OPTIMAL TRANSPORT, OMNI-POTENTIAL FLOW AND COSMOLOGICAL RECONSTRUCTION

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Based on the joint work: **U. Frisch, O. Podvigina, B. Villone, V. Zheligovsky**
Optimal transport by omni-potential flow and cosmological reconstruction.
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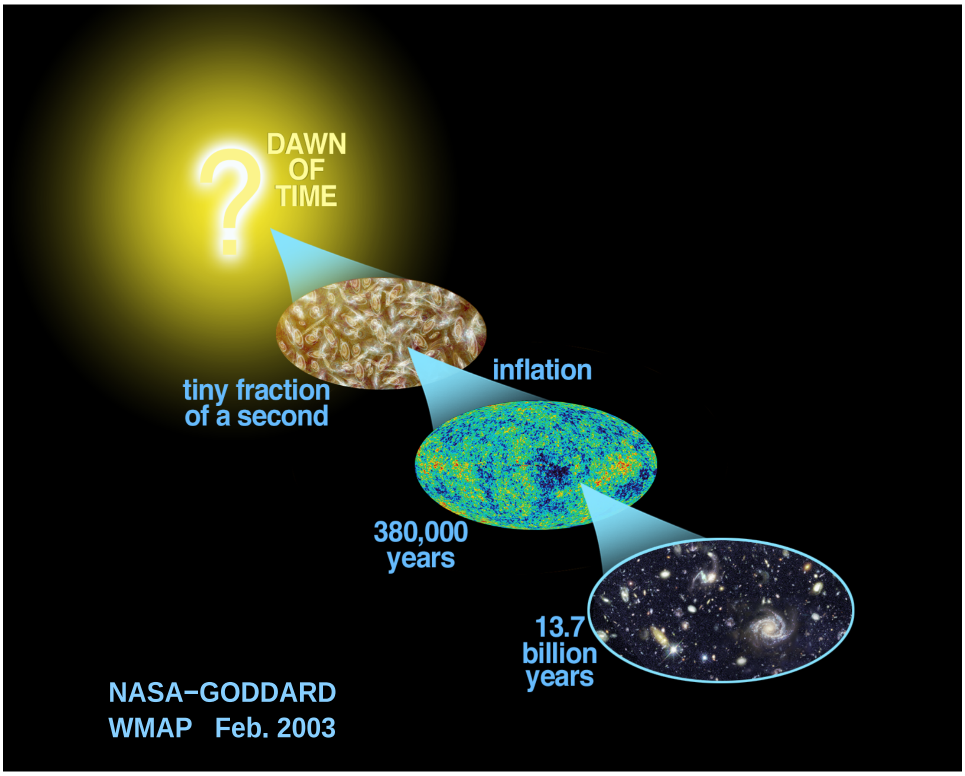
with additional contributions from: **J. Bec, A. Sobolevski**

THE PROBLEM OF RECONSTRUCTION



Find (reconstruct) the dynamical history of the Universe from the initial and present mass distribution.

Find the trajectory $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q}, t)$ of each point mass initially at $\boldsymbol{q}$, and the velocity $\partial_t \boldsymbol{x}(\boldsymbol{q}, t)$.

$\boldsymbol{q}$: **Lagrangian** coordinates;
$\boldsymbol{x}$: **Eulerian** coordinates.

The restricted problem:
Find the Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q})$ from the initial position $\boldsymbol{q}$ at $t = t_{\mathrm{in}}$ to the present one, $\boldsymbol{x}(\boldsymbol{q})$, at $t = t_0$, and its inverse.

FORMER WORK

• Statement of the problem of reconstruction (for a small number of Local Group galaxies) and application of variational methods:

P.J.E. Peebles. Astrophys. J. **344L**, 53-56 (1989)
P.J.E. Peebles. Astrophys. J. **362**, 1-13 (1990)

• Existence and uniqueness of solutions to the reconstruction BVP by action minimization:

Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee, A. Sobolevskii. Mon. Not. R. Astron. Soc. **346**, 501-524 (2003) [astro-ph/0304214]
G. Loeper. Arch. Rational Mech. Anal. **179**, 153-216 (2006)

• Statement of the problem of the optimal mass transport:

G. Monge. Hist. Acad. R. Sci. Paris, 666-704 (1781)

• Existence and uniqueness of solutions to the optimal mass transport problem:

Y. Brenier. C.R. Acad. Sci. Paris I, **305**, 805-808 (1987)
Y. Brenier. Comm. Pure Appl. Math. **44**, 375-417 (1991)
J.-D. Benamou, Y. Brenier. Numer. Math. **84**, 375-393 (2000)

• Development and application of the MAK algorithm:

U. Frisch, S. Matarrese, R. Mohayaee, A. Sobolevski. Nature, **417**, 260-262 (2002)
R. Mohayaee, U. Frisch, S. Matarrese, A. Sobolevski. Astron. Astrophys. **406**, 393-401 (2003)

• Asymptotic analysis of solutions to the Euler–Poisson equations:

Ya.B. Zeldovich. Astrophys. J. **5**, 84-89 (1970)
F. Moutarde, J.M. Alimi, F.R. Bouchet, R. Pellat, A. Raman. Astrophys. J. **382**, 377-381 (1991)
T. Buchert. Mon. Not. R. Astron. Soc. **254**, 729-737 (1992)
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T. Buchert. In Proc. IOP Enrico Fermi, Course CXXXII, *Dark Matter in the Universe*, Varenna 1995, eds.: S. Bonometto, J. Primack, A. Provenzale, IOS Press Amsterdam, pp. 543-564 (1995)
T. Buchert, J. Ehlers. Mon. Not. R. Astron. Soc. **264**, 375-387 (1993)
P. Catelan. Mon. Not. R. Astron. Soc. **276**, 115-124 (1995)
F. Bernardeau, S. Colombi, E. Gaztañaga, R. Scoccimarro. Phys. Rep. **367**, 1-309 (2002)

Newtonian statistical mechanics description of the condensation through self-gravitating dynamics of barionic matter.

- Particle of mass m and velocity $\boldsymbol{v}$ has the impulse $\boldsymbol{p} = m\boldsymbol{v}$.
- Particles are identical; their distribution function: $f(\boldsymbol{x}, \boldsymbol{p}, t)$.
- The matter density: $\rho(\boldsymbol{x}, t) = m \displaystyle\int f(\boldsymbol{x}, \boldsymbol{p}, t)\, \mathrm{d}\boldsymbol{p}$.
- Pressure is neglected; no diffusion (for simplicity).
- **The Liouville equation:** $\partial_t f + (m^{-1}\boldsymbol{p} \cdot \nabla_{\boldsymbol{x}} - \nabla_{\boldsymbol{x}} \varphi \cdot \nabla_{\boldsymbol{p}}) f = 0$.
- **The Poisson equation** for the gravity potential $\varphi(\boldsymbol{x}, t)$: $\nabla^2_{\boldsymbol{x}} \varphi = 4\pi G(\rho(\boldsymbol{x}, t) - \overline{\rho})$.

