Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations

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Outline

- Introduction : Statistical Mechanics and Turbulence
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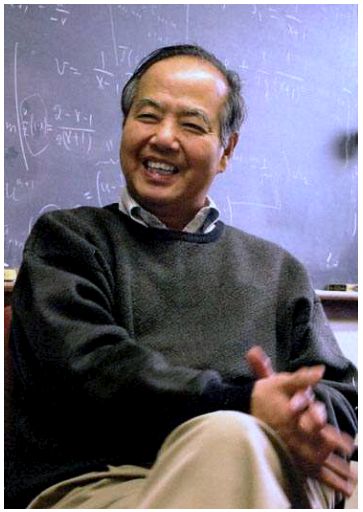
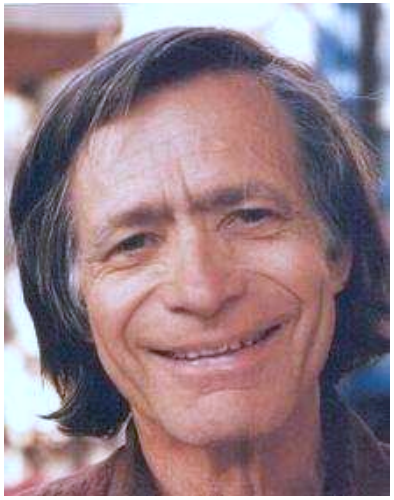
- The Tyger Phenomenon : 2D Euler Equation
- The Birth of Tygers : 1D Burgers Equation
- Conclusions and Perspective

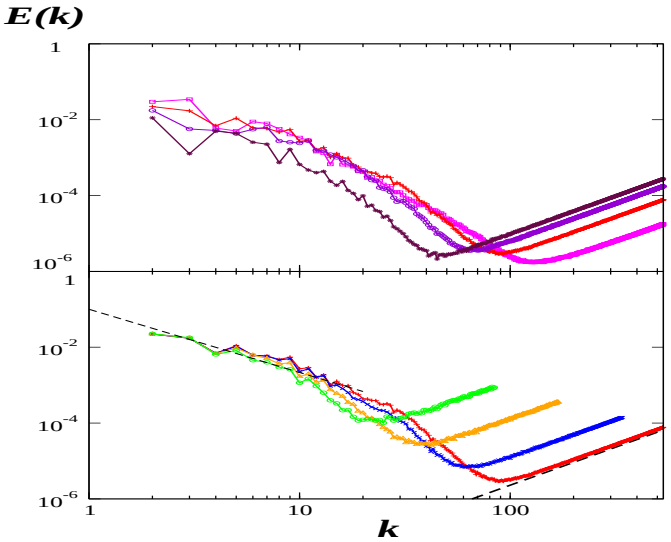
Equilibrium Statistical Mechanics and Turbulence

- Equilibrium statistical mechanics is concerned with conservative Hamiltonian dynamics, Gibbs states, ...
- Turbulence is about dissipative out-of-equilibrium systems.
- In 1952 Hopf and Lee apply equilibrium statistical mechanics to the 3D Euler equation and obtain the equipartition energy spectrum which is very different from the Kolmogorov spectrum.
- In 1967 Kraichnan uses equilibrium statistical mechanics as one of the tools to predict the existence of an inverse energy cascade in 2D turbulence.

Equilibrium Statistical Mechanics and Turbulence

- In 1989 Kraichnan remarks *the truncated Euler system can imitate NS fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wave-number modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared with the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity.*
- In 2005 Cichowlas, Bonaiti, Debbasch, and Brachet discovered long-lasting, partially thermalized, transients similar to high-Reynolds number flow.



The Galerkin-truncated 1D Burgers equation

- The (untruncated) inviscid Burgers equation, written in conservation form, is

$$\partial_t u + \partial_x (u^2/2) = 0; \qquad u(x,0) = u_0(x).$$

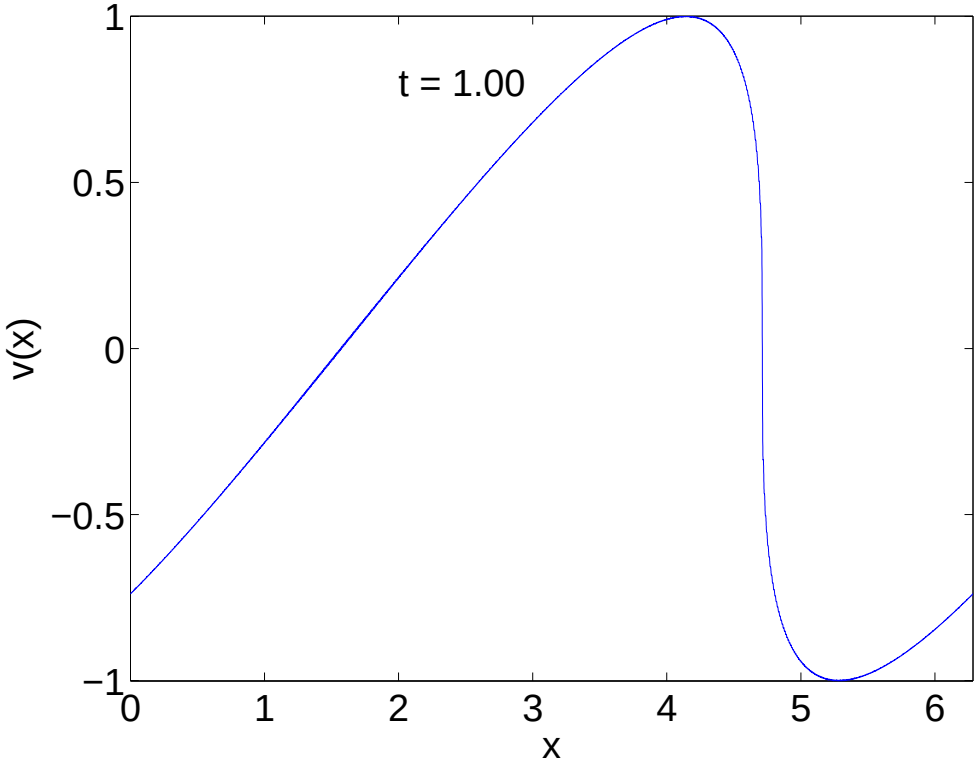
- Let K_G be a positive integer, here called the Galerkin truncation wavenumber, such that the action of the projector $\mathrm{P}_{K_\mathrm{G}}$:

$$\mathrm{P}_{K_\mathrm{G}} u(x) = \sum_{|k| \leq K_\mathrm{G}} \mathrm{e}^{\mathrm{i}kx} \hat{u}_k.$$

