Timescales of Turbulent Relative Dispersion

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Tracers in a turbulent flow separate according to the celebrated $t^{3/2}$ Richardson–Obukhov law, which is usually explained by a scale-dependent effective diffusivity. Here, supported by state-of-the-art numerics, we revisit this argument. The Lagrangian correlation time of velocity differences is found to increase too quickly for validating this approach, but acceleration differences decorrelate on dissipative timescales. This results in an asymptotic diffusion $\propto t^{1/2}$ of velocity differences, so that the long-time behavior of distances is that of the integral of Brownian motion. The time of convergence to this regime is shown to be that of deviations from Batchelor's initial ballistic regime, given by a scale-dependent energy dissipation time rather than the usual turnover time. It is finally argued that the fluid flow intermittency should not affect this long-time behavior of relative motion.

Turbulence has the feature of strongly enhancing the dispersion and mixing of the species it transports. It is known since the work of Richardson [1] that tracer particles separate in an explosive manner $\propto t^{3/2}$ that is much faster and less predictable than in any chaotic system. While little doubt remains about its validity in three-dimensional homogeneous isotropic turbulence, observations of this law in numerics and experiments are difficult, as they require a huge scale separation between the dissipative lengths, the initial separation of tracers, the observation range and the integral scale of the flow [2, 3]. Much effort has been devoted to test the universality of this law, which was actually retrieved in various turbulent settings, such as the two-dimensional inverse cascade [4], buoyancy-driven flows [5], and magnetohydrodynamics [6]. At the same time, breakthroughs on transport by time-uncorrelated scale-invariant flows have strenghtened the original idea of Richardson that this law originates from the diffusion of tracer separation in a scale-dependent environment [7]. As a result, the physical mechanisms leading to Richardson–Obukhov $t^{3/2}$ law are still rather poorly understood and many questions remain open on the nature of subleading terms, the rate of convergence and on the effects of the intermittent nature of turbulent velocity fluctuations [8, 9].

Turbulent relative dispersion consists in understanding the evolution of the separation $\delta\boldsymbol{x}(t) = \boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)$ between two tracers. Richardson's argument can be reinterpreted by assuming that the velocity difference $\delta\boldsymbol{u}(t) = \boldsymbol{u}(\boldsymbol{X}_1, t) - \boldsymbol{u}(\boldsymbol{X}_2, t)$ has a short correlation time. This means that the central-limit theorem applies and that, for sufficiently large timescales,

$$\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{d}t} = \delta\boldsymbol{u} \simeq \sqrt{\tau_L}\, \boldsymbol{U}(\delta\boldsymbol{x})\, \boldsymbol{\xi}(t), \qquad (1)$$

where $\boldsymbol{\xi}$ is the standard three-dimensional white noise, $\boldsymbol{U}^{\mathsf{T}}\boldsymbol{U} = \langle \delta\boldsymbol{u} \otimes \delta\boldsymbol{u}\rangle$ the Eulerian velocity difference correlation tensor, and τ_L the Lagrangian correlation time of velocity differences between pair separated by $\delta x = |\delta\boldsymbol{x}|$. As stressed by Obukhov [10], when assuming Kolmogorov 1941 scaling, $\tau_L \sim \delta x^{2/3}$, $\boldsymbol{U} \sim \delta x^{1/3}$, and the Fokker–Planck equation associated to (1) exactly corresponds to that derived by Richardson for the probability density $p(\delta x, t)$. It predicts in particular that the squared distance $\langle |\delta\boldsymbol{x}(t)|^2\rangle_{r_0}$ averaged over all pairs that are initially at a distance $|\delta\boldsymbol{x}(0)| = r_0$ has a long-time behavior $\propto t^3$ that is independent on r_0. This memory lost on the initial separation can only occur on time scales longer than the correlation time $\tau_L(r_0) \sim r_0^{2/3}$ of the initial velocity difference. For times $t \ll \tau_L(r_0)$, one cannot make use of the approximation (1) as the velocity difference almost keeps its initial value. This corresponds to the ballistic regime $\langle |\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(0)|^2\rangle_{r_0} \simeq t^2 S_2(r_0)$, where $S_2(r) = \langle |\delta\boldsymbol{u}|^2\rangle$ is the Eulerian second-order structure function over a separation r, introduced by Batchelor [11]. The diffusive approach (1) can however be modified to account for the ballisitic regime [12]. Nevertheless a short-time correlation of velocity differences can hardly been derived from first principles and seems to contradict turbulence phenomenology. Indeed, as stressed in [7], if δx grows like $t^{3/2}$, the Lagrangian correlation time τ_L is of the order of $\delta x^{2/3} \sim t$, so that the velocity difference correlation time is always of the order of the observation time. Despite such apparent contradictions, Richardson diffusive approach might be relevant to describe some intermediate regime valid for large-enough times and typical separations. Several measurements show that the separations distribute with a probability that is fairly close to that obtained from an eddy-diffusivity approach [9, 13, 14].

