S}} \mathbf{M}^\bullet$ with any $\mathcal{C}$-coacyclic complex of left $\mathbf{S}$-semimodules $\mathbf{M}^\bullet$ is acyclic.

The functor mapping the quotient category of the homotopy category of semiflat complexes of left $\mathbf{S}$*-semimodules by its intersection with the thick subcategory of* $\mathcal{C}$*-coacyclic complexes of left* $\mathbf{S}$*-semimodules into the semiderived category of left* $\mathbf{S}$*-semimodules is an equivalence of triangulated categories.* Using this theorem, one can define the double-sided derived functor $\mathrm{SemiTor}^{\mathbf{S}}(\mathbf{N}^\bullet, \mathbf{M}^\bullet)$ on the Cartesian product of the semiderived categories of left and right $\mathbf{S}$-semimodules by restricting the functor of semitensor product either to the Carthesian product of the homotopy category of semiflat complexes of right $\mathbf{S}$-semimodules and the homotopy category of arbitrary complexes of left $\mathbf{S}$-semimodules, or to the Carthesian product of the homotopy category of arbitrary complexes of right $\mathbf{S}$-semimodules and the homotopy category of semiflat complexes of left $\mathcal{S}$-semimodules. This is a particular case of a general categorical definition of double-sided derived functors.

The above results were first explained (in the case of semialgebras over coalgebras over fields rather than corings over rings) in a series of letters to S. Arkhipov and R. Bezrukavnikov which were distributed privately for some time and later made available on the web [P2]. The influence of [P2] is acknowledged, e. g., in the paper [7]. Subsequently this was written up (in the more general setting above) in the long paper [P3]. It was shown in [P3] that the above construction of double-sided derived functor $\mathrm{SemiTor}^{\mathcal{S}}$ and an analogous construction of double-sided derived functor $\mathrm{SemiExt}_{\mathcal{S}}$ include as particular cases the definitions of semi-infinite Tor and Ext by S. Arkhipov [1, 2] and A. Sevostyanov [10].

1.4. **Comodule-contramodule correspondence.** There are two kinds of modules over a coalgebra: in addition to the more widely known comodules, there are also contramodules. A *left contramodule* $\mathfrak{P}$ over a coring $\mathcal{C}$ is a left A-module endowed with a left A-module map $\mathrm{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying certain contraassociativity and counity equations. The category of left $\mathcal{C}$-contramodules is abelian whenever $\mathcal{C}$ is a projective left A-module; we will presume that $\mathcal{C}$ is a projective left and a flat right A-module. The group of *cohomomorphisms* $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{M},\mathfrak{P})$ from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is defined as the cokernel of the pair of maps $\mathrm{Hom}_A(\mathcal{C}\otimes_A\mathcal{M},\mathfrak{P}) \rightrightarrows \mathrm{Hom}_A(\mathcal{C},\mathfrak{P})$, one of which comes from the $\mathcal{C}$-coaction in $\mathcal{M}$ and the other from the $\mathcal{C}$-contraaction in $\mathfrak{P}$. A left $\mathcal{C}$-comodule is called *coprojective* if the functor of cohomomorphisms from it is exact on the category of left $\mathcal{C}$-contramodules.

Let $\mathcal{S}$ be a semialgebra; we will presume that $\mathcal{S}$ is a coprojective left and a coflat right $\mathcal{C}$-comodule. A *left semicontramodule* $\boldsymbol{\mathfrak{P}}$ over $\mathcal{S}$ is a left $\mathcal{C}$-contramodule endowed with a left $\mathcal{C}$-contramodule map $\boldsymbol{\mathfrak{P}} \longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\boldsymbol{\mathfrak{P}})$ satisfying certain associativity and unity equations. Assume that the ring A has a finite left homological dimension. The *semiderived category* of left $\mathcal{S}$-semicontramodules is defined analogously to the semiderived category of left $\mathcal{S}$-semimodules except that one uses infinite products instead of infinite direct sums.