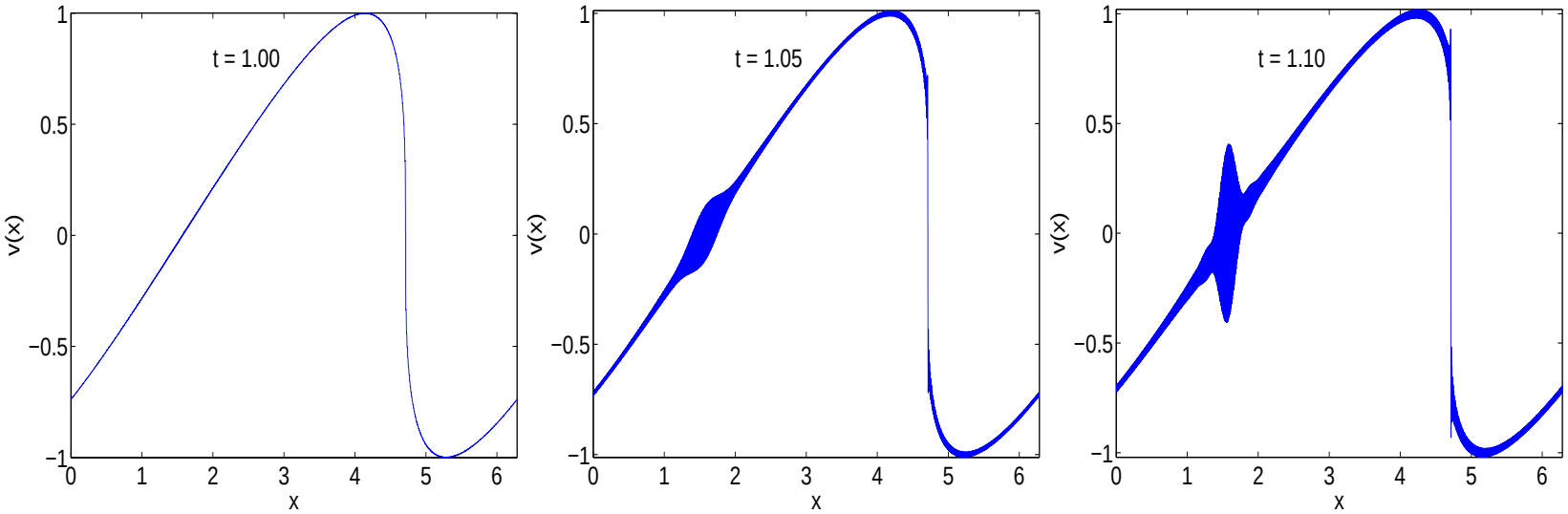
- The associated Galerkin-truncated (inviscid) Burgers equation

$$\partial_t v + \mathrm{P}_{K_\mathrm{G}} \partial_x (v^2/2) = 0; \qquad v_0 = \mathrm{P}_{K_\mathrm{G}} u_0.$$

Time Evolution of the Truncated Equation



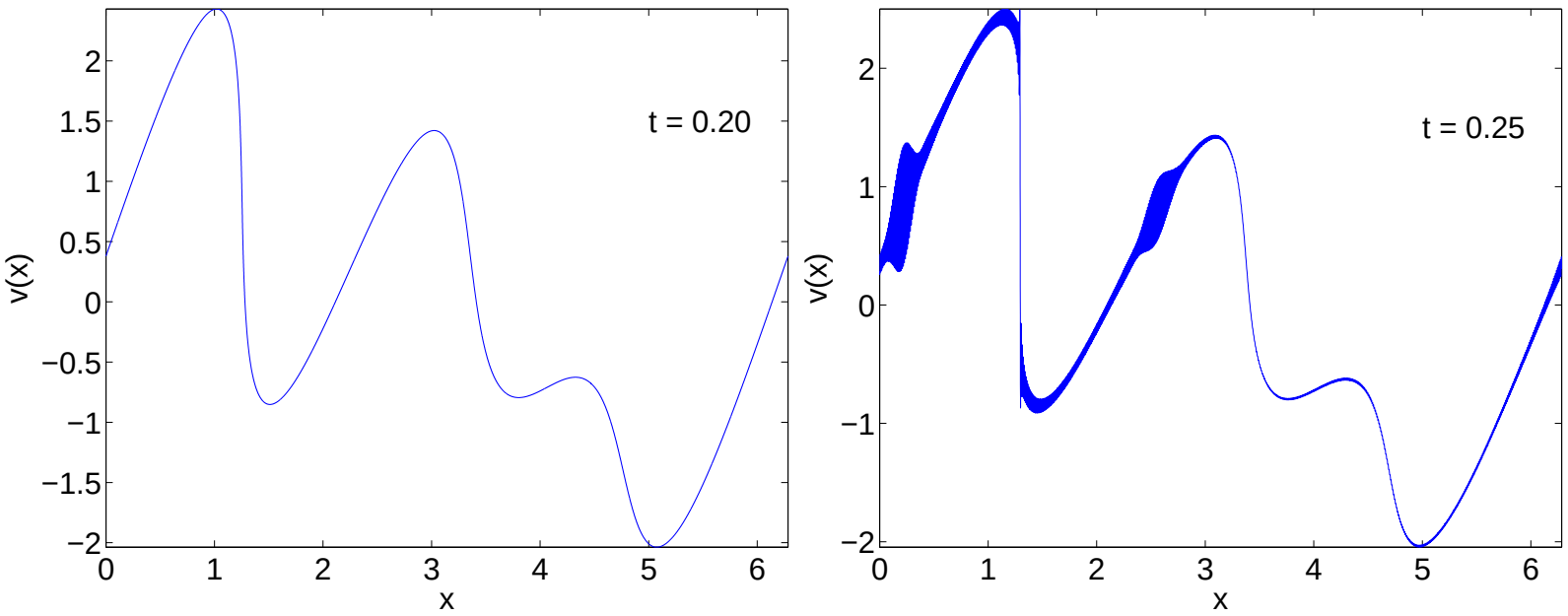
Tygers in the Galerkin-truncated 1D Burgers equation



Growth of a tyger in the solution of the inviscid Burgers equation with initial condition $v_0(x) = \sin(x - \pi/2)$. Galerkin truncation at $K_{\mathrm{G}} = 700$. Number of collocation points $N = 16,384$. Observe that the bulge appears far from the place of birth of the shock.

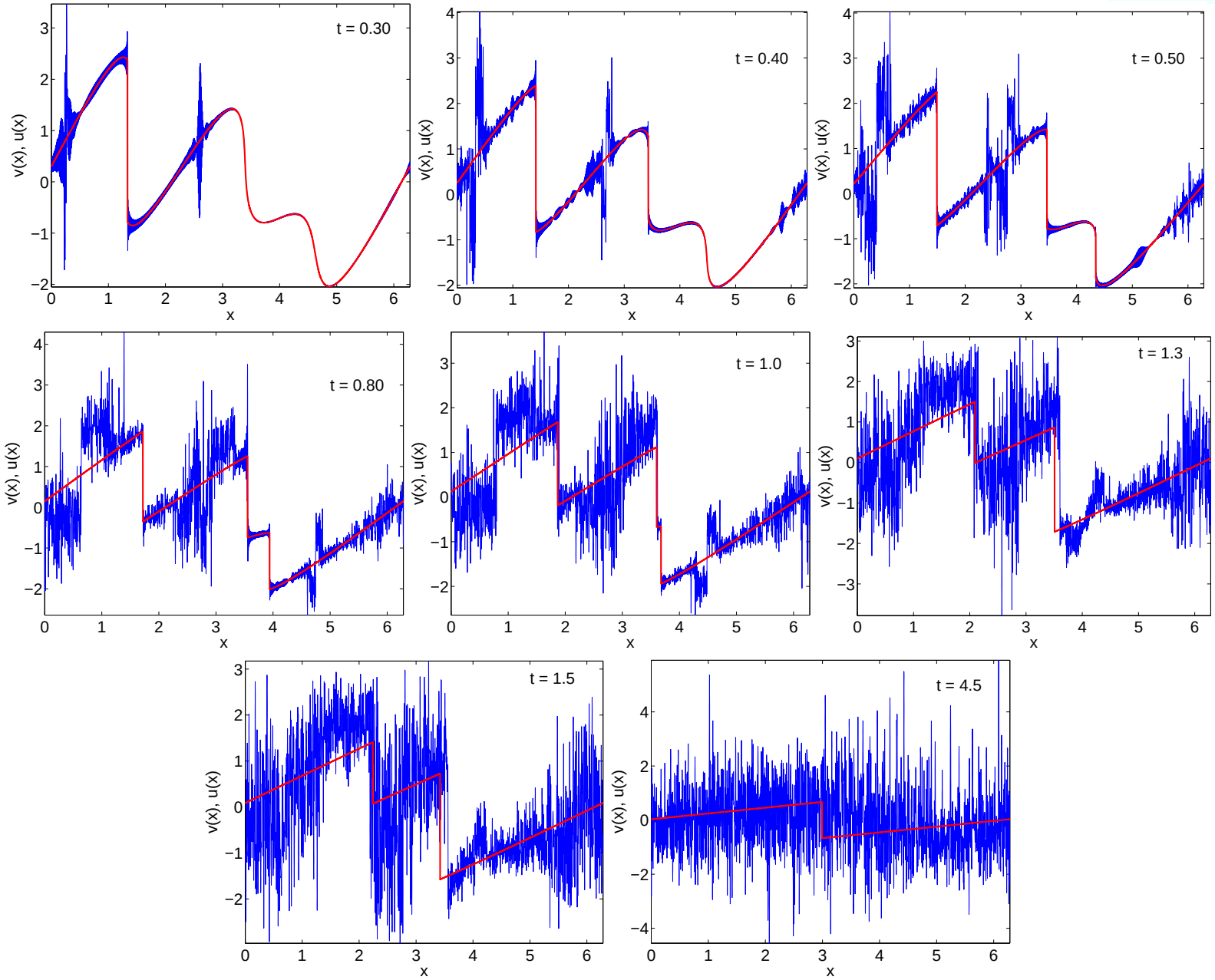
Tygers only at regions of positive strain