  In $\mathbb{R}^3$, $\varphi(\boldsymbol{x}, t) = -Gm \displaystyle\iint \frac{f(\boldsymbol{y}, \boldsymbol{p}, t)}{|\boldsymbol{y} - \boldsymbol{x}|}\, \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{y}$.

However, solving the Liouville equation in $\mathbb{R}^6$ is too numerically intensive a problem (although the unknown function is just a scalar field).

Single-speed solutions to hydrodynamic-like equations.

- Density is rescaled. Initially, at $\tau = 0$, it is uniform: $\rho_{\text{in}}(\boldsymbol{q}) = 1$.
- The "linear growth factor" $\tau \propto t^{2/3}$ is used instead of time t.
- Equations are in the spatial coordinate system co-moving with the expansion.
- **The Euler equation:** $\partial_\tau \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v} = -\dfrac{3}{2\tau}(\boldsymbol{v} + \nabla_{\boldsymbol{x}}\,\varphi)$.
- **Mass conservation:** $\partial_\tau \rho + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0$.
- **The Poisson equation** for the gravity potential $\varphi(\boldsymbol{x}, t)$: $\nabla^2_{\boldsymbol{x}}\,\varphi = (\rho - 1)/\tau$.
- The solution is non-singular near $\tau = 0$ only if *slaving* occurs: $\boldsymbol{v}_{\text{in}}(\boldsymbol{q}) = -\nabla_{\boldsymbol{x}}\,\varphi_{\text{in}}$.

Slaving implies that **for any $\tau \geq 0$ the flow velocity $\boldsymbol{v}(\boldsymbol{x}, \tau)$ is potential in the Eulerian coordinates.**

$\boldsymbol{v}(\boldsymbol{x}, \tau) = \nabla_{\boldsymbol{x}}\Psi$, the potential satisfying $\partial_\tau \Psi + \dfrac{1}{2}\,|\nabla_{\boldsymbol{x}}\Psi|^2 = -\dfrac{3}{2\tau}(\Psi + \varphi)$ and $\Psi_{\text{in}} = -\varphi_{\text{in}}$.

THE ZELDOVICH APPROXIMATION

• Solutions to the EP problem can be expanded in a power series in τ.

• The leading term of the expansion is the **Zeldovich approximation**, satisfying $\partial_\tau \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{v} = 0$.

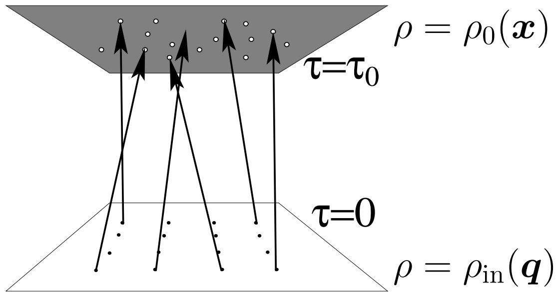
• In the Lagrangian formulation, the Zeldovich approximation amounts to $\mathrm{D}_\tau \boldsymbol{v} = 0$, i.e. particles move with constant speed along straight lines.

In the Zeldovich approximation, for any $\tau \geq 0$, **the flow velocity is potential both in the Lagrangian and Eulerian coordinates;**
$\Rightarrow$ **the map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q})$ is potential.**

• Actually, the second term in the short-time Lagrangian expansion yields a map, which is **potential in Lagrangian — but not Eulerian — coordinates** (Moutarde et al., 1991).

OPTIMAL MASS TRANSPORT PROBLEM

Consider the **restricted reconstruction problem**:



- **Optimal mass transport problem with quadratic cost** (Brenier 1987, 1991):

$$\int |\boldsymbol{x}(\boldsymbol{q}) - \boldsymbol{q}|^2 \rho_{\rm in}(\boldsymbol{q})\, \mathrm{d}\boldsymbol{q} = \int |\boldsymbol{x} - \boldsymbol{q}(\boldsymbol{x})|^2 \rho_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \ \to \ \text{minimum.}$$

- **Mass conservation**: $\rho_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \rho_{\rm in}(\boldsymbol{q})\, \mathrm{d}\boldsymbol{q}$.
- The optimal map is potential: $\boldsymbol{x}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q})$. The potential $\Phi(\boldsymbol{q})$ is convex.

$\Rightarrow$ The inverse mapping $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q})$ is well-defined and has a convex potential

$\Theta(\boldsymbol{x}) = \max_{\boldsymbol{q}}(\boldsymbol{x} \cdot \boldsymbol{q} - \Phi(\boldsymbol{q}))$ (the Legendre transform of Φ).

- Numerical algorithm: **The Monge-Ampère-Kantorovich (MAK) method**.

$$\left.\begin{array}{l}\text{Mass conservation: } \rho_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \rho_{\rm in}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} \\[2ex] \text{Map potentiality: } \boldsymbol{x}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q})\end{array}\right\} \quad \Rightarrow$$

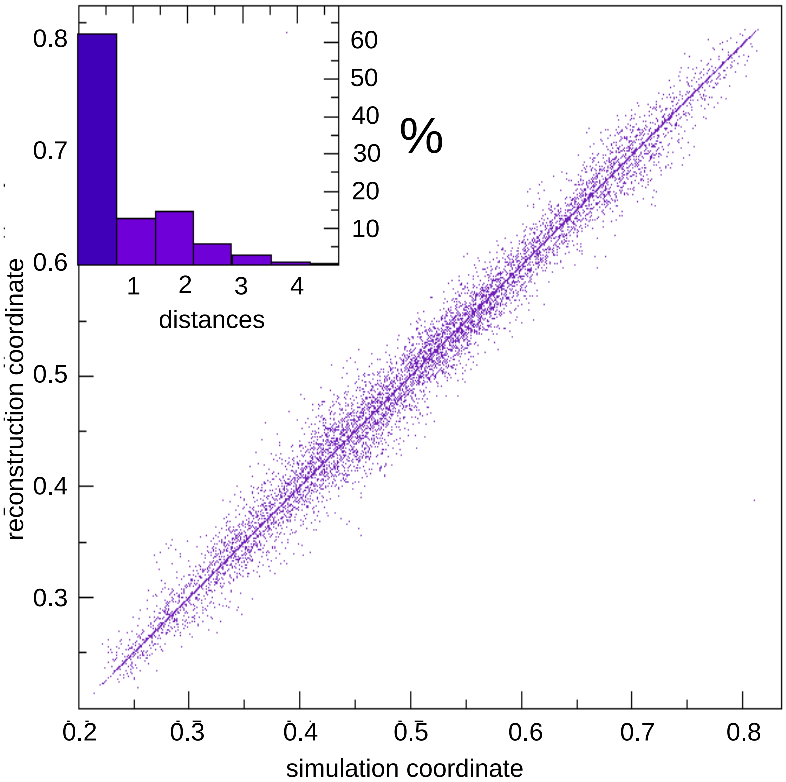
The Monge–Ampère equation: $\det \mathcal{H}(\Phi) = \dfrac{\rho_{\rm in}(\boldsymbol{q})}{\rho_0(\nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q}))}$,

where the matrix $\mathcal{H}(\Phi) \equiv |\partial^2_{q_i q_j}\Phi|$ is the **Hessian** of the potential $\Phi(\boldsymbol{q})$.