The semiderived categories of left $\mathcal{S}$-semimodules and left $\mathcal{S}$-semicontramodules are naturally equivalent. This result was proven modulo a certain conjecture in [P2] and completely proven in [P3].

2. Current Research and Future Plans

Let $(\mathfrak{g},H)$ be a Tate Harish-Chandra pair, i. e., $\mathfrak{g}$ is a Tate (locally linearly compact) Lie algebra [4] and H is a proalgebraic group corresponding to a linearly compact open Lie subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Let $c\colon(\mathfrak{g}',H)\longrightarrow(\mathfrak{g},H)$ be a morphism of Tate Harish-Chandra pairs with the same subgroup H such that $\mathfrak{g}'\longrightarrow\mathfrak{g}$ is a central extension of Lie algebras whose kernel is identified with the ground field k. One example of such a central extension of Tate Harish-Chandra pairs comes from the canonical central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$; we denote the corresponding morphism by c_0. It is not difficult to see that the tensor product $\mathcal{S}_c(\mathfrak{g},H) = U_c(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C}(H)$ has a natural structure of semialgebra over the coalgebra $\mathcal{C}=\mathcal{C}(H)$ of functions on H; the category of left semimodules over $\mathcal{S}_c=\mathcal{S}_c(\mathfrak{g},H)$ is isomorphic to the category of $(\mathfrak{g}',H)$-modules where the unit central element of $\mathfrak{g}'$ acts by the identity. It is expected that the semialgebra $\mathcal{S}_c^{\mathrm{op}}$ opposite to $\mathcal{S}_c$ is isomorphic to $\mathcal{S}_{-c_0-c}$. Now assume that the proalgebraic group H is prounipotent. Then it is expected that for any complexes $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ of right and left $\mathcal{S}_c$-semimodules the complex of semi-infinite forms with coefficients in $\mathcal{N}^\bullet \otimes_k \mathcal{M}^\bullet$ computes $\mathrm{SemiTor}^{\mathcal{S}_c}(\mathcal{N}^\bullet, \mathcal{M}^\bullet)$.

I am now working together with S. Arkhipov on this problem and it seems that the key ideas of the proof are present. The argument is based on a relative version of nonhomogeneous quadratic duality; basically, in order to obtain an isomorphism $\mathcal{S}_c^{\mathrm{op}} \simeq \mathcal{S}_{-c_0-c}$ one constructs a natural isomorphism between the CDG-coalgebras corresponding to these filtered semialgebras.

I plan to study the example of a semialgebra $\mathcal{S}$ over a coring $\mathcal{C}$ over a commutative ring A for which the right and the left actions of A in $\mathcal{C}$ are different that is coming from a pair consisting of a smooth affine grouppoid and a closed subgrouppoid with the same variety of objects. In particular, an equivalence between the categories of left and right $\mathcal{S}$-semimodules is expected.

I am working together with R. Bezrukavnikov on a generalization of the results of his paper [5]. The expected result in this direction would claim that under certain rather strict conditions on a finite-dimensional graded semialgebra $\mathcal{S}$ over a graded coalgebra $\mathcal{C}$ the graded version of the functor $\mathrm{SemiExt}_{\mathcal{S}}$ for bounded complexes of finite-dimensional semimodules and semicontramodules can be interpreted as morphisms from objects of the derived category of left $\mathcal{S}$-semimodules into objects of the derived category of left $\mathcal{S}$-semicontramodules through their common subcategory of bounded complexes of finite-dimensional $\mathcal{C}$-injective $\mathcal{S}$-semimodules, or, which is equivalent, $\mathcal{C}$-projective $\mathcal{S}$-semicontramodules.

For a semialgebra $\mathcal{S}$ over a coalgebra $\mathcal{C}$, it is expected that the relative Koszul duality should transform the functors $\mathrm{SemiTor}^{\mathcal{S}}$ and $\mathrm{SemiExt}_{\mathcal{S}}$ into the functors Cotor and Coext over the CDG-coalgebra obtained by applying the relative bar construction to $\mathcal{S}$; the absolute Koszul duality should then transform these functors into the conventional Tor and Ext over a DG-algebra. The same should hold for a semialgebra over a coring, except that the dual object would not be a CDG-coring, but rather have a structure analogous to the structure on the coring of polyvector fields dual to the de Rham differential on forms.