$$u_0(x) = \sin(x) + \sin(2x + 0.9) + \sin(3x)$$

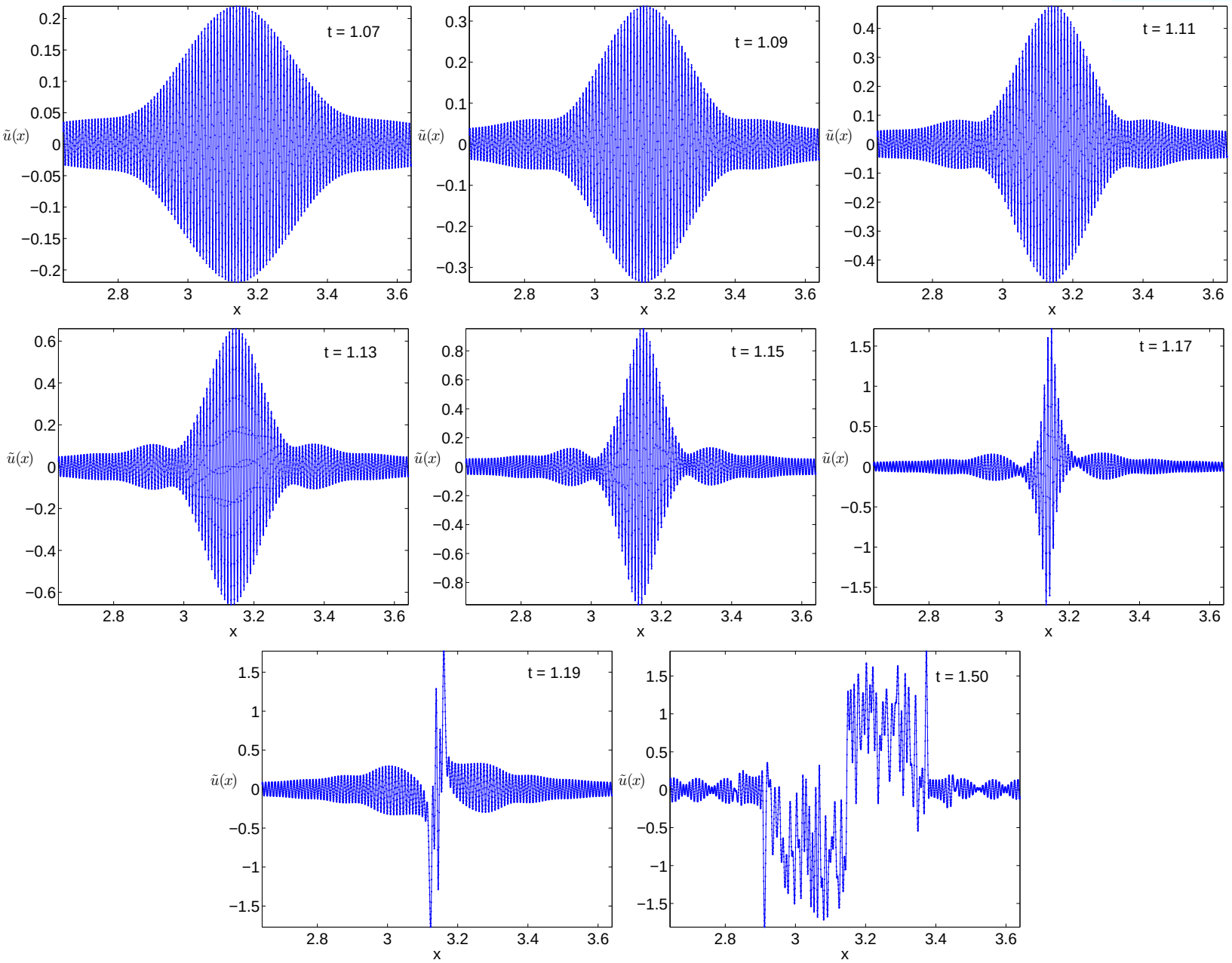


Three-mode initial condition. Tygers appear at the points having the same velocity as the shock and positive strain.

From tygers to thermalization



From tygers to thermalization



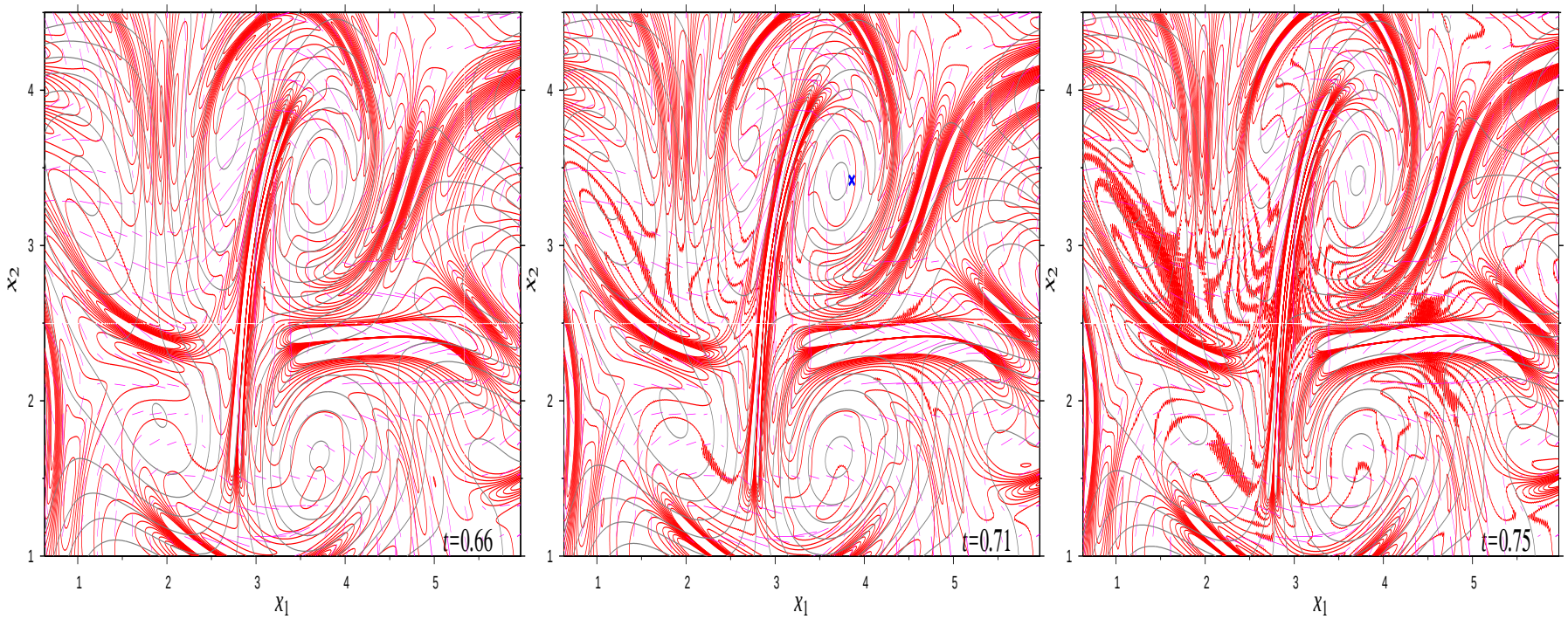
Phenomenological Explanation : 1D Burgers