- $\rho_0(\boldsymbol{x})$, $\rho_{\rm in}(\boldsymbol{q}) > 0 \;\Rightarrow$ the potential $\Phi(\boldsymbol{q})$ is convex.

$\Rightarrow$ The inverse mapping $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q})$ is well-defined and has a convex potential $\Theta(\boldsymbol{x})$ satisfying, for $\rho_{\rm in} = 1$, the MAE $\det \mathcal{H}(\Theta) = \rho_0(\boldsymbol{x})$.

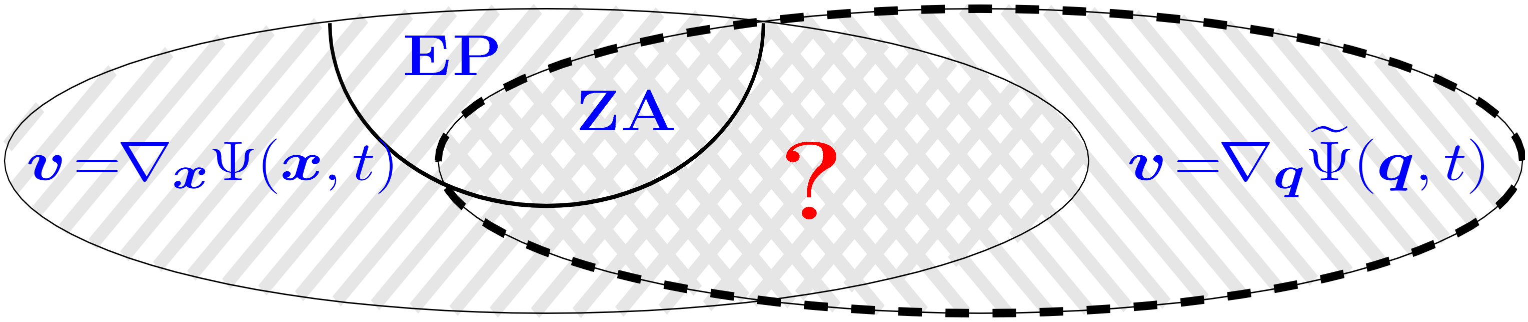
MAK RECONSTRUCTION TEST

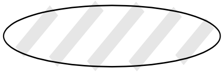


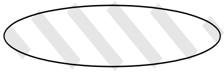
Test of the MAK reconstruction for a sample of N=17178 points initially situated on a cubic grid with $\Delta x = 6.25\ h^{-1}$Mpc. The scatter diagram plots true versus reconstructed initial positions using a quasiperiodic projection which ensures one-to-one correspondence with points on the cubic grid. The histogram inset gives the distribution (in percentages) of distances between true and reconstructed initial positions; the horizontal unit is the sample mesh. The width of the first bin is less than unity to ensure that only exactly reconstructed points fall in it.

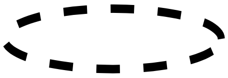
Brenier et al., MNRAS (2003);
Frisch et al., Nature (2002)

Why is the accuracy good?



 Flows, potential in the Eulerian coordinates

 Flows, potential in the Lagrangian coordinates

 $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}} \Phi(\boldsymbol{q}, t)$ (MAK $\Leftrightarrow$ MAE)

Actually, any optimal mass transport can be realized by a potential Euler flow [Benamou, Brenier, 2000].

EP: solutions to the Euler-Poisson equations (+ slaving); **ZA**: Zeldovich Approximation to EP; **MAE**: the Monge–Ampère equation; **MAK**: the Monge-Ampère-Kantorovich method.

Part II. EXAMPLES OF FLOWS, POTENTIAL IN LAGRANGIAN AND EULERIAN COORDINATES IN $\mathbb{R}^2$

Yes, such flows in $\mathbb{R}^2$ do exist!

- **Bi-potential and omni-potential flows in $\mathbb{R}^d$, $d \geq 2$**
- **Criteria for omni-potentiality of flows in $\mathbb{R}^d$, $d \geq 2$**
- **Zeldovich-type flows**
- **2D Hessian codiagonalizability PDE (HCE)**
- **Construction of omni-potential flows in $\mathbb{R}^2$, whose potentials are linear combinations of infinitely many homogeneous polynomials, by application of the 2D HCE.**

BI-POTENTIAL AND OMNI-POTENTIAL FLOWS

Let flows $\boldsymbol{v}(\boldsymbol{x}, t)$ be defined in $\mathbb{R}^d \times [0, T]$ and be sufficiently smooth.

Definition. If the flow velocity is potential in both Eulerian and Lagrangian coordinates, the flow is called **bi-potential**.

Definition. For any two times t and τ, such that $0 \le t < \tau \le T$, the mapping from fluid particle positions at time t to their positions at time τ is called the **(t, τ)-mapping**.

• The $(0, \tau)$-mapping is the standard Lagrangian map.

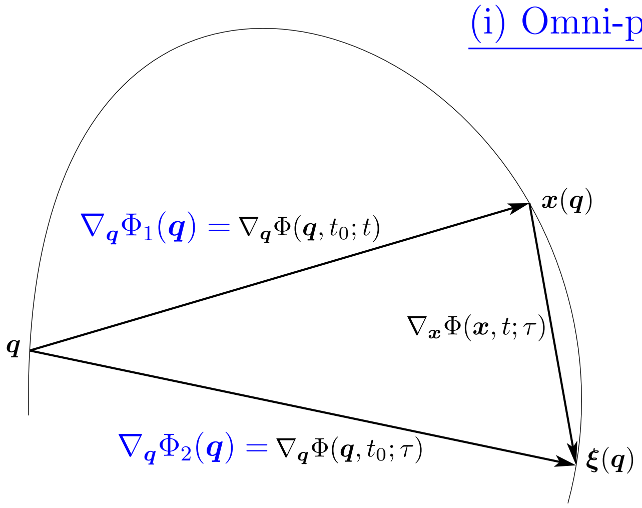
Definition. When the flow-induced (t, τ)-mapping between any two times t and τ is potential, $\boldsymbol{q} \mapsto \boldsymbol{x} = \nabla_{\boldsymbol{q}} \Phi(\boldsymbol{q}, t; \tau)$, the flow is called **omni-potential**.

