Lyapunov Exponents for Stochastic Differential Equations with Infinite Memory. Applications to Stochastic Navier-Stokes System in 2D

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Abstract

We give an Oseledec type theorem concerning Lyapunov spectrum for linear stochastic differential equations with infinite memory and apply this result to the Stochastic Navier–Stokes system on 2D-torus as well as to a couple of simple examples.

Key words: Stochastic differential equations, infinite memory, Lyapunov exponents, Stochastic Navier–Stokes system

1 Introduction and main result

In this note we give an Oseledec type theorem concerning Lyapunov exponents for linear stochastic differential equations with memory (SDEM). Oseledec proved his multiplicative ergodic theorem for linear cocycles in finite-dimensional spaces in his seminal work [7]. Infinite-dimensional versions of this result can be found in [9], [10] and [11].

Lyapunov spectrum for SDEMs was studied e.g. in [3], [4], [5], [6] under very general assumptions on the coefficients.

The novelty of the main result of this paper compared to those mentioned above is infiniteness of memory: the drift coefficient in the equation under consideration will be a functional of the history of the solution up to $-\infty$. Lyapunov spectrum for deterministic equations with infinite memory was studied

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in [8], [2]. We introduce the necessary definitions and notation in Section 1 and prove the main theorem in Section 2. Two simple examples are given in Section 3. In Section 4 we consider a linear SDEM related to Stochastic Navier-Stokes system on 2D torus and show that our result is applicable.

Consider the equation

$$\frac{dX(t)}{dt} = A_t(\pi_t X). \tag{1}$$

The projection and shift π_t is a map from the space C of $\mathbb{R}^d$-valued continuous functions defined on $\mathbb{R}$ to the space C^- of continuous functions defined on $\mathbb{R}_- = (-\infty, 0]$:

$$\pi_t X(s) = X(s+t), \quad s \in \mathbb{R}_-.$$

This map gives the past of a continuous process up to time $t \in \mathbb{R}$.

The coefficient A_t is a stationary random process taking values in the space of linear continuous operators from $\mathcal{X}_\gamma$ to $\mathbb{R}^d$, the space $\mathcal{X}_\gamma$ ($\gamma > 0$) consists of continuous functions $x : \mathbb{R}_- \to \mathbb{R}^d$ such that $\lim_{s\to-\infty} |x(s)|e^{\gamma s} = 0$. Here $|\cdot|$ means the Euclidean norm in $\mathbb{R}^d$. The space $\mathcal{X}_\gamma$ is a separable Banach space if equipped with the norm

$$\|x\|_\gamma = \sup_{s\in\mathbb{R}_-} |x(s)|e^{\gamma s}. \tag{2}$$

We will assume that our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the space of continuous paths in the space of bounded linear operators from $\mathcal{X}_\gamma$ to $\mathbb{R}^d$ equipped with locally uniform topology defined by the metric

$$d(A, B) = \sum_{N=1}^{\infty} \frac{\sup_{t\in[-N,N]} \|A - B\| \wedge 1}{2^N},$$

where $\|\cdot\|$ denotes the operator norm, and $\mathcal{F}$ is the completion of the Borel σ-field corresponding to this topology with respect to probability measure $\mathbb{P}$.

We will also require that

$$\mathsf{E} \sup_{t\in[0,T]} \|A_t\| < \infty \quad \text{for all} \quad T > 0. \tag{3}$$

By solution to Cauchy problem for equation (1) with initial value $x \in \mathcal{X}_\gamma$ on an interval $[T_1, T_2] \subset \mathbb{R}$ we mean a continuous $\mathbb{R}^d$-valued random process X adapted to the natural filtration generated by A such that $\pi_{T_1} X = x$ and equation (1) is satisfied on $[T_1, T_2]$. We say that equation (1) possesses the property of pathwise uniquness if there exists a functional Φ such that if X is a solution to Cauchy problem on $[T_1, T_2]$ subject to initial value x then

$$X\Big|_{[T_1,T_2]} = \Phi(x, A_{[T_1,T_2]}) \tag{4}$$

with probability 1.

The following theorem can be proved using a fixed point argument.