- A localized strong nonlinearity, such as is present at a preshock or a shock, acts as a source of a *truncation wave*.
- Away from the source this *truncation wave* is mostly a plane wave with wavenumber close to K_{G}.
- The radiation of truncation waves begins only at or close to the time of formation of a preshock.
- Resonant interactions are confined to particles such that $\tau \Delta v \equiv \tau |v - v_{\mathrm{s}}| \lesssim \lambda_{\mathrm{G}}$.
- If τ is small the region of resonance will be confined to a small neighborhood of widths $\sim {K_{\mathrm{G}}}^{-1/3}$ around the point of resonance.
- In a region of negative strain a wave of wavenumber close to K_{G} will be squeezed and thus disappearing beyond the *truncation horizon* which acts as a kind of black hole.

Truncated 2D Euler

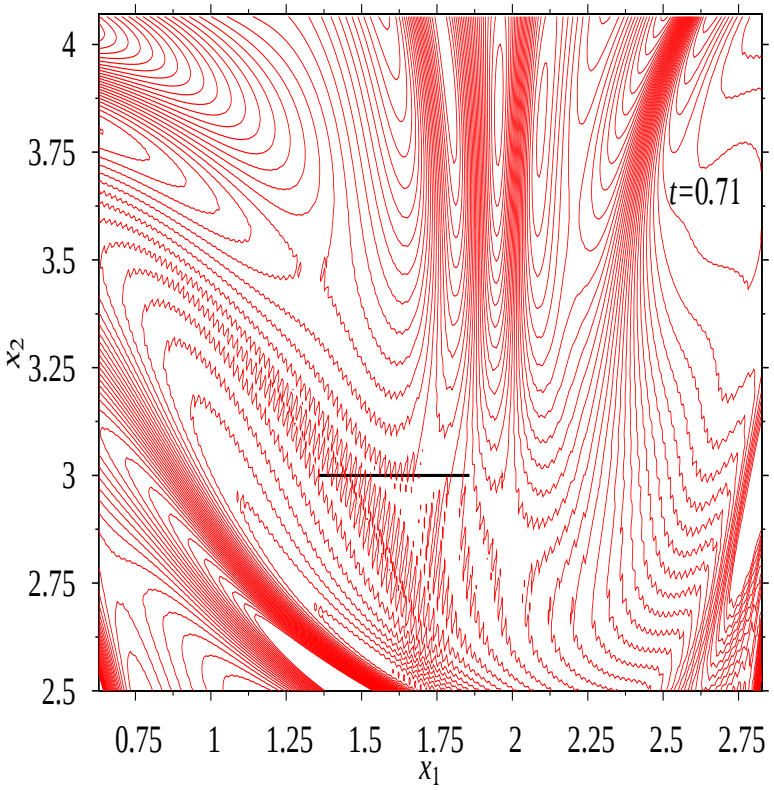
- Numerical integration of the truncated 2D incompressible Euler equation with random initial conditions and resolutions between 512^2 and 8192^2.
- Although for the untruncated solution real singularities are ruled out at any finite time, there is strong enhancement of spatial derivatives of the vorticity.
- The highest values of the Laplacian is found in the straight cigar-like structure.

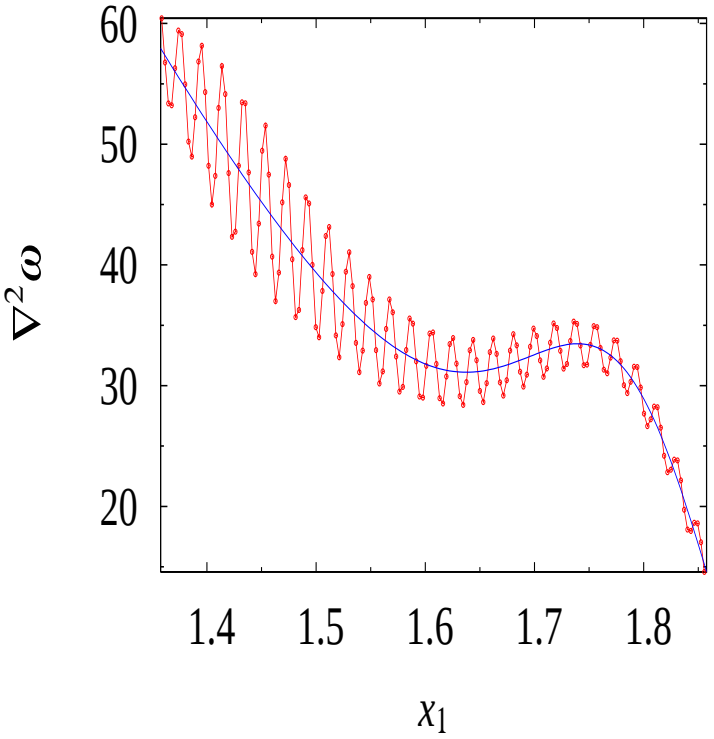
2D Euler



A 2D tyger: before ($t = 0.66$), early ($t = 0.71$) and later ($t = 0.75$). Figures, moderately zoomed, centered on the main cigar. Contours of the Laplacian of vorticity in red, ranging from -200 to 200 by increments of 25, streamlines in gray, ranging from -1.6 to 1.6 by increments of 2 and positive strain eigendirections in pink segments.