• Here it is required that **all potentials $\Phi(\boldsymbol{q}, t; \tau)$ be convex in $\boldsymbol{q}$** $\Rightarrow$ the (t, τ)-mappings have inverses that are also potential. **Invertibility and continuity of the Hessians $\mathcal{H}(\Phi(\boldsymbol{q}, t; \tau))$ in t and τ imply convexity of the potentials.** The (t, t)-mapping is the identity mapping having the convex potential $\Phi(\boldsymbol{q}, t; t) = |\boldsymbol{q}|^2/2$, whose Hessian is the identity. For $\tau > t$, convexity is lost when an eigenvalue of the Hessian goes through zero; $\Rightarrow$ the Jacobian matrix (for a potential mapping, equal to the Hessian of the potential) becomes degenerate; $\Rightarrow$ generically, the inverse mapping ceases to exist.

CRITERIA FOR OMNI-POTENTIALITY

***Theorem*. (i) A flow is omni-potential $\Leftrightarrow$ the Hessians $\mathcal{H}(\Phi(q,t;\tau))$, calculated at the same trajectory for any two pairs of times, t and τ, commute; (ii) $\Leftrightarrow$ $\dot{\mathcal{H}}\mathcal{H} = \mathcal{H}\dot{\mathcal{H}}$, where we have denoted $\mathcal{H}(t) = \mathcal{H}(\Phi(\boldsymbol{q};0,t))$; (iii) Omni-potentiality of a flow is equivalent to its bi-potentiality.**

(i) Omni-potentiality $\Leftrightarrow$ the Hessians commute (same start time)



Potential composition of two potential maps.

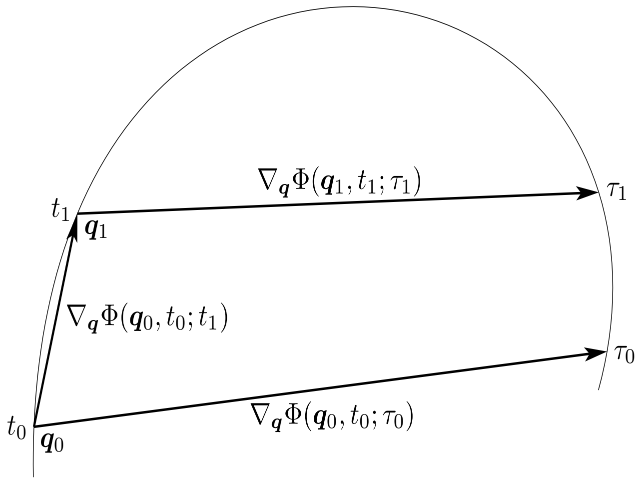
The (t,τ)-mapping is potential $\Leftrightarrow$ $\partial_{x_j}\xi_i = \partial_{x_i}\xi_j \;\; \forall\, 1 \le i,j \le d$. By the chain rule,

$$\mathcal{H}_{mn}(\Phi_2) = \partial_{q_n}\xi_m = \sum_{k=1}^{d} \partial_{x_k}\xi_m\,\partial_{q_n}x_k$$

$$= \sum_{k=1}^{d} \partial_{x_k}\xi_m\,\mathcal{H}_{kn}(\Phi_1)$$

$$\Leftrightarrow \quad \left\|\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\right\| = \mathcal{H}(\Phi_2)\mathcal{H}^{-1}(\Phi_1).$$

The r.h.s. is a symmetric matrix $\Leftrightarrow$ the Hessians $\mathcal{H}(\Phi_2)$ and $\mathcal{H}(\Phi_1)$ commute.



A sketch of a trajectory and flow-induced mappings from times t_0 and t_1 to times τ_0 and τ_1.

- Along a given trajectory, the Hessians of the potentials for the same start time commute: e.g., for the (t_0, t_1)-mapping and the (t_0, τ_0)-mapping.

- Similarly, the Hessians of the potentials of two mappings, such that the end time of one of them coincides with the start time of the second one, commute: e.g., for the (t_0, t_1)-mapping and the (t_1, τ_1)-mapping.

⇒ By the theorem on codiagonalizability of symmetric commuting matrices (commutativity of symmetric matrices with distinct eigenvalues is associative), the Hessians of the potentials of the (t_0, τ_0)-mapping and of the (t_1, τ_1)-mapping, calculated for the same trajectory, commute, for any $t_0 \le \tau_0$ and $t_1 \le \tau_1$.

Proof of (ii). Commutation of the Hessian and its time derivative

$H(t)H(t') = H(t')H(t) \;\; \forall t, t' \;\Rightarrow\; H(t)\dot{H}(t) = \dot{H}(t)H(t)$,
where $H(t)$ is any differentiable family of symmetric matrices, e.g. $H(t) = \mathcal{H}(\Phi(\boldsymbol{q}, 0; t))$.
The converse is also true.

Suppose (for simplicity) all eigenvalues λ_i of the symmetric matrix $H(t)$ are distinct.

$H(t) = U^t(t)\Lambda(t)U(t)$, where U is a unitary matrix, and Λ is diagonal.

$U(t)U^t(t) = I \;\Rightarrow\; U\dot{U}^t = -\dot{U}U^t \;\Rightarrow\; X \equiv U\dot{U}^t$ is antisymmetric.

$H(t)\dot{H}(t) = \dot{H}(t)H(t) \;\Rightarrow\; \Lambda(X\Lambda - \Lambda X) = (X\Lambda - \Lambda X)\Lambda$. By the theorem on codiagonalizability,

$X\Lambda - \Lambda X$ and Λ are simultaneously diagonalizable $\Rightarrow X\Lambda - \Lambda X$ is diagonal.

The entries of $X\Lambda - \Lambda X$ are $(\lambda_j - \lambda_i)X_{ij} \;\Rightarrow\; X_{ij} = 0 \;\forall i \neq j$.

By antisymmetry of X, $X = 0 \;\Rightarrow\; \dot{U} = -XU = 0$. QED.

Proof of (iii). Omni-potentiality $\Rightarrow$ bi-potentiality

• Let the (t, τ)-mappings be the gradients of convex potentials, $\boldsymbol{q} \mapsto \boldsymbol{x} = \nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q}, t; \tau)$.
Differentiation in τ yields $\boldsymbol{v}(\boldsymbol{q}, t; \tau) = \nabla_{\boldsymbol{q}}\Psi(\boldsymbol{q}, t; \tau)$, where $\Psi(\boldsymbol{q}, t; \tau) = \partial_\tau\Phi(\boldsymbol{q}, t; \tau)$.

$\Rightarrow$ In an omni-potential flow, the Lagrangian velocity $\boldsymbol{v}(\boldsymbol{q}, 0; t) = \nabla_{\boldsymbol{q}}\Psi(\boldsymbol{q}, 0; t)$
and the Eulerian velocity $\boldsymbol{v}(\boldsymbol{x}, t; t) = \nabla_{\boldsymbol{x}}\Psi(\boldsymbol{x}, t; t)$ are both potential.

Converse: Bi-potentiality ⇒ omni-potentiality

Denote by $\boldsymbol{v}^{\mathrm{L}}(\boldsymbol{q},t)$ and $\boldsymbol{v}^{\mathrm{E}}(\boldsymbol{x},t)$ the Lagrangian and Eulerian velocity, respectively.