Theorem 1 *Under the above assumptions for all $x \in \mathcal{X}_\gamma$ there exists a solution to Cauchy problem on $[0, +\infty)$ subject to initial value x. This solution is unique. With probability 1 the following cocycle property for the functional Φ given in (4) is true for any $T_1 < T_2 < T_3$:*

$$\Phi(x, A_{[T_1,T_3]}) = \Phi\left(x, A_{[T_1,T_2]}\right) : \Phi\left(x{:}\Phi(x, A_{[T_1,T_2]}), A_{[T_2,T_3]}\right) \qquad (5)$$

where the semicolon means concatenation. The evolution operator Φ is linear and continuous in x.

For our purposes it is more convenient to work with operators

$$\Psi(x, t, A) = \pi_t \Phi(x, A_{[0,t]}),$$
$$(\theta^t A)_s = (A_{t+s}).$$

The evolution defined by Ψ takes place in $\mathcal{X}_\gamma$ and with the group of time shift operators θ^t defines a linear cocycle in $\mathcal{X}_\gamma$.

Now we can state the main result of this note.

Theorem 2 *Under the conditions stated above almost all realizations of A are regular in the following sense. There exists a set $\mathcal{K}(A)$, a family of θ-invariant functions $(\lambda_k(A))_{k\in\mathcal{K}}$ and two families $(F_k(A))_{k\in\mathcal{K}}$, $(E_k(A))_{k\in\mathcal{K}}$ of subspaces of $\mathcal{X}_\gamma$ with the following properties.*

(1) The set $\mathcal{K}$ is one of the following: $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $\mathbb{N}$ or $\mathbb{N} \cup \{\infty\}$.
(2) If $m > k$ then $\lambda_m < \lambda_k$; $\inf_k \lambda_k = -\gamma$.
(3) For any $\lambda > -\gamma$ inequality $\lambda_k(A) > \lambda$ is true for finitely many values of k.
(4) If $x \in F_k(A) \setminus F_{k+1}(A)$ for some $k \in \mathcal{K}$ then

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Psi(x, t, A)\|_\gamma = \lambda_k.$$

If $\lambda_k = -\gamma$ for some $k \in \mathcal{K}$ and $x \in F_k(A)$ then

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Psi(x, t, A)\|_\gamma = -\gamma.$$

(5) $\Psi(F_k(A), t, A) \subset F_k(\theta^t A)$.
(6) $\Psi(E_k(A), t, A) \subset E_k(\theta^t A)$.
(7) $F_1(A) = \mathcal{X}_\gamma$; if $k < m$ then $F_k(A) \supset F_m(A)$. All the subspaces $F_k(A)$ are finite-codimensional.

(8) $E_1(A) = \{0\}$; *if* $k < m$ *then* $E_k(A) \subset E_m(A)$. *All the subspaces* $E_k(A)$ *are finite-dimensional.*

(9) $F_k(A) \oplus E_k(A) = \mathcal{X}_\gamma$ *for all* k.

(10) *Subspaces* $F_k(A)$ *and* $E_k(A)$ *depend on* A *continuously (in Grassman topology, see [11]) for all* k.

(11) *For any* $\varepsilon > 0$ *and* $k \in \mathbb{N}$

$$\lim_{t\to\infty} \frac{1}{t} \log C_{\varepsilon,k}(\theta^t A) = 0$$

where

$$C_{\varepsilon,k}(A) = \sup_{n\in\mathbb{Z}} \left[e^{-|n|\varepsilon} \|\pi_{E_k(\theta^n A)\|F_k(\theta^n A)}\| \right].$$

Remark: The values λ_k defined in this theorem are usually called Lyapunov exponents.

The following theorem is an obvious corollary of Theorem 2.

Theorem 3 *If* $\lambda > -\gamma$ *then the space*

$$F(A, \lambda) = \left\{ x \in \mathcal{X}_\gamma : \limsup_{t\to\infty} \frac{1}{t} \log |\Psi(x, t, A)(0)| \geq \lambda \right\}$$

has finite codimension with probability 1.

Remark: The parameter γ plays a threshold role. Though generically there may be a solution X for the equation (1) such that $|X(t)| < e^{\lambda t}$ for some $\lambda < -\gamma$ and all $t > 0$, nevertheless $\|\pi_t(X)\|_\gamma \geq \|\pi_0(X)\|_\gamma e^{-\gamma t}$ for $t > 0$ and the Lyapunov exponent corresponding to such a solution is $-\gamma$.