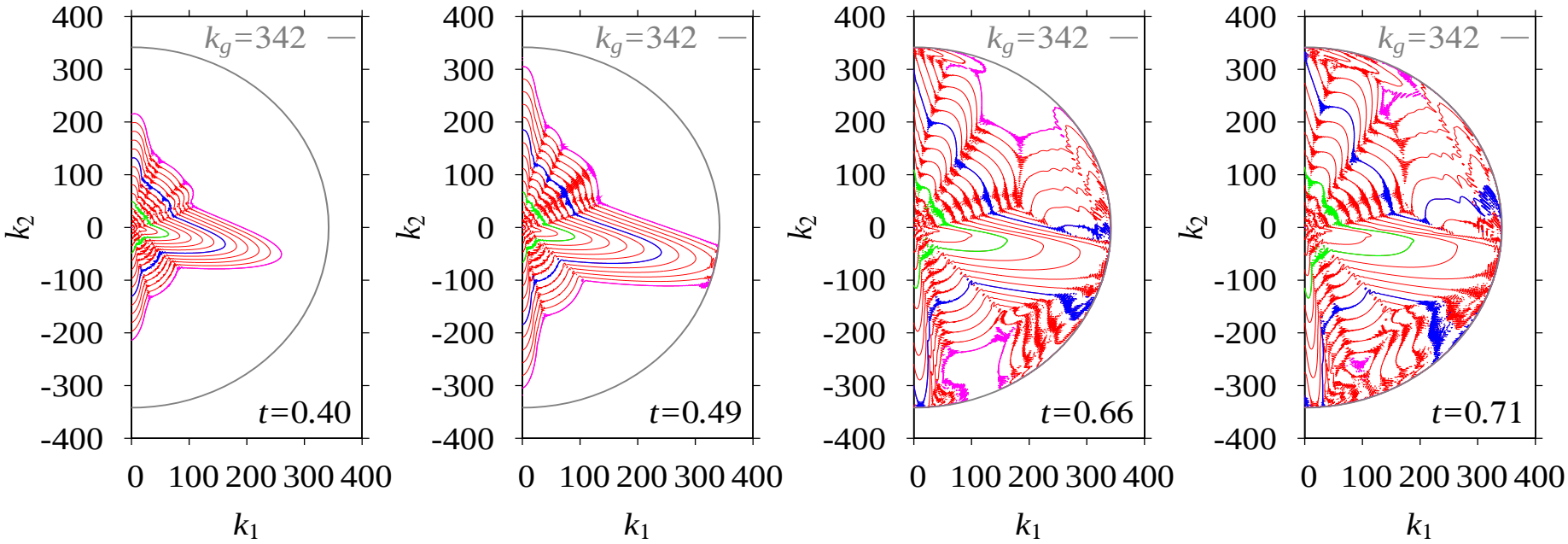
2D Euler tygers : Physical space





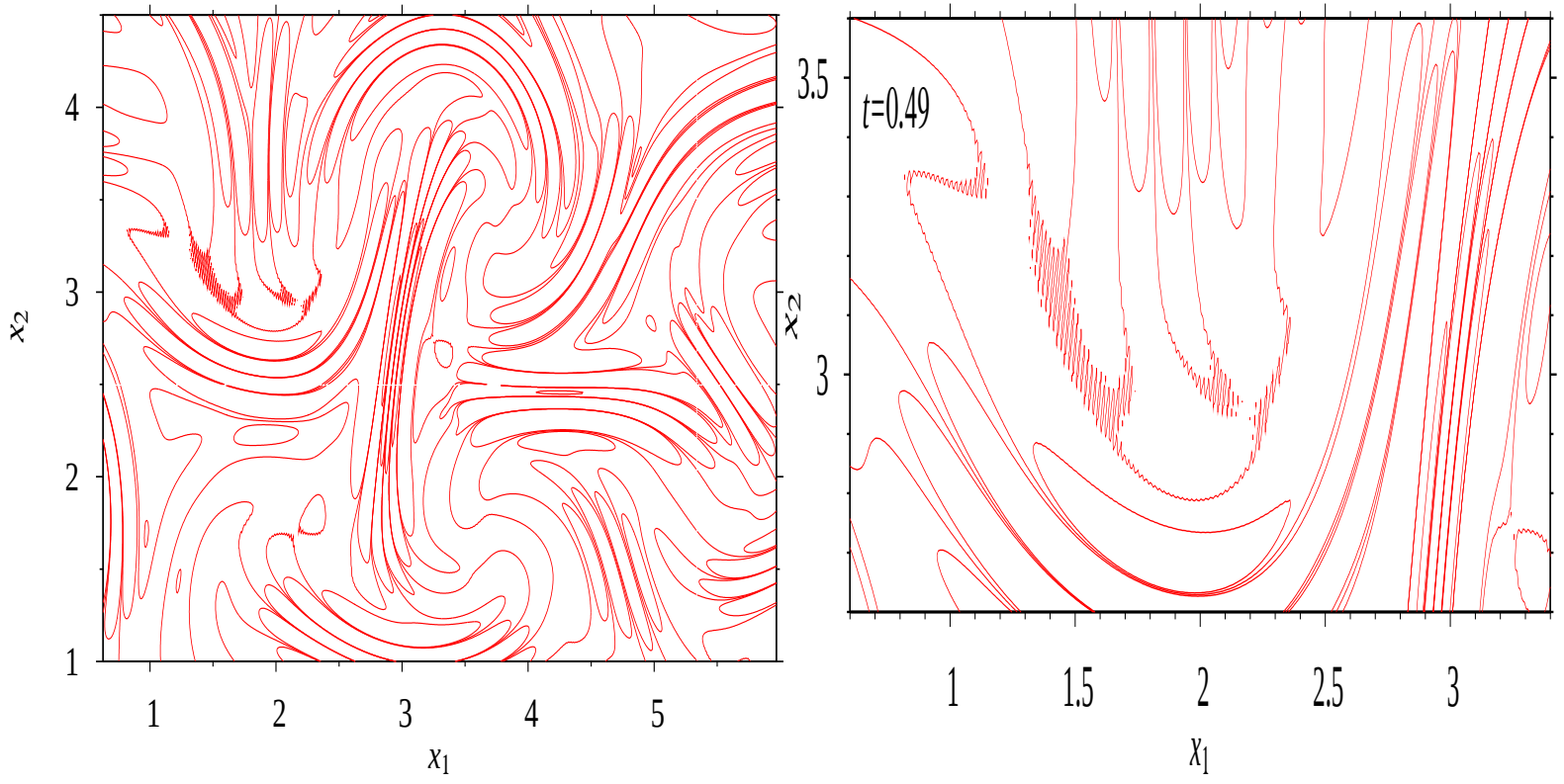
Left: zoomed version of contours of the Laplacian of vorticity at $t = 0.71$.
Right: plot of the Laplacian of vorticity along the horizontal segment near $x_2 = 3$, shown in the left panel.

2D Euler : Fourier space



Contours of the modulus of the vorticity Fourier coefficients at various times. Negative k_1 values not shown because of Hermitian symmetry. Contour values are 10^{-1}, $10^{-2}, \ldots, 10^{-15}$ from inner to outer (green, blue and pink highlight the values $10^{-5}, 10^{-10}$, and 10^{-15}, respectively). Galerkin truncation effects are visible above the rounding level already at $t = 0.49$ and become more and more invasive.