To clarify when and where Richardson's approach might be valid, it is important to understand the timescale of convergence to the explosive t^3 law. Much work has recently been devoted to this issue: it was for instance proposed to make use of fractional diffusion with memory [15], to introduce random delay times of convergence to Richardson scaling [16], or to estimate the influence of extreme events in particle separation [17]. All these approaches consider as granted that the final behavior of separations is diffusive. As we will see here, many aspects of the convergence to Richardson's law for pair dispersion can be clarified in terms of a diffusive behavior of velocity differences.

To address such issues, we make use of direct numerical simulations. For this, the Navier–Stokes equation with a large-scale-forcing is integrated in a periodic domain using a massively parallel spectral solver at two different resolutions. Table I summarizes the parameters of the simulations (see [18] for more details). In each case, the flow is seeded with 10^7 Lagrangian tracers. Their positions, velocities, and accelerations are then stored with enough frequency to study relative motion.

N	R_λ	ν	ϵ	$u_{\rm rms}$	η	τ_η	L	T
2048^3	460	$2.5\cdot10^{-5}$	$3.6\cdot10^{-3}$	0.19	$1.4\cdot10^{-3}$	0.083	1.85	9.9
4096^3	730	$1.0\cdot10^{-5}$	$3.8\cdot10^{-3}$	0.19	$7.2\cdot10^{-4}$	0.05	1.85	9.6

TABLE I: Parameters of the numerical simulations. N is the number of grid points, R_λ the Taylor-based Reynolds number, ν the kinematic viscosity, ϵ the averaged energy dissipation rate, $u_{\rm rms}$ the root-mean square velocity, $\eta=(\nu^3/\epsilon)^{1/4}$ the Kolmogorov dissipative scale, $\tau_\eta=(\nu/\epsilon)^{1/2}$ the associated turnover time, $L=u_{\rm rms}^3/\epsilon$ the integral scale and $T=L/u_{\rm rms}$ the associated large-scale turnover time.

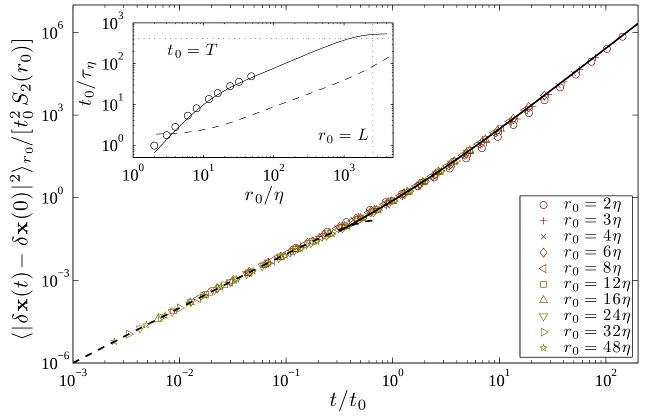

FIG. 1: Time-evolution of the mean-square separation for $R_\lambda = 730$ and different initial separations. The dashed line represents the two leading terms of the ballisitic behavior (2). The solid line is a fit to the Richardson regime (4) with $g = 0.52$ and $C = 1.6$. Inset: t_0 as a function of r_0 in dissipative-scale units. The solid line is an Eulerian average, the circles are Lagrangian measurements and the dashed line is the turnover time $\tau(r_0)$.