2 Proof of the main result

Proof: To prove Theorem 2 we verify the conditions of the ergodic multiplicative theorem by Thieullen–Schaumlöffel–Flandoli [11], [10]:

(1) Time shifts θ^t are homeomorphisms of Ω.

(2) Linear operator $\Psi(\,\cdot\,, t, A)$ is injective.

(3) Operator $\Psi(\,\cdot\,, t, A)$ depends continuously on A.

(4) Integrability condition holds:

$$\mathsf{E} \sup_{0\leq t\leq 1} \log^+ \|\Psi(\,\cdot\,, t, A)\|_\gamma < \infty$$

$$\mathsf{E} \sup_{0\leq t\leq 1} \log^+ \|\Psi(\,\cdot\,, 1-t, \theta^t A)\|_\gamma < \infty$$

(5) The asymptotic logarithmic index of noncompactness $\varkappa(A)$ for the linear cocycle is equal to $-\gamma$. Here

$$\varkappa(A)=\lim_{t\to\infty}\frac{1}{t}\log\rho(\Psi(\,\cdot\,,t,A))$$

where $\rho(L)$ denotes Kuratowski's measure of noncompactness i.e. the infimum of all reals $\varepsilon>0$ such that the image of the unit ball under linear operator L can be covered by a finite number of balls of radius ε.

Condition 1 is obvious since if a sequence of trajectories in the operator space converges uniformly in the operator norm on every compact subset of $\mathbb{R}$ then the same holds for the shifts of these trajectories.

Condition 2 is obvious since if two trajectories in the past do not coincide whatever trajectories in the future are glued to them the resulting trajectories also do not coincide.

To prove condition 3 let's fix some constant $M>0$ and a path $x\in\mathcal{X}_\gamma$ with $\|x\|_\gamma\le 1$. Consider two realizations $A^{(1)},A^{(2)}:\mathbb{R}\times\mathcal{X}_\gamma\to\mathbb{R}^d$ such that $\sup_{s\in[0,t]}\|A_s^{(i)}\|\le M, i=1,2$. Let $X^{(i)}$ be the solution to (1) subject to initial data x and realization of coefficient $A^{(i)}, i=1,2$. Since norms of $A^{(i)}$ are bounded by M a straightforward estimate implies

$$\|\pi_s X^{(i)}\|_\gamma\le e^{Ms}\|x\|_\gamma,\quad s\in[0,t]. \tag{6}$$

Denote $Y(s)=X^{(1)}(s)-X^{(2)}(s)$. Then

$$\dot{Y}(s)=A^{(1)}(s)\left(\pi_sX^{(1)}\right)-A_s^{(2)}\left(\pi_sX^{(2)}\right)=A_s^{(1)}\left(\pi_sY\right)+\left(A_s^{(1)}-A_s^{(2)}\right)\left(\pi_sX^{(2)}\right).$$

Since $Y(0)=0$, the last relation and (6) imply

$$\|\pi_sY\|_\gamma\le M\int_0^s\|\pi_rY\|_\gamma dr+\sup_{r\in[0,s]}\|A_r^{(1)}-A_r^{(2)}\|se^{Ms}\|x\|_\gamma. \tag{7}$$

Now the Gronwall inequality implies

$$\|\pi_tY\|_\gamma\le te^{2Mt}\sup_{r\in[0,s]}\|A_r^{(1)}-A_r^{(2)}\|\|x\|_\gamma$$

which proves condition 3.

Condition 4 follows from the Gronwall inequality and (3).

Finally we will estimate $\varkappa(A)$. Let's denote $B=\{x:\ \|x\|_\gamma\le 1\}$ the unit ball in $\mathcal{X}_\gamma$.