2D Euler : Magnification of tyger effects

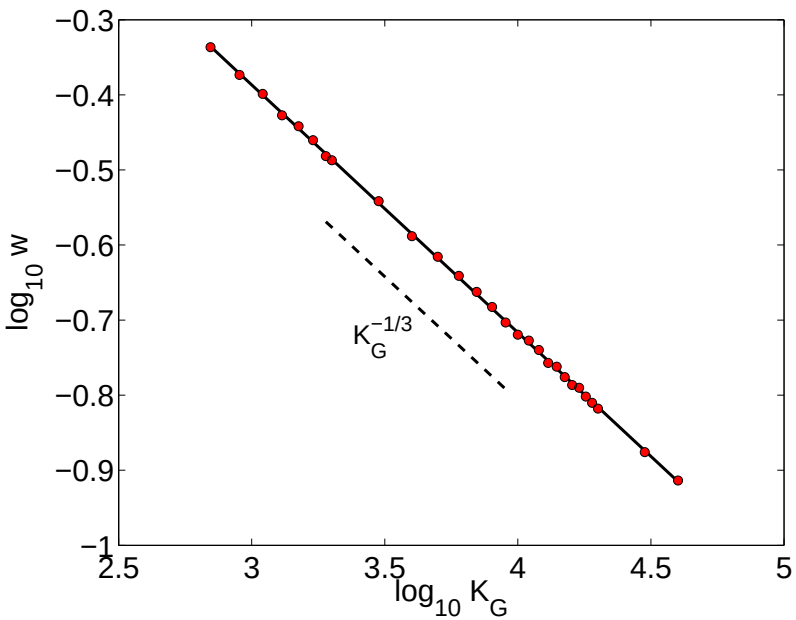


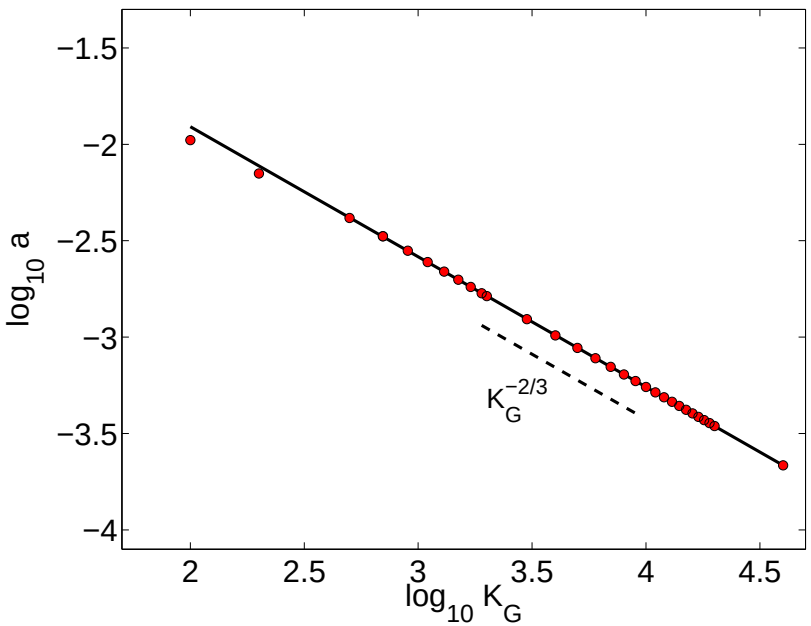
Contours of tri-Laplacian of the vorticity showing a tyger already at $t = 0.49$.

2D Euler : How similar is it to the Burgers equation?

- Most of these tygers appear at places which had no preexisting small-scale activity.
- The streamlines indicate that tyger activity appears at places where the velocity is roughly parallel to the central cigar.
- Considering the cigar as a one-dimensional straight object, the truncation waves generated by the cigar will have crests parallel to the cigar and those fluid particles which move parallel to the crest keep a constant phase and thus have resonant interactions with the truncation waves.
- If we now consider the one-parameter family of straight lines perpendicular to a given cigar, each such line will have some number (possibly zero) of resonance points; altogether they form the tygers.
- There are points where this kind of resonance condition holds but no tyger is seen; this can be interpreted in terms of strain.

Back to 1D : Scaling properties of the early tygers





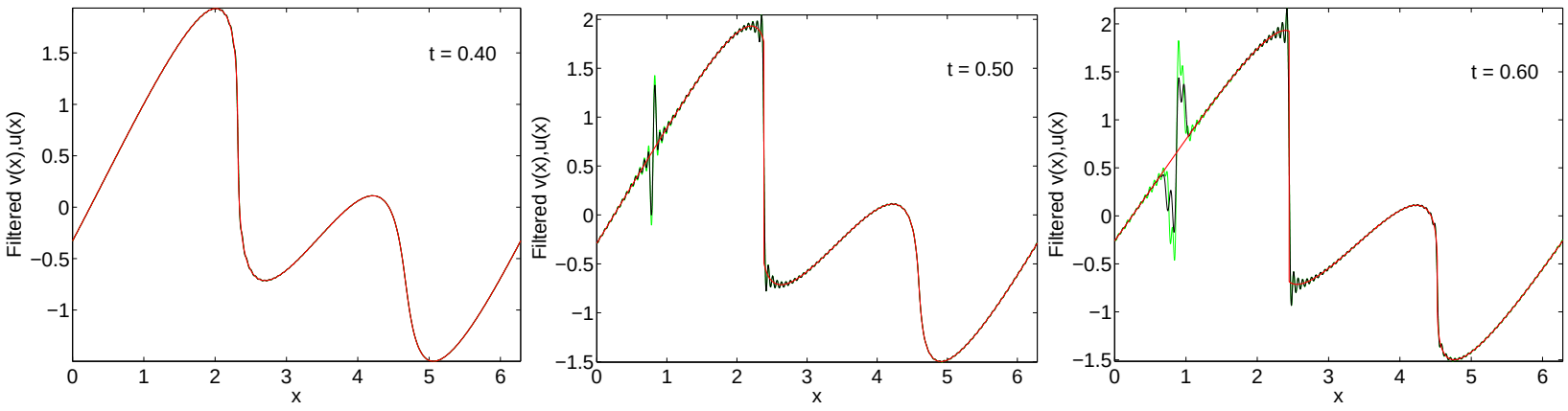
$$\text{width} \propto K_{\mathrm{G}}{}^{-1/3} \qquad \text{(using phase mixing arguments)}$$

$$\text{amplitude} \propto K_{\mathrm{G}}{}^{-2/3} \qquad \text{(using energy conservation arguments)}$$

Scaling properties of the early tygers

- Scaling of the tyger widths :
  - By the time $t_\star$, truncation is significant only for a lapse of time $O({K_{\mathrm{G}}}^{-2/3})$.
  - The phase mixing argument tells us that the coherent build up of a tyger will affect only those locations whose velocity differs from that at resonance by an amount $\Delta v \lesssim \frac{2\pi}{{K_{\mathrm{G}}}^{-2/3} K_{\mathrm{G}}} \propto {K_{\mathrm{G}}}^{-1/3}$.
  - Since at such times, the velocity v of the truncated solution is expected to stay close to the velocity u of the untruncated solution and the latter varies linearly with x near the resonance point, the width of the $t_\star$ tyger is itself proportional to ${K_{\mathrm{G}}}^{-1/3}$.
- Scaling of the tyger amplitudes :
  - The Galerkin-truncated Burgers equation conserves energy.
  - The apparent energy loss due to truncation $\sim \int_0^{\lambda_{\mathrm{G}}} x^{2/3} dx \sim {K_{\mathrm{G}}}^{-5/3}$.
  - Conservation demands that this energy-loss is transferred to the tygers which gives the tyger-amplitude scaling as $\propto {K_{\mathrm{G}}}^{-2/3}$.
  - The above argument is appealing but not rigorous.

Weak solutions?



Plots of solution of the Galerkin-truncated Burgers equation, with $K_G = 5,461$ (green) and $K_G = 21,845$ (black), low-pass filtered at wavenumber $K = 100$, at various times. Initial condition $v_0(x) = \sin(x) + \sin(2x - 0.741)$. The untruncated solution is shown in red.