We first report results on the behavior of the separation $\delta\boldsymbol{x}(t)$ as a function of time. Following [13], a Taylor expansion at short times leads to

$$\left\langle|\delta\boldsymbol{x}(t)-\delta\boldsymbol{x}(0)|^2\right\rangle_{r_0} = t^2 S_2(r_0)+t^3\left\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\right\rangle+\mathrm{O}(t^4)\,, \quad (2)$$

where $S_2(r) = \langle|\delta\boldsymbol{u}|^2\rangle$ is the second-order structure function, $\langle\cdot\rangle$ denote Eulerian averages, and $\delta\boldsymbol{a}(t)=\boldsymbol{a}(\boldsymbol{X}_1,t)-\boldsymbol{a}(\boldsymbol{X}_2,t)$ is the difference of the fluid acceleration sampled by the two tracers (where the notation $\boldsymbol{a}=\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}$). As long as the term $\propto t^2$ is dominant, the tracers separate ballistically. Expansion (2) clearly fails for $t\approx t_0 = S_2(r_0)/|\left\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\right\rangle|$. It is known [7, 19] that for separations in the inertial range $\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\rangle = -2\epsilon$, which is nothing but a Lagrangian version of the 4/5 law. This implies that the ballistic regime ends up at times of the order of

$$t_0 = S_2(r_0)/(2\epsilon). \quad (3)$$

This timescale can be interpreted as the time required to dissipate the kinetic energy contained at the scale r_0. We thus expect it to be equal to the correlation time of the initial velocity difference. t_0 differs from the turnover time $\tau(r_0) = r_0/[S_2(r_0)]^{1/2}$ defined as the ratio between the separation r_0 and the typical turbulent velocity at that scale. When Kolmogorov 1941 scaling is assumed, these two time scales have the same dependency on r_0. However, usual estimates of the Kolmogorov constant lead to $t_0/\tau(r_0) \approx 20$. Also, note that intermittency corrections to the scaling behavior of S_2 should in principle decrease this ratio. Figure 1 represents the mean-squared displacement rescaled by $t_0^2 S_2(r_0)$ as a function of t/t_0, for various values of the initial separation r_0. In such units and when r_0 is far in the inertial range, all measurements collapse onto a single curve. The subleading term $\propto t^3$ in (2) is relevant for times $t \lesssim 0.01\,t_0$.

The data collapse extends to times larger than t_0 when the mean squared separation tends to Richardson t^3 regime. This unexpected fact implies that t_0 is not only the timescale of departure from the ballistic regime, but also that of convergence to Richardson's law. More precisely, numerical data suggest that for $t \gg t_0$

$$\left\langle|\delta\boldsymbol{x}(t)-\delta\boldsymbol{x}(0)|^2\right\rangle_{r_0} = g\,\epsilon\,t^3\left[1+C(t_0/t)\right]+\text{h.o.t.}. \quad (4)$$

The constant C does not strongly depend on the Reynolds number. Systematic measurements as a function of the initial separation show that C is negative when r_0 is of the order of the Kolmogorov scale η. The convergence to Richardson law is then from below and is thus contaminated by tracer pairs which spend long times close together before sampling the inertial range; this is consistent with the findings of [17]. When r_0 is far-enough in the inertial range, $C \approx 1.6$ becomes independent on the initial separation and the convergence to Richardson law is from above. One finds that $C = 0$ for $r_0 \approx 4\eta$; the only subleading terms in (4) are then of higher order, so that the mean-squared separation converges faster to Richardson regime. Such an initial separation could be an "optimal choice" to observe the t^3 behavior in experimental settings.