Lemma 1 *With probability* 1 *the set*

$$\Phi_{[T_1,T_2]}(B) = \left\{ y \in C[T_1,T_2] : \text{there is } x \in \mathcal{X}_\gamma, \Phi(x, A_{[T_1,T_2]})\Big|_{[T_1,T_2]} \equiv y \right\}$$

is precompact in $C[T_1,T_2]$ *(the space of continuous functions defined on the segment* $[T_1,T_2]$*) equipped with uniform topology.*

Proof: First note that if X is a solution to Cauchy problem for (1) on $[T_1,T_2]$ with initial data $x \in B$ then

$$\|\pi_t X\|_\gamma \leq \sup_{t\in[T_1,T_2]} |X(t)|.$$

Let $\tau_R = T_2 \wedge \inf\{t \geq T_1 : |X(t)| \geq R\}$. Denote $X^*(t) = \sup_{s\in[T_1,t]} |X(s)|$. If $t < \tau_R$ then

$$X^*(t) \leq |X(T_1)| + \sup_{s\in[T_1,T_2]} \|A_s\| \int_{T_1}^t X^*(s)ds$$

and the Gronwall inequality implies

$$X^*(t) \leq e^{\sup_{s\in[T_1,T_2]} \|A_s\|(t-T_1)}.$$

Since this estimate does not depend on R and $\tau_R = T_2$ for large enough R uniformly in $x \in B$ with probability 1 it holds for all $t \in [T_1,T_2]$. So, $|X(s)|$ and $\|\pi_s X\|_\gamma$ are uniformly bounded on $[T_1,T_2]$ by some constant K depending on realization of A on this interval. Then the equation (1) implies that

$$|X(t) - X(s)| \leq K \sup_{s\in[T_1,T_2]} \|A_s\||t-s|$$

and

$$\alpha(X,\delta) \leq K \sup_{s\in[T_1,T_2]} \|A_s\|\delta$$

where $\alpha(f,\delta) = \sup\{|f(s)-f(t)| : t,s \in [T_1,T_2], |t-s| \leq \delta\}$ is the modulus of continuity of a function f. The last inequality and the Arzela-Ascoli theorem imply the conclusion of the theorem. □

Proof of condition 5: Fix an $r > 0$. The previous lemma allows to find a finite r-cover of $\Psi(B,t,A)\Big|_{[-t,0]}$ by balls $\{B_{uniform}(x_i,r),\ i = 1,\dots,n\}$ in uniform norm with $x_i \in \Psi(B,t,A), i = 1,\dots,n$. Consider a new family of balls $\{B_\gamma(0_{(-\infty,-t]}{:}x_i, 2r), i = 1,\dots,n\}$. They cover $\Psi(B,t,A)$ if $2re^{\gamma t} \geq 2$. This implies $\rho(\Psi(\,\cdot\,,t,A)) \leq 2e^{-\gamma t}$ and $\varkappa(A) \leq -\gamma$.

Suppose now that $r < e^{-\gamma t}/2$. Then for every ball of radius r in $\mathcal{X}_\gamma$ one can pick a trajectory $x \in \Psi(B,t,A)$ with $x(s) = x(-t)e^{\gamma(t-|s|)}$ for $s \leq -t$ which does not belong to that ball. So, for any finite family of such balls one can use segments of these trajectories to construct a trajectory which does not belong to any ball of the family. This proves that $\rho(\Psi(\,\cdot\,,t,A)) > e^{-\gamma t}/2$ and $\varkappa(A) \geq -\gamma$. This finishes the proof of condition 5 and Theorem 2. $\square$

Remark: Another simple way to prove relation $\varkappa(A) \geq -\gamma$ is to notice that two distinct trajectories of our system in $\mathcal{X}_\gamma$ cannot approach each other faster than with the rate $e^{-\gamma t}$.

3 Two simple examples

In this section we give two simple examples where the situation described by Theorem 2 is clearly seen.

In our first example $d = 1$ and $A_t(x) = a_t x(0)$ where a_t is an ergodic $\mathbb{R}$-valued stationary continuous process such that $\mathsf{E}\sup_{s\in[0,1]}|a_s| < \infty$. For $t \geq 0$ one has

$$X(t) = X(0)e^{\int_0^t a_s ds}.$$

If $X(0) = 0$ then $X(t) = 0$ for $t \geq 0$. So, the subspace $N = \{x : x(0) = 0\}$ has Lyapunov exponent $-\gamma$ and codimension 1 (we say that a set B has Lyapunov exponent λ if $\limsup_{t\to\infty} \frac{1}{t}\log\|\Psi(x,t,A)\|_\gamma = \lambda$ for all nonzero elements x of B)

The Birkhoff–Khinchin ergodic theorem implies that $\lim_{t\to\infty} \frac{1}{t}\log|X(t)| = \mathsf{E}a_0$. So, if $-\gamma < \mathsf{E}a_0$ then the set $\mathcal{X}_\gamma \setminus N$ has Lyapunov exponent equal to $\mathsf{E}a_0$. If $-\gamma \geq \mathsf{E}a_0$ then the whole space $\mathcal{X}_\gamma$ has Lyapunov eponent $-\gamma$.