Birth of tygers : Systematic theory

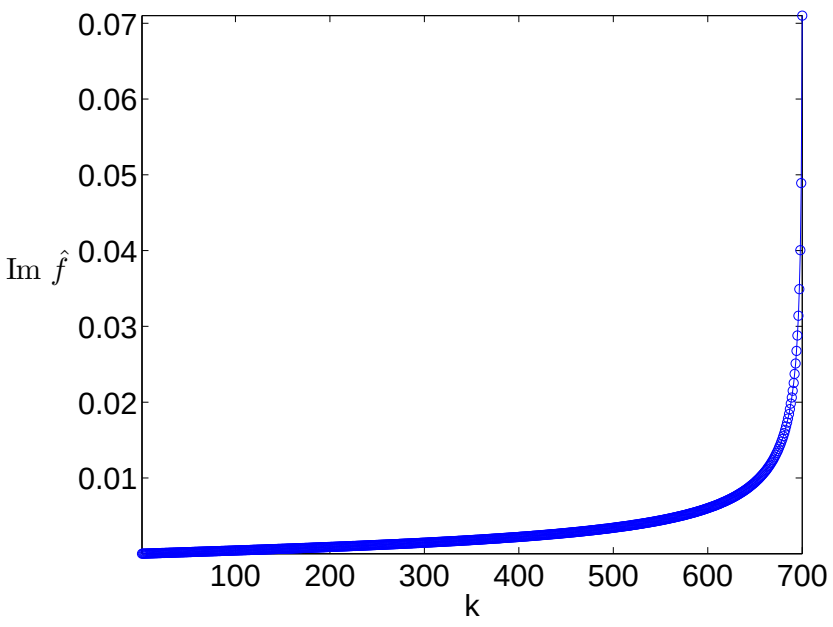
- Define discrepancy $\tilde{u} \equiv v - u$ to obtain

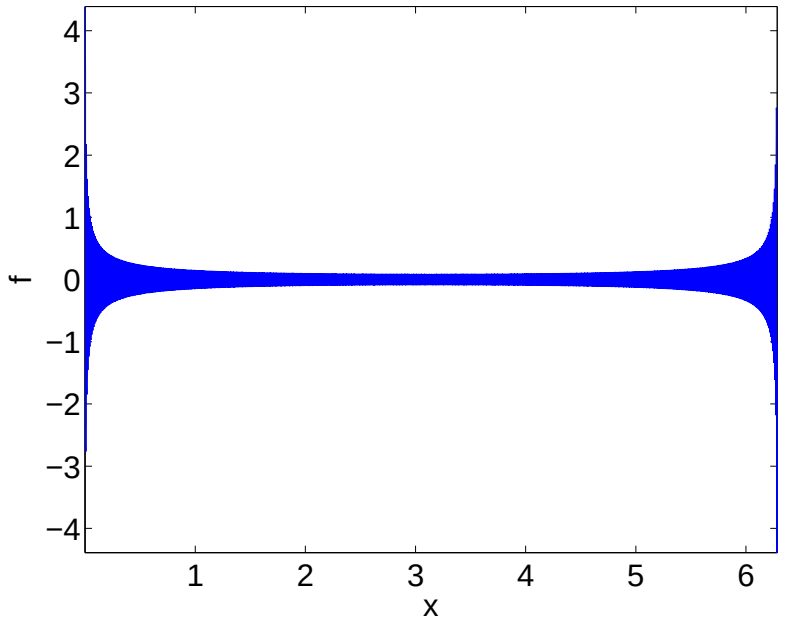
$$\partial_t \tilde{u} + \mathrm{P}_{k_\mathrm{G}} \partial_x \left(u\tilde{u} + \frac{\tilde{u}^2}{2} \right) = \left(\mathrm{I} - \mathrm{P}_{k_\mathrm{G}} \right) \partial_x \frac{u^2}{2}, \quad \tilde{u}(0) = 0.$$

- Decompose $u = u^< + u^>$, where $u^< \equiv \mathrm{P}_{k_\mathrm{G}} u$ and $u^> \equiv (\mathrm{I} - \mathrm{P}_{k_\mathrm{G}})u$.
- Similarly the perturbation $u' \equiv \mathrm{P}_{k_\mathrm{G}} \tilde{u}$.
- Finally we obtain :

$$\partial_t u' + \mathrm{P}_{k_\mathrm{G}} \partial_x \left(u^< u' + \frac{(u')^2}{2} \right) = \mathrm{P}_{k_\mathrm{G}} \partial_x \left(u^< u^> + \frac{(u^>)^2}{2} \right).$$

The beating input





Birth of tygers : Approximations

$$\partial_t u' + \mathrm{P}_{K_\mathrm{G}} \partial_x \left(u^< u' + \frac{(u')^2}{2} \right) = \mathrm{P}_{K_\mathrm{G}} \partial_x \left(u^< u^> + \frac{(u^>)^2}{2} \right).$$

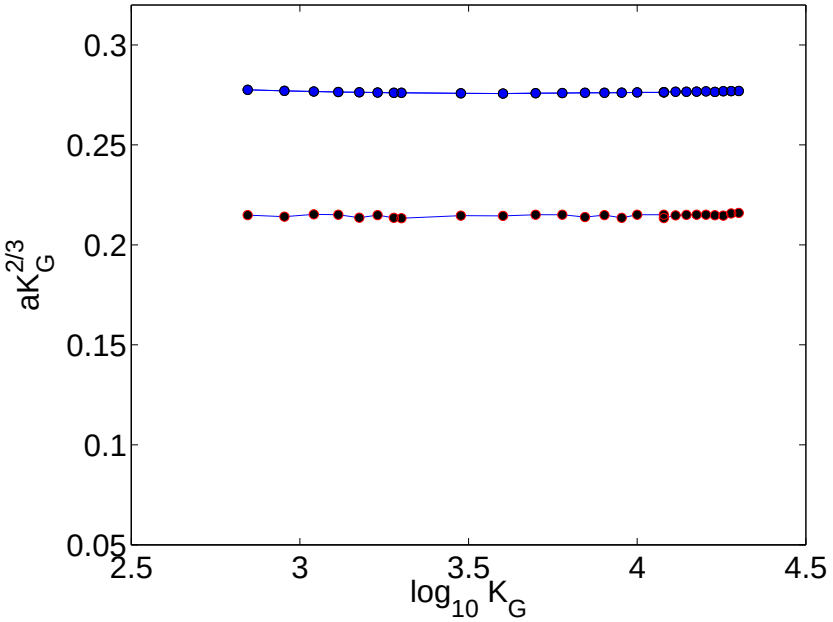
- Strategy :
  1. The term $(u')^2$ is discarded;
  2. The perturbation u' is set to zero at time t_G;
  3. The untruncated solution is frozen to its $t_\star$ value.
- With the three approximations the temporal dynamics of the perturbation near $t_\star$ is

$$\frac{d}{d\tau} \hat{u}'_k = \sum_{k'=-K_\mathrm{G}}^{K_\mathrm{G}} A_{kk'}\, \hat{u}'_{k'} + \hat{f}_k\,, \qquad \hat{u}'_k(0) = 0,$$