To understand why the timescale of convergence to Richardson law is of the order of t_0, let us examine the timescales entering the relative dispersion process. As already stated, the velocity difference $\delta\boldsymbol{u}$ between the two tracers stays correlated over a time that increases too fast with the separation, making difficult to justify the diffusive approach (1). However, it is known that turbulent acceleration, which is a small-scale quantity, is correlated over times that are of the order of τ_η the Kolmogorov turnover time [20]. Its amplitude is rather correlated on times of the order of the forcing correlation time, but this does not alter the argument below. Figure 2 repre-

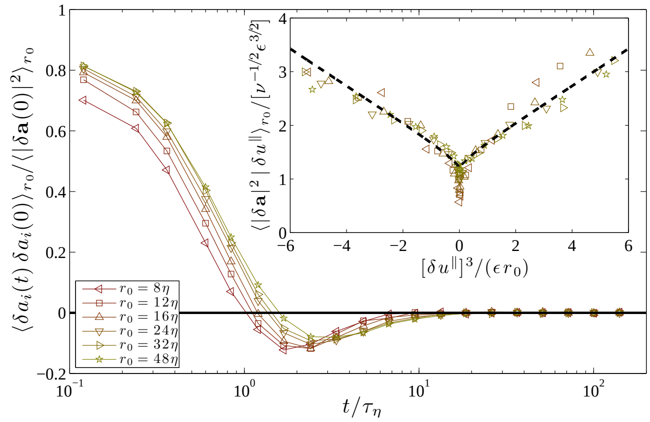

FIG. 2: Lagrangian time autocorrelation of the acceleration difference $\delta \boldsymbol{a}$ for various r_0 and $R_\lambda = 730$. Inset: for the same separations r_0, variance of the acceleration difference amplitude conditioned on the longitudinal velocity difference $\delta u^\parallel$ as a function of the local dissipation rate $[\delta u^\parallel]^3/r_0$.

sents the Lagrangian autocorrelation of the difference of acceleration $\delta \boldsymbol{a}$ between two tracers. We clearly see that the components of this quantity decorrelate on times of the order of τ_η. This suggests applying the central-limit theorem, so that for separations in the inertial range and on timescales much longer than the τ_η, the difference of acceleration between two tracers can be approximated by a delta-correlated-in-time random process. We thus have

$$\frac{\mathrm{d}\delta \boldsymbol{x}}{\mathrm{d}t} = \delta \boldsymbol{u}, \ \text{with} \ \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} = \delta \boldsymbol{a} \simeq \sqrt{\tau_\eta^{\rm loc}}\, \boldsymbol{A}(\delta \boldsymbol{x}, \delta \boldsymbol{u})\, \boldsymbol{\xi}(t), \quad (5)$$

where $\boldsymbol{A}$ is defined as $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A} = \langle \delta \boldsymbol{a} \otimes \delta \boldsymbol{a} \,|\, \delta \boldsymbol{x}, \delta \boldsymbol{u}\rangle$, $\boldsymbol{\xi}$ is the three-dimensional white noise, and the product is here understood in the Stratonovich sense. The idea of assuming uncorrelated accelerations is common to many stochastic models for turbulent dispersion (see, e.g., [2, 21]). However, this model does not require any Eulerian input and involves a multiplicative noise ($\boldsymbol{A}$ depends on $\delta \boldsymbol{u}$). Dimensional arguments indicate that the local Kolmogorov time $\tau_\eta^{\rm loc}$ and the acceleration amplitude $A = |\boldsymbol{A}|$ depends only on the viscosity ν and on the local energy dissipation rate $\epsilon_{\rm loc}$. We thus have $\tau_\eta^{\rm loc} \sim \nu^{1/2}\, \epsilon_{\rm loc}^{-1/2}$ and $A \sim \nu^{-1/4}\, \epsilon_{\rm loc}^{3/4}$. These estimates predict that the multiplicative term in (5) behaves as $[\tau_\eta^{\rm loc}]^{1/2} A \sim \epsilon_{\rm loc}^{1/2}$. Interestingly this quantity is independent on ν and is thus expected to have a finite limit at infinite Reynolds numbers. Phenomenological arguments suggest that for typical values of the velocity difference $\delta \boldsymbol{u}$, the local dissipation rate can be written as $\epsilon_{\rm loc} \sim [\delta u^\parallel]^3/\delta x$, where $\delta u^\parallel = \delta \boldsymbol{x} \cdot \delta \boldsymbol{u}/\delta x$ is the longitudinal velocity difference between the tracers. When $\delta u^\parallel = 0$, the local dissipation rate does not vanish but can be estimated through an averaged contribution of larger eddies, leading to $\epsilon_{\rm loc} \simeq \epsilon$, the averaged energy dissipation rate. These estimations have been tested against numerical simulations: the inset of Fig. 2 shows the variance of the acceleration differences conditioned on the longitudinal velocity difference for various separations. Up to some statistical errors, it seems that data are in rather good agreement with the phenomenological prediction which is shown as a dashed line. Finally such dimensional considerations lead to model the large-time evolution of tracer separation as