Let now $d = 1$ and

$$A_t(x) = a_t \int_{-\infty}^0 e^{\nu s} x(s)ds$$

where $\nu > \gamma$ and a_t is as in the previous example. Introduce a new variable $Y(t) = \int_{-\infty}^t e^{\nu s} X(s)ds$ and notice that $X(t)$ and $Y(t)$ solve the following system of ordinary linear equations:

$$\frac{d}{dt}X(t) = a_t e^{-\nu t} Y(t)$$
$$\frac{d}{dt}Y(t) = e^{\nu t} X(t).$$

This means that if $X(0) = Y(0) = 0$ then $X(t) = Y(t) = 0$ for all $t \geq 0$. So, for every nonzero element x of the subspace

$$N = \left\{x : \int_{-\infty}^0 e^{\nu s} x(s)ds = 0, x(0) = 0\right\}$$

its Lyapunov exponent in $\mathcal{X}_\gamma$ is equal to $-\gamma$ which means that codimension of subspace corresponding to Lyapunov exponent $-\gamma$ does not exceed 2.

More detailed analysis can be done by studying equation

$$\frac{d^2}{dt^2}Y(t) - \nu\frac{d}{dt}Y(t) - a_t Y(t) = 0.$$

4 Stochastic Navier–Stokes in 2D

Consider incompressible Navier-Stokes system on 2-dimensional torus with stochastic forcing:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu\Delta u - \nabla P + f \\ &\nabla \cdot u = 0. \end{aligned} \tag{8}$$

This equation admits unique stationary solution with finite energy under suitable conditions on the stochastic forcing f (see [1]). Let u be a typical trajectory of this stationary solution defined on $\mathbb{R}$. Let us linearize the system (8) along u:

$$\frac{\partial v}{\partial t} = \nu\Delta v - P_{div}(u \cdot \nabla)v - P_{div}(v \cdot \nabla)u \tag{9}$$

Here v is a linear perturbation of u and P_{div} is the L^2 orthogonal projector on the subspace $\mathbb{L}^2$ of divergence-free vector fields.

Take an $N \in \mathbb{N}$ and define subspaces of low and high modes as

$$\mathbb{L}^2_l = \text{span}\{e_k, |k| \le N\}, \quad \mathbb{L}^2_h = \text{span}\{e_k, |k| > N\}.$$

where $\{e_k\}$ is the exponential basis in $\mathbb{L}^2$. Denote P_l and P_h orthogonal projectors on these two spaces respectively. Let $h = P_h v$, $l = P_l v$. Then the system (9) can be written as

$$\frac{\partial l}{\partial t} = \nu\Delta l - B_l(u, l) - B_l(l, u) - B_l(u, h) - B_l(h, u) \tag{10}$$

$$\frac{\partial h}{\partial t} = \nu\Delta h - B_h(u, h) - B_h(h, u) - B_h(u, l) - B_h(l, u) \tag{11}$$

where $B_l(w_1, w_2) = P_l[(w_1 \cdot \nabla)w_2]$ and $B_h(w_1, w_2) = P_h[(w_1 \cdot \nabla)w_2]$.

From now on $|v|$ denotes L^2-norm of $v \in L^2$. Let the linear space

$$\mathcal{X}_\gamma(\mathbb{L}^2) = \{v : \mathbb{R}_- \to \mathbb{L}^2 \mid \lim_{s\to\infty} e^{\gamma s}|v(s)| = 0|\}$$

be equipped with the norm

$$\|v\|_\gamma = \sup_{s\in\mathbb{R}_-} |v(s)|e^{\gamma s}.$$

Define also

$$\mathcal{X}_{\gamma,l} = \mathcal{X}_\gamma(\mathbb{L}^2) \cap \{v : \mathbb{R}_- \to \mathbb{L}^2_l\},$$
$$\mathcal{X}_{\gamma,h} = \mathcal{X}_\gamma(\mathbb{L}^2) \cap \{v : \mathbb{R}_- \to \mathbb{L}^2_h\}$$