$\boldsymbol{v}^{\mathrm{L}}(\boldsymbol{q},t)$ is potential ⇒ the Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q},t)$ has a convex potential $\Phi(\boldsymbol{q},0;t)$.

$\boldsymbol{v}^{\mathrm{E}}(\boldsymbol{x},t) = \boldsymbol{v}^{\mathrm{L}}(\boldsymbol{q}(\boldsymbol{x},t),t)$;
$\boldsymbol{x} \mapsto \boldsymbol{q}(\boldsymbol{x},t)$ is the inverse Lagrangian map;
its Jacobian is $\mathcal{H}^{-1}$, where $\mathcal{H} = \mathcal{H}(\Phi(\boldsymbol{q},0;t))$.

By the chain rule, $\forall i,j$,

$$\partial_{x_i} v_j^{\mathrm{E}}(\boldsymbol{x},t) = \sum_{m=1}^{d} (\mathcal{H}^{-1})_{im}\, \partial_{q_m} v_j^{\mathrm{L}}(\boldsymbol{q},t) = \sum_{m=1}^{d} (\mathcal{H}^{-1})_{im}\, \partial^2_{q_m q_j} \dot{\Phi}(\boldsymbol{q},0;t) = \sum_{m=1}^{d} (\mathcal{H}^{-1})_{im} \dot{\mathcal{H}}_{mj}.$$

$\boldsymbol{v}^{\mathrm{E}}(\boldsymbol{x},t)$ is potential
$\Leftrightarrow \partial_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{E}}(\boldsymbol{x},t) = \mathcal{H}^{-1}\dot{\mathcal{H}}$ is a symmetric matrix
$\Leftrightarrow \mathcal{H}^{-1}$ (and $\mathcal{H}$) commute with $\dot{\mathcal{H}}$.

QED

ZELDOVICH-TYPE FLOWS

• In the Zeldovich approximation, the Lagrangian map is $\boldsymbol{q} \mapsto \boldsymbol{x} = \nabla_{\boldsymbol{q}} \left(\frac{|\boldsymbol{q}|^2}{2} + t\Psi_0(\boldsymbol{q}) \right)$.
Here $\Psi_0(\boldsymbol{q})$ is the velocity potential at $t = 0$.

• Clearly, the Hessians $\mathcal{H}(\boldsymbol{q}, t) = I + t\mathcal{H}(\Psi_0)$ commute $\Rightarrow$ **the flow is omni-potential**.
Here I is the identity matrix.

• Similarly, the maps defined by the potentials $\Phi(\boldsymbol{q}, 0; t) = \mu(t)\dfrac{|\boldsymbol{q}|^2}{2} + \eta(t)\Psi_0(\boldsymbol{q})$,
are associated with omni-potential flows. Here $\mu(t)$ and $\eta(t)$ are arbitrary.

• After the zooming factor $1/\mu(t)$ is applied, and the new time $t' = \eta(t)/\mu(t)$ is introduced, particles move along straight lines with a constant velocity, like in Zeldovich flow.
We call such flows **Zeldovich-type flows**.

Do omni-potential flows exist that are not of this type?

Another "uninteresting" spherically-symmetric flow: $\Phi(\boldsymbol{q}, 0; t) = \Phi(|\boldsymbol{q}|, t)$.
$\mathcal{H}_{ij}(\Phi(|\boldsymbol{q}|, t)) = \Phi'|\boldsymbol{q}|^{-1}\delta_i^j + (\Phi''|\boldsymbol{q}|^{-2} - \Phi'|\boldsymbol{q}|^{-3})q_i q_j$.
Particles move along straight lines in radial directions.

A PDE FOR TWO-DIMENSIONAL OMNI-POTENTIAL FLOW

Let an eigenvector of a symmetric 2×2 matrix H make angle θ with the cartesian axis:

$$H_{11}\cos\theta + H_{12}\sin\theta = \lambda\cos\theta, \qquad H_{12}\cos\theta + H_{22}\sin\theta = \lambda\sin\theta$$

$$\Rightarrow \qquad \frac{H_{11} - H_{22}}{H_{12}} = \cot 2\theta$$

The r.h.s. uniquely defines the orthogonal frame of the two eigendirections.

The values of $\cot 2\theta$ define θ modulo $\pi/2$;

changing $\theta \to \theta + \pi/2$ swaps the eigendirections, but does not affect the set of eigendirections.

In an omni-potential flow, the eigendirections of the Hessians of the $(0,t)$-potentials should depend only on the Lagrangian position $\boldsymbol{q}$ and not on the time t.

In $\mathbb{R}^2$, omni-potential flow with the potential $\Phi(\boldsymbol{q},t)$ satisfies the 2D HCE:

$$(\partial^2_{q_1q_1} - \partial^2_{q_2q_2})\Phi = g(\boldsymbol{q})\,\partial^2_{q_1q_2}\Phi.$$

The search for non-Zeldovich-type omni-potential flow in $\mathbb{R}^2$ is reduced to solving the "2D Hessian codiagonalizability equation" (2D HCE) for suitably prescribed functions $g(\boldsymbol{q})$.

The strategy: use the 2D HCE $(\partial^2_{q_1q_1} - \partial^2_{q_2q_2})\Phi = g(\boldsymbol{q})\,\partial^2_{q_1q_2}\Phi$.

• Find linearly independent solutions, $\Phi_k(\boldsymbol{q})$; $\Phi_0(\boldsymbol{q}) = |\boldsymbol{q}|^2/2$.
By linearity, $\Phi = \sum_k \mu_k(t)\Phi_k(\boldsymbol{q})$ is a solution.

• The potential **Φ gives rise to an omni-potential flow** that is of non-Zeldovich type, if $\mu_k(t)$ are linearly independent. (Smallness of μ_k for $k > 0$ ensures convexity of Φ is inherited from $|\boldsymbol{q}|^2/2$).

The algebraic approach

• Let $g(\boldsymbol{q})$ be a ratio of homogeneous polynomials of degree m ($2m + 1$ independent coefficients).

• Seek a homogeneous polynomial solution, $p_n^{(2)}(\boldsymbol{q})$, of degree $n \geq m+2$ (n independent coefficients).

• The 2D HCE reduces to $m + n - 1$ equations in $2m + n + 1$ coefficients.

$\Rightarrow$ A family of $p_n^{(2)}(\boldsymbol{q})$ parameterized by $m + 2$ coefficients of $g(\boldsymbol{q})$ is expected to exist (however, the equations for the coefficients are, in general, nonlinear).

• When $g(\boldsymbol{q})$ is the ratio of linear functions, the equations for the coefficients of $p_n^{(2)}(\boldsymbol{q})$ are linear, and can be solved for any prescribed coefficients of $g(\boldsymbol{q})$.