$$\frac{\mathrm{d}\delta x}{\mathrm{d}t} = \delta u^\parallel, \quad \frac{\mathrm{d}\delta u^\parallel}{\mathrm{d}t} \sim \left[\epsilon + \alpha\, \frac{[\delta u^\parallel]^3}{\delta x}\right]^{1/2} \xi(t), \qquad (6)$$

where α is a positive parameter. Again here the multiplicative noise is understood with Stratonovich convention. When rewriting it in the Itō sense, the additional drift that appears introduces a "correlation time" equal to the instantaneous turnover time $\delta x/\delta u^\parallel$. Preliminary studies of (6) showed that its solutions follow a ballistic regime at short times and behave according to Richardson law, i.e. $\langle \delta x^2\rangle \sim t^3$ at large times. In this stochastic model, the local dissipation $[\delta u^\parallel]^3/\delta x$ tends to a constant at large times, so that in the asymptotic regime, the velocity difference obeys an equation of the form $\mathrm{d}\delta u^\parallel/\mathrm{d}t \propto \xi(t)$ and thus diffuses. So far, we have only investigated the one-dimensional version (6) of the model. Extension to higher dimensions requires accounting for incompressibility and is the subject of ongoing work.

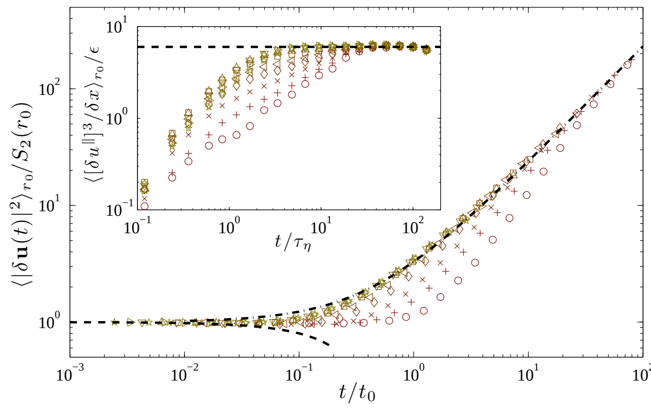

FIG. 3: Time evolution of the averaged longitudinal velocity for $R_\lambda = 730$ and different r_0 (same symbols as in Fig. 1). The short-time prediction (7) is shown as a dashed line. The diffusive behavior $\langle |\delta \boldsymbol{u}|^2\rangle_{r_0} \simeq S_2(r_0) + 2.3\, \epsilon\, t$ is represented as a dash-dotted line. Inset: time evolution of $\langle [\delta u^\parallel]^3/\delta x\rangle_{r_0}$; the dashed line corresponds to the value 6ϵ.