$$A_{kk'} \equiv -\mathrm{i}k\, \hat{u}^<_{\star,\, k-k'}\,,$$

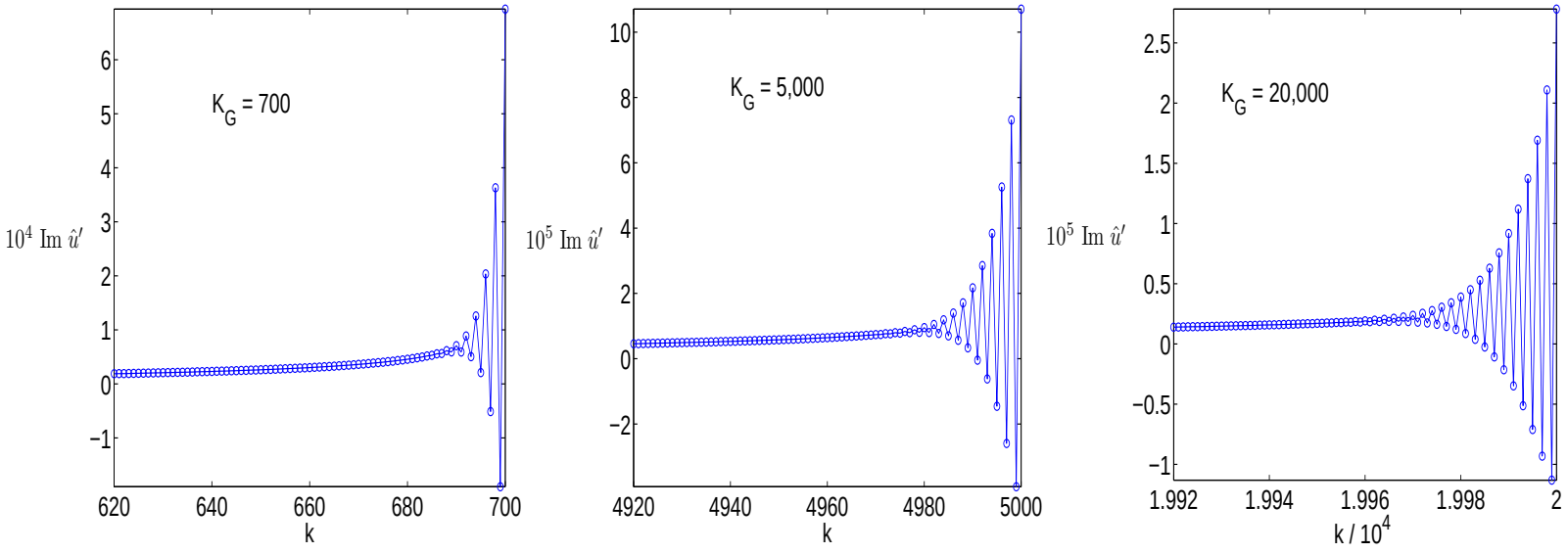
$$\hat{f}_k \equiv \mathrm{i}k \sum_{p+q=k} (\hat{u}^<_{\star p}\, \hat{u}^>_{\star q} + \frac{1}{2} \hat{u}^>_{\star p}\, \hat{u}^>_{\star q}).$$

Are the approximations justified?



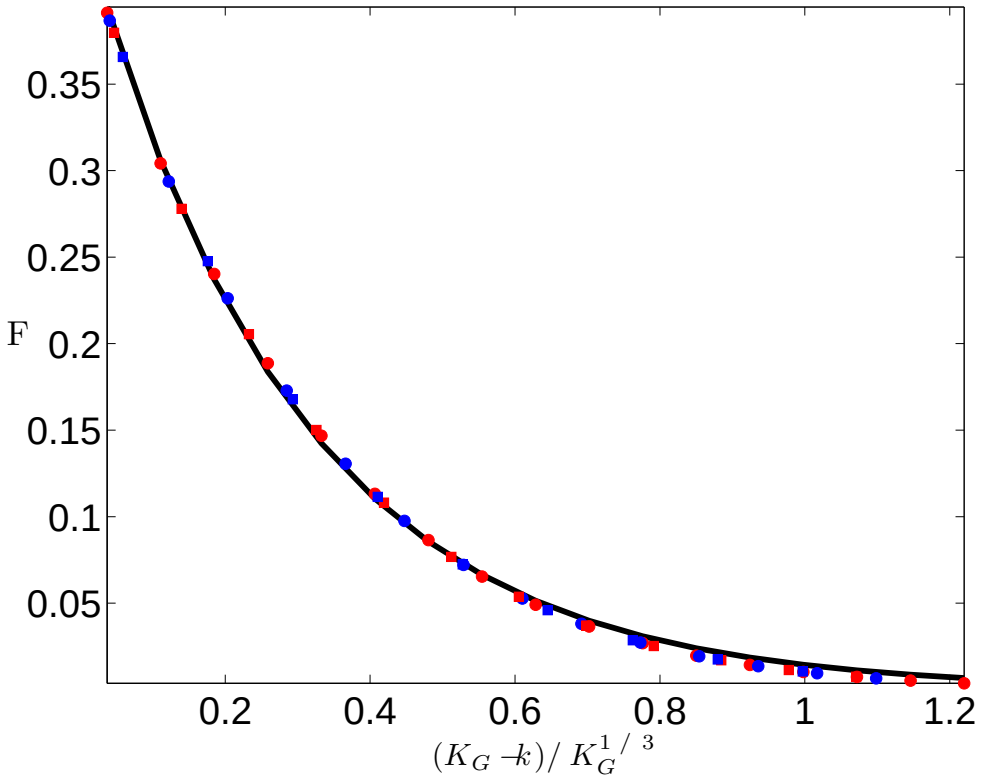
Log-linear plot of the compensated amplitude of the tyger $K_G^{2/3}\, a(K_{\mathrm{G}})$, calculated (i) from $\tilde{u}$, (ii) from the linearised approximation for u', and (iii) from the freezing plus reinitialization approximation, all versus K_{G}.

Fourier space solution of the perturbation



The boundary layer in Fourier space near K_{G}. Shown are the imaginary parts of $\hat{u}'(t_\star)$ for three values of K_{G}. The origin is at the preshock. The even-odd oscillations indicate that most of the activity is at the tyger, a distance π away.

Scaling function for the boundary layer



The envelopes of the various boundary layers shown in earlier (with preshock contributions subtracted out), collapsed into a single curve after rescaling. Red circles: $K_{\mathrm{G}} = 20,000$, blue circles: $K_{\mathrm{G}} = 15,000$, red squares: $K_{\mathrm{G}} = 10,000$, blue squares: $K_{\mathrm{G}} = 5,000$. The thick black line is the exponential fit.

Tygers can be reduced to a problem in linear algebra

- We can solve

$$\frac{d}{d\tau}\hat{u}'_k = \sum_{k'=-K_{\mathrm{G}}}^{K_{\mathrm{G}}} A_{kk'}\,\hat{u}'_{k'} + \hat{f}_k\,, \qquad \hat{u}'_k(0) = 0$$

at time $\tau_\star = t_\star - t_{\mathrm{G}}$:

$$u'(\tau_\star) = A^{-1}\left(\mathrm{e}^{\tau_\star A} - \mathrm{I}\right) f = \left(\sum_{n=0}^{\infty} A^n \frac{\tau_\star^{n+1}}{(n+1)!}\right) f\,.$$

- From this it becomes clear that much will be controlled by the *spectral* properties of the operator A.
- We are thus led to consider the associated eigenvalue/eigenvector equation $A\psi = \lambda\psi$.

Conclusions and Perspective

- Tygers provide a clue as to the onset of thermalization.
- We do not have a complete understanding of the phenomenon.
- Tygers do not modify shock dynamics but modify the flow elsewhere because the tygers induce Reynolds stresses on scales much larger than the Galerkin wavelength; hence the weak limit of the Galerkin-truncated solution as $K_{\mathrm{G}} \to \infty$ is NOT the inviscid limit of the untruncated solution.
- There is good evidence that the key phenomena associated to tygers are also present in the two-dimensional incompressible Euler equation (and also perhaps in three dimensions).
- It is clear that complex-space singularities approaching the real domain within one Galerkin wavelength are the triggering factor in both the 2D Euler and the 1D Burgers case.
- Can we "purge tygers away" and thereby obtain a subgrid-scale method which describes the inviscid-limit solution right down to the Galerkin wavelength?

Purging ?