One would like to generalize the results of 1.3–1.4 from the case of a ring A of finite homological dimension to the case of a ring of arbitrary size endowed with a good enough topology, like the ring of functions on a good ind-affine ind-scheme. Notice that it is not difficult to define contramodules over good enough topological rings. I also plan to consider semialgebras in the categories of pro-vector spaces and ind-pro-vector spaces in the spirit of [7].

I plan to write a detailed paper on relative Koszul duality at some point.

3. Teaching Experience and Plans

My teaching experience ranges from teaching informal advanced courses on various mathematical subjects in Moscow in the beginning of '90s to working as a Course Assistant during my study as a Graduate Student at Harvard University to teaching courses on Galois Theory and Inverse Galois Problem at the Independent University of Moscow in 1999–2000 and 2003–2004. I plan to give a series of lectures or teach a course on semi-infinite cohomology of associative algebraic structures in IUM, provided that there is an interested audience.

References

[P1] L. Positselski. Nonhomogeneous quadratic duality and curvature. *Funct. Anal. Appl.* **27**, #3, p. 197–204, 1993.

[P2] L. Positselski. A series of letters about semi-infinite (co)homology of associative algebras (Russian). 2000, 2002. Available from `http://positselski.livejournal.com/314.html` .

[PP] A. Polishchuk, L. Positselski. Quadratic Algebras. University Lecture Series, 37. American Mathematical Society, Providence, RI, 2005. xii+159 pp.

[P3] L. Positselski. Homological algebra of semimodules and semicontramodules. Electronic preprint `arXiv:0709.3398v1 [math.CT]`, 190 pp, 2007.

[1] S. Arkhipov. Semi-infinite cohomology of associative algebras and bar-duality. *Internat. Math. Research Notices* **1997**, #17, p. 833–863. `arXiv:q-alg/9602013`

[2] S. Arkhipov. Semi-infinite cohomology of quantum groups II. Topics in quantum groups and finite-type invariants, p. 3–42, *Amer. Math. Soc. Translations, Ser. 2* **185**, AMS, Providence, RI, 1998. `arXiv:q-alg/9610020`

[3] A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory. *Journ. Amer. Math. Soc.* **9**, #2, p. 473–527, 1996.

[4] A. Beilinson, V. Drinfeld. Chiral algebras. AMS Colloquium Publications, 51. AMS, Providence, RI, 2004. See also: Quantization of Hitchin's integrable system and Hecke eigensheaves. February 2000. Available from `http://www.math.utexas.edu/~benzvi/Langlands.html` .

[5] R. Bezrukavnikov. On semi-infinite cohomology of finite dimensional algebras. Electronic preprint `arXiv:math.RT/0005148`.

[6] A. Braverman, D. Gaitsgory. Poincare–Birkhoff-Witt theorem for quadratic algebras of Koszul type. *Journ. Algebra* **181**, #2, p. 315–328, 1996. `arXiv:hep-th/9411113`

[7] D. Gaitsgory, D. Kazhdan. Algebraic groups over a 2-dimensional local field: some further constructions. Studies in Lie theory, p. 97-130, *Progress in Math.* **243**, Birkhaeuser Boston, Boston, MA, 2006. `arXiv:math.RT/0406282`

[8] B. Keller. Koszul duality and coderived categories (after K. Lefèvre). October 2003. Available from `http://www.math.jussieu.fr/~keller/publ/index.html` .

[9] K. Lefèvre-Hasegawa. Sur les A_∞-catégories. Thèse de doctorat, Université Denis Diderot – Paris 7, November 2003. Electronic preprint `arXiv:math.CT/0310337`.

[10] A. Sevostyanov. Semi-infinite cohomology and Hecke algebras. *Advances Math.* **159** #1, p. 83–141, 2001. `arXiv:math.RA/0004139`