To address the relevance of such a model to real flows, we turn back to the analysis of simulation data. Figure 3 shows the time evolution of $\langle |\delta \boldsymbol{u}(t)|^2\rangle_{r_0}$ for various values of r_0. At small times this quantity slightly decreases because the subleading term is negative. We indeed have

$\delta\boldsymbol{u}(t) \simeq \delta\boldsymbol{u}(0) + t\delta\boldsymbol{a}(0)$, so that the ballistic regime reads

$$\langle|\delta\boldsymbol{u}(t)|^2\rangle_{r_0} = S_2(r_0)\,(1 - 2\,t/t_0) + \text{h.o.t.} \qquad (7)$$

Again, the subleading terms are relevat for times $t \lesssim 0.01\,t_0$. Figure 3 also shows that at large times the mean-squared velocity difference looses dependence on r_0 and grows $\propto \epsilon\, t$. In addition, as seen from the inset of Fig. 3, the averaged local dissipation rate $\langle[\delta u^{\|}]^3/\delta x\rangle_{r_0}$ along particle pairs approaches a positive constant $\simeq 6\,\epsilon$ (independently on R_λ) on times of the order of τ_η. This confirms the relevance of the mechanisms described above in terms of a stochastic equation for the velocity differences.

Numerical results indicate that the time t_0 controls the convergence to a diffusive regime for initial separations r_0 far enough in the inertial range. This can be explained by the following argument. As $\langle[\delta u^{\|}]^3/\delta x\rangle_{r_0}$ becomes constant on a short timescale, one expects that

$$\langle|\delta\boldsymbol{u}(t)|^2\rangle_{r_0} \simeq S_2(r_0) + D\,\epsilon\, t \quad \text{for } t \gg \tau_\eta, \qquad (8)$$

where D is a positive constant (for both Reynolds numbers, we observe $D \approx 2.1$). By balancing the diffusive term with the initial mean-squared velocity difference $\langle|\delta\boldsymbol{u}(0)|^2\rangle_{r_0} = S_2(r_0)$, we find again that the former is dominant for times t much larger than t_0. The diffusive behavior of velocity differences is thus reached at times of the order of t_0 and this explains in turn why this timescale is that of convergence to Richardson's regime.

Let us summarize here our findings. In this work we give some evidence that the Richardson explosive regime $\langle|\delta\boldsymbol{x}|^2\rangle \propto t^3$ for the separation between two tracers in a turbulent flow originates from a diffusive behavior of their velocity difference rather than from dimensional arguments or equivalently a scale-dependent eddy diffusivity for their distance. This leads on to reinterpret the t^3 law as that of the integral of Brownian motion. Such an argument is supported by two observations. First, the acceleration difference has a short correlation time (of the order of the Kolmogorov dissipative timescale) and can be approximated as a white noise. Second, the amplitude of this noise solely depends on the local dissipation rate $\langle[\delta u^{\|}]^3/\delta x\rangle_{r_0}$, which becomes constant also on short timescales. These considerations allow us to show that the time t_0 of convergence to Richardson's law is equal to that of deviations from Batchelor's ballistic regime. This time, which reads $t_0 = S_2(r_0)/(2\epsilon)$, is the time required to dissipate the kinetic energy contained at a scale equal to the initial separation between tracers.

The interpretation of Richardson's law as the diffusion of velocity differences strongly questions possible effects of fluid-flow intermittency on trajectory separation. Indeed, considerations on velocity scaling, which are primordial in approaches based on eddy diffusivity, are absent from the arguments leading to a diffusive behavior of $\delta\boldsymbol{u}$. Hence, we expect the separation $\delta\boldsymbol{x}$ to follow a self-similar evolution in time, independently on the order of the statistics. Intermittency will however affect s directly the time of convergence to such a regime. More frequent violent events (of tracer pairs approaching or fleeing away in an anomalously strong manner) will result in longer times for being absorbed by the average. Such arguments do not rule out the possibility of having intermittency corrections when interested in other observables than moments of the separation, as it is for instance the case for exit times [8]. Such issues will certainly gain much from a systematic study of multi-dimensional generalizations of the stochastic model introduced here.

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