Theorem 4 *There exist positive constants $C_1, \mathcal{E}$ such that if $\gamma, \nu > 0$ and $N \in \mathbb{N}$ satisfy*

$$\delta_0 := \nu N^2 - \frac{C_1^2 \mathcal{E}}{2\nu^2} - 2\gamma > 0 \tag{12}$$

then every low-mode trajectory in $\mathcal{X}_{\gamma,l}$ defines uniquely corresponding high-mode trajectory in $\mathcal{X}_{\gamma,h}$ More precisely, if (l, h_1) and (l, h_2) are both solutions to (9) *on $(-\infty, 0]$ and $h_1, h_2 \in \mathcal{X}_{\gamma,h}$ then $h_1(0) = h_2(0)$.*

Proof: Due to linearity of equation (9) it is sufficient to show that if $(0, h)$ solves (9) on $(-\infty, 0]$ with $h \in \mathcal{X}_{\gamma,h}$ then $h(0) = 0$. Let's rewrite (11) as

$$\frac{\partial h}{\partial t} = \nu\Delta h - B_h(u, h) - B_h(h, u).$$

Take the inner product (denoted by $\langle\cdot,\cdot\rangle$) of both sides with h in L^2 to obtain

$$\frac{1}{2}\frac{d|h|^2}{dt} = -\nu|\Lambda h|^2 - \langle B_h(h, u), h\rangle$$

where Λ is the self-adjoint positive operator defined by $\Lambda^2 u = -P_{div}\Delta u$. Here we used the equality

$$\langle B_h(u, h), h\rangle = 0.$$

Inequality

$$|\langle B_h(h, u), h\rangle| \le C_1|\Lambda h||h||\Lambda u| \le \frac{\nu}{2}|\Lambda h|^2 + \frac{C_1^2}{2\nu}|h|^2|\Lambda u|^2$$

implies the estimate

$$\frac{1}{2}\frac{d|h|^2}{dt} \le -\frac{\nu}{2}|\Lambda h|^2 + \frac{C_1^2}{2\nu}|h|^2|\Lambda u|^2.$$

Since $h(s) \in \mathbb{L}_h$ for all s the Poincarè inequality implies

$$\frac{d|h|^2}{dt} \le \left(-\nu N^2 + \frac{C_1^2}{\nu}|\Lambda u|^2\right)|h|^2.$$

So, for $t < 0$

$$|h(0)|^2 \le |h(t)|^2 \exp\left\{-\nu N^2|t| + \frac{C_1^2}{\nu}\int_t^0 |\Lambda u(s)|^2 ds\right\}. \tag{13}$$

Since the trajectory u is typical,

$$\lim_{t\to-\infty}\frac{1}{t}\int_t^0 |\Lambda u(s)|^2 ds = \frac{\mathcal{E}}{2\nu}$$

where $\mathcal{E}$ is the average energy supplied to the system by the exterior stochastic force. Hence there exist positive T and $\delta \in (0, \delta_0)$ such that

$$-\nu N^2|t| + \frac{C_1^2}{\nu}\int_t^0 |\Lambda u(s)|^2 ds + 2\gamma|t| \leq -\delta|t|$$

if $|t| > T$. This relation with the condition $h \in \mathcal{X}_{\gamma,h}$ and inequality (13) implies $h(0) = 0$. □

Our next step will be to define dynamics on low-mode trajectories.

Theorem 5 *Suppose that*

$$\delta_1 := \nu N^2 - \frac{C_1^2 \mathcal{E}}{2\nu^2} - 4\gamma > 0. \tag{14}$$

Fix $l \in \mathcal{X}_{\gamma,l}$ and let h_n be solution to (11) *on the interval* $[-n, 0]$ *with initial data $h_n(-n) = 0$. Then there exist an L^2-limit*

$$\Psi(l) = \lim_{n\to\infty} h_n(0). \tag{15}$$

The linear operator $\Psi : \mathcal{X}_{\gamma,l} \to \mathbb{L}_h$ is continuous. Its norm is a locally bounded function of