Solving $(\partial^2_{q_1q_1} - \partial^2_{q_2q_2})\Phi = g(\boldsymbol{q})\,\partial^2_{q_1q_2}\Phi$ for $g(\boldsymbol{q}) = (aq_1^2 - bq_2^2)/(q_1q_2)$

- Homogeneous polynomial solutions involving only even powers of q_i (to enforce convexity in $\mathbb{R}^2$):

$$p^{(2)}_{2k}(q_1,q_2) = \sum_{i=0}^{k}\left(\prod_{j=0}^{i-1}(2k-1+2j(a-1))\prod_{j=0}^{k-1-i}(2k-1+2j(b-1))\right)\frac{k!\; q_1^{2i}q_2^{2(k-i)}}{i!(k-i)!(2k-1)}.$$

- Small-n examples: $p^{(2)}_4(q_1,q_2) = (2a+1)q_1^4 + 6q_1^2q_2^2 + (2b+1)q_2^4$,

$p^{(2)}_6(q_1,q_2) = (4a+1)(2a+3)q_1^6 + 15(2a+3)q_1^4q_2^2 + 15(2b+3)q_1^2q_2^4 + (4b+1)(2b+3)q_2^6$.

- $p^{(2)}_{2k} = 0$ identically for $\widehat{a} = 1 - \dfrac{2k-1}{2\widehat{j}}$ and $\widehat{b} = 1 - \dfrac{2k-1}{2j}$, where $j \geq 1$ and $\widehat{j} \geq 1$ are integer, and $j + \widehat{j} \leq k-1$.

For such $\widehat{a}$ and $\widehat{b}$, two independent solutions are $\dfrac{\partial}{\partial a}\,p^{(2)}_{2k}\Big|_{a=\widehat{a},\,b=\widehat{b}}$ and $\dfrac{\partial}{\partial b}\,p^{(2)}_{2k}\Big|_{a=\widehat{a},\,b=\widehat{b}}$.

- $p^{(2)}_{2k}(\boldsymbol{q})$ is convex, if all coefficients are positive: $\min(a,b) \geq -1/(2k-2)$.

The potentials $$\boxed{\Phi(\boldsymbol{q},t) = \mu_2(t)\frac{|\boldsymbol{q}|^2}{2} + \sum_{k\geq 2}\mu_{2k}(t)p^{(2)}_{2k}(q_1,q_2)}$$ are convex if $\min(a,b) \geq 0$ and all $\mu_{2k}(t) \geq 0$. Also need $\mu_{2k}(t) \to 0$ fast enough to ensure the convergence.

- The initial condition is satisfied, if $\mu_2(0) = 1$ and $\mu_{2k}(0) = 0\ \forall k > 1$.

Part III. EXAMPLES OF SYMMETRIC OMNI-POTENTIAL FLOWS IN $\mathbb{R}^3$

Yes, omni-potential flows in $\mathbb{R}^d$ for $d \geq 3$ do exist!
But all our examples of such flows are symmetric in q_i.

- **Invariants of $d \times d$ real symmetric matrices under variation of eigenvalues (for $d \geq 2$)**
- **A set of PDEs for omni-potential flows in $\mathbb{R}^3$**
- **Construction of omni-potential flows in $\mathbb{R}^d$ (for $d \geq 3$), whose potentials are linear combinations of three symmetric homogeneous polynomials of degree up to 6**
- **Construction of omni-potential flows in $\mathbb{R}^3$, whose potentials are linear combinations of infinitely many symmetric homogeneous polynomials**

Part IV. OPEN PROBLEMS

INVARIANTS OF SYMMETRIC MATRICES UNDER VARIATION OF EIGENVALUES

How to characterize the linear subspace of $d \times d$ symmetric matrices (spanned by $\{H^k \,|\, 0 \le k \le d-1\}$)**, whose frame of eigendirections coincides with that of a given H?**

This must have been done in the XIX century, but we have not found references.

• The general problem can be tackled by using **Plücker coordinates**.
For $d > 3$, our characterization involves fewer invariants.

Let (for simplicity) all eigenvalues λ_i of the symmetric $d \times d$ matrix H be distinct
$\Leftrightarrow$ all eigendirections be uniquely defined. Denote by $\mathbf{h}(\lambda_i)$ an eigenvector associated with λ_i.

• For $1 \le m \ne n \le d$ and $k \le d$, set $\beta_{mn,i} = h_m(\lambda_i)/h_n(\lambda_i)$ and $\gamma_{mn}^{(d,k)} = P^{(d,k)}(\beta_{mn,1}, ..., \beta_{mn,d})$,
where $P^{(d,k)}$ are symmetric homogeneous polynomials of degree $k \le d$:
for $\boldsymbol{y} \in \mathbb{R}^d$, $P^{(d,k)}(\boldsymbol{y}) \equiv \sum_{1 \le j_1 < ... < j_l < ... < j_k \le d} y_{j_1} ... y_{j_l} ... y_{j_k}$.

By construction, $\gamma_{mn}^{(d,k)}$ **are invariant** — they depend only on the set of eigendirections.

• Expressing $\mathbf{h}(\lambda_i)$ in terms of λ_i and H_{ij}, represent $\gamma_{mn}^{(d,k)}$ as a rational function of H_{ij} and λ_i.

• λ_i enter only through symmetric polynomial combinations, that are
known functions of H_{ij} by Viète's theorem applied to the characteristic polynomial.

• An arbitrary set of d orthogonal directions in $\mathbb{R}^d$ is described by $d(d-1)/2$ parameters. The $d^2(d-1)$ invariants $\gamma_{mn}^{(d,k)}$ are clearly too numerous to be independent.
E.g., for any $1 \le m \ne n \ne l \le d$ and $0 < k < d$, $\gamma_{mn}^{(d,d)}\gamma_{nm}^{(d,d)} = 1$, $\gamma_{ml}^{(d,d)}\gamma_{ln}^{(d,d)} = \gamma_{mn}^{(d,d)}$, $\gamma_{mn}^{(d,k)} = \gamma_{mn}^{(d,d)}\gamma_{nm}^{(d,d-k)}$.

• Do $d(d-1)/2$ suitably chosen **invariants uniquely define the frame of eigendirections?**

• In $\mathbb{R}^2$, $\gamma_{12}^{(2,1)} = (H_{11} - H_{22})/H_{12}$ is the only non-trivial invariant.

INVARIANTS IN $\mathbb{R}^3$

• In $\mathbb{R}^3$, $\mathbf{h}(\lambda_i) = (H_{12}H_{23} + H_{13}(\lambda_i - H_{22}),\ H_{12}H_{13} + H_{23}(\lambda_i - H_{11}),\ (\lambda_i - H_{11})(\lambda_i - H_{22}) - H_{12}^2)$

$$\Rightarrow \quad \gamma_{21}^{(3,1)} = \frac{H_{22} - H_{11}}{H_{12}} + \frac{H_{13}}{H_{12}} \frac{(H_{11} - H_{22})H_{13}H_{23} + (H_{23}^2 - H_{13}^2)H_{12}}{(H_{22} - H_{33})H_{12}H_{13} + (H_{13}^2 - H_{12}^2)H_{23}}$$

$$+ \frac{(H_{11} - H_{33})H_{12}H_{23} + (H_{23}^2 - H_{12}^2)H_{13}}{(H_{22} - H_{33})H_{12}H_{13} + (H_{13}^2 - H_{12}^2)H_{23}}.$$

• $\gamma_{21}^{(3,2)} = \gamma_{21}^{(3,3)}\gamma_{12}^{(3,1)}$; $\gamma_{21}^{(3,3)}$ is the ratio of two polynomials $\prod_{i=1}^{d}(\lambda_i + c) = \det \|H + cI\|$:

$$\gamma_{21}^{(3,3)} = -\frac{(H_{11} - H_{33})H_{12}H_{23} + (H_{23}^2 - H_{12}^2)H_{13}}{(H_{22} - H_{33})H_{12}H_{13} + (H_{13}^2 - H_{12}^2)H_{23}}.$$

• The invariants $\gamma_{21}^{(3,k)}$ for $1 \leq k \leq 3$ uniquely define $\beta_i = h_2(\lambda_i)/h_1(\lambda_i)$:
by Viète's theorem, β_i are roots of

$$\beta^3 - \gamma_{21}^{(3,1)}\beta^2 + \gamma_{21}^{(3,2)}\beta^2 - \gamma_{21}^{(3,3)} = 0.$$

• Eigenvectors are recovered as $\mathbf{h}(\lambda_i) = (1, \beta_i, c_i)$,
where c_i are determined from the orthogonality relations.

• This yields two solutions: $\{c_i\}$ and $\{-c_i\}$.
$\Rightarrow$ The invariants $\gamma_{21}^{(3,k)}$, $1 \leq k \leq 3$, define two distinct sets of eigendirections.
The non-uniqueness is eliminated, if in addition we know any of $\gamma_{j3}^{(3,i)}$ or $\gamma_{3j}^{(3,i)}$ for $i = 1, 3$ and $j = 1, 2$.

• The invariants $\gamma_{21}^{(3,k)}$, $1 \leq k \leq 3$, admit real values γ_k, respectively, whenever
(i) The equation for β_i has three real roots:

$$4(3\gamma_2 - \gamma_1^2)^3 + (2\gamma_1^3 - 9\gamma_1\gamma_2 + 27\gamma_3)^2 \leq 0.$$

(ii) The orthogonality relations are solvable in c_i:

$$(1 + \beta_1\beta_2)(1 + \beta_2\beta_3)(1 + \beta_3\beta_1) \leq 0 \qquad \Leftrightarrow \qquad \gamma_2 + \gamma_1\gamma_3 + \gamma_3^2 \leq -1.$$

$$\frac{\partial^2_{q_1,q_3}\Phi}{\partial^2_{q_1,q_2}\Phi}\left(\frac{(\partial^2_{q_1,q_1}-\partial^2_{q_2,q_2})\Phi}{\partial^2_{q_1,q_2}\Phi}+\frac{\partial^2_{q_2,q_3}\Phi}{\partial^2_{q_1,q_3}\Phi}-\frac{\partial^2_{q_1,q_3}\Phi}{\partial^2_{q_2,q_3}\Phi}\right)$$

$$=\left(g_1(\boldsymbol{q})+\frac{(\partial^2_{q_1,q_1}-\partial^2_{q_2,q_2})\Phi}{\partial^2_{q_1,q_2}\Phi}\right)\left(\frac{(\partial^2_{q_2,q_2}-\partial^2_{q_3,q_3})\Phi}{\partial^2_{q_2,q_3}\Phi}+\frac{\partial^2_{q_1,q_3}\Phi}{\partial^2_{q_1,q_2}\Phi}-\frac{\partial^2_{q_1,q_2}\Phi}{\partial^2_{q_1,q_3}\Phi}\right),$$

$$\frac{\partial^2_{q_2,q_3}\Phi}{\partial^2_{q_1,q_3}\Phi}\left(g_1(\boldsymbol{q})+\frac{(\partial^2_{q_1,q_1}-\partial^2_{q_2,q_2})\Phi}{\partial^2_{q_1,q_2}\Phi}\right)=g_2(\boldsymbol{q})-g_3(\boldsymbol{q})\frac{(\partial^2_{q_1,q_1}-\partial^2_{q_2,q_2})\Phi}{\partial^2_{q_1,q_2}\Phi},$$

$$g_3(\boldsymbol{q})\left(\frac{(\partial^2_{q_2,q_2}-\partial^2_{q_3,q_3})\Phi}{\partial^2_{q_2,q_3}\Phi}+\frac{\partial^2_{q_1,q_3}\Phi}{\partial^2_{q_1,q_2}\Phi}-\frac{\partial^2_{q_1,q_2}\Phi}{\partial^2_{q_1,q_3}\Phi}\right)=\frac{(\partial^2_{q_3,q_3}-\partial^2_{q_1,q_1})\Phi}{\partial^2_{q_1,q_3}\Phi}+\frac{\partial^2_{q_1,q_2}\Phi}{\partial^2_{q_2,q_3}\Phi}-\frac{\partial^2_{q_2,q_3}\Phi}{\partial^2_{q_1,q_2}\Phi},$$

where g_k are **modified invariants**: $g_1(\boldsymbol{q})=\gamma_{21}^{(3,1)}+\gamma_{21}^{(3,3)},\ g_2(\boldsymbol{q})=\gamma_{21}^{(3,2)}+1,\ g_3(\boldsymbol{q})=\gamma_{21}^{(3,3)}$.

AN OPEN PROBLEM: What are the solvability conditions in terms of $g_k(\boldsymbol{q})$?

The strategy

• The potential is a linear combination of homogeneous polynomials, $p_n^{(d)}(\boldsymbol{q})$ (of degree n), with time-dependent coefficients. One polynomial, $p_m^{(d)}(\boldsymbol{q})$, is prescribed. For any other polynomial the commutator of the two Hessians,

$$C(p_m^{(d)}, p_n^{(d)}) \equiv \mathcal{H}(p_m^{(d)})\mathcal{H}(p_n^{(d)}) - \mathcal{H}(p_n^{(d)})\mathcal{H}(p_m^{(d)})$$

must vanish. This implies the required commutation of the Hessians. $|\boldsymbol{q}|^2$ is a trivial solution.

• **In general, this strategy fails**: $p_n^{(d)}(\boldsymbol{q})$ has $\dfrac{(n+d-1)!}{n!(d-1)!}$ coefficients. C is antisymmetric $\Rightarrow$ we must consider the $\dfrac{d(d-1)}{2}$ non-diagonal entries of C; they are homogeneous polynomials of degree $m+n-4$. $\Rightarrow$ The number of equations, $\dfrac{d(m+n+d-5)!}{2(m+n-4)!(d-2)!}$ exceeds the number of coefficients, $\dfrac{(m+d-1)!}{m!(d-1)!} + \dfrac{(n+d-1)!}{n!(d-1)!}$.

• The **strategy works, if the homogeneous polynomials are symmetric** in their arguments (i.e., invariant under any permutation $q_i \leftrightarrow q_j$): it **suffices to consider one equation** arising from any non-diagonal entry of C (all such equations are equivalent).

An example in $\mathbb{R}^d$ for $d \geq 3$ involving one unknown homogeneous polynomial

• We seek convex potentials ⇒ we consider polynomials involving only even powers of q_j:

$$p_4^{(d)}(\boldsymbol{q}) = \sum_{i=1}^{d} q_i^4 + \widetilde{c} \sum_{i=2}^{d} \sum_{j=1}^{i-1} q_i^2 q_j^2, \qquad p_6^{(d)}(\boldsymbol{q}) = \sum_{i=1}^{d} q_i^6 + \widetilde{a} \sum_{i=1}^{d} \sum_{j=1}^{d} q_i^4 q_j^2 + \widetilde{b} \sum_{1 \leq i < j < k \leq d} q_i^2 q_j^2 q_k^2.$$

• The degree of $C_{12}(p_6^{(d)}, p_4^{(d)})$ is 6. In $p_4^{(d)}$ and $p_6^{(d)}$ any power of q_1 and q_2 is even ⇒ $C_{12} \propto q_1 q_2$, and the polynomial $C_{12}/(q_1 q_2)$ involves each q_i only in even powers. By symmetry, $C_{12} = 0$ for $q_1 = q_2$ ⇒ $C_{12} \propto (q_1^2 - q_2^2)$ ⇒ $C_{12} = q_1 q_2 (q_1^2 - q_2^2) \left(\alpha_1 (q_1^2 + q_2^2) + \alpha_2 \sum_{j=3}^{d} q_j^2 \right)$.

Three independent parameters, $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ enter just two equations, $\alpha_1 = \alpha_2 = 0$.

For $\widetilde{a} = \dfrac{15\,\widetilde{c}}{12 - \widetilde{c}}$ and $\widetilde{b} = \dfrac{75\,\widetilde{c}^2}{(12 - \widetilde{c})(3 + \widetilde{c})}$, $\Phi(\boldsymbol{q}, t) = \mu_2(t) \dfrac{|\boldsymbol{q}|^2}{2} + \mu_4(t) p_4^{(d)}(\boldsymbol{q}) + \mu_6(t) p_6^{(d)}(\boldsymbol{q})$

is the potential of a non-Zeldovich-type omni-potential flow in $\mathbb{R}^d$ for any $d \geq 3$.

• $\Phi(\boldsymbol{q}, t)$ is **convex** if all $\mu_i(t) \geq 0$ and $0 \leq \widetilde{c} < 12$.

• For $\widetilde{c} \neq 2$, $p_4^{(d)}$ and $p_6^{(d)}$ (and hence $\Phi(\boldsymbol{q}, t)$) do not have spherical symmetry.

An example in $\mathbb{R}^3$ involving infinitely many homogeneous polynomials

$$p_{2n}^{(3)}(\boldsymbol{q}) = \sum_{i,j,k\geq 0,\, i+j+k=n} \widetilde{a}_{i,j,k}\, q_1^{2i} q_2^{2j} q_3^{2k} \quad \text{is symmetric} \;\Leftrightarrow\; \begin{cases} \widetilde{a}_{i,j,k} \text{ does not change under any} \\ \text{permutations of subscripts } i, j, k. \end{cases}$$

$$C_{12}(p_{2n}^{(3)}, p_4^{(3)}) = 8q_1q_2 \sum_{i,j,k\geq 0,\, i+j+k=n} \widetilde{a}_{i,j,k}\, q_1^{2i-2} q_2^{2j-2} q_3^{2k} \left(ij(\widetilde{c}-6)(q_1^2-q_2^2) + \widetilde{c}\,(-j(2j-1+2k)q_1^2 + i(2i-1+2k)q_2^2)\right)$$

- $C_{12}(p_{2n}^{(3)}, p_4^{(3)}) = 0 \;\; \Rightarrow$ the Hessians of any two polynomials from this family commute.
- $C_{12} = 0 \;\Leftrightarrow\; \widetilde{a}_{i,j,k} = \widetilde{a}_{i+1,j-1,k}\, \chi_j/\chi_{i+1} \;\; \forall i,\, j$ and k, where $\chi_m = (\widetilde{c}\,(2n+2-3m)+6(m-1))/m$.
- For each k, this as a recurrence for $\widetilde{a}_{i,j,k}$.

For $k=0$, set $\widetilde{a}_{n,0,0} = 1 \;\;\Rightarrow\;\; \widetilde{a}_{i,n-1,0} = \prod_{m=1}^{n-i}\chi_m \prod_{m=1}^{i}\chi_m \big/ \prod_{m=1}^{n}\chi_m\,.$

For $k>0$, set $\widetilde{a}_{n-k,0,k} = \widetilde{a}_{n-k,k,0} \;\;\Rightarrow\;\; \widetilde{a}_{i,j,k} = \prod_{m=1}^{i}\chi_m \prod_{m=1}^{j}\chi_m \prod_{m=1}^{k}\chi_m \big/ \prod_{m=1}^{n}\chi_m\,.$

- Clearly, such polynomial $p_{2n}^{(3)}$ is symmetric.
- The potential $\boxed{\widetilde{\Phi}(\boldsymbol{q},t) = \mu_2(t)\dfrac{|\boldsymbol{q}|^2}{2} + \sum_{n\geq 2} \mu_{2n}(t) p_{2n}^{(3)}(\boldsymbol{q})}$ defines a non-Zeldovich-type

omni-potential flow in $\mathbb{R}^3$, if $\mu_{2n}(t)$ are linearly independent and decay sufficiently fast.

- $p_{2n}^{(3)}(\boldsymbol{q})$ is convex for $0 \leq \widetilde{c} < \dfrac{6(n-1)}{n-2} \;\Rightarrow\; \widetilde{\Phi}(\boldsymbol{q},t)$ is convex for $\mu_{2n}(t) \geq 0$ and $0 \leq \widetilde{c} \leq 6$.

OPEN PROBLEMS

• How general are omni-potential flows in $\mathbb{R}^3$?

In $\mathbb{R}^2$, any initial flow can be accommodated for small enough τ.
This was shown by a WKB technique.

• Find all relations between the invariants in $\mathbb{R}^d$ for $d > 3$.

In $\mathbb{R}^d$, we have introduced $d^2(d-1)$ invariants $\gamma_{mn}^{(d,k)}$ — too many,
since a frame of eigendirections is described by just $d(d-1)/2$ parameters.
For $d = 3$, we have derived 15 relations between the 18 invariants.

• What are the solvability conditions for the set of PDEs for three-dimensional omni-potential flow in terms of the invariants $g_k(\boldsymbol{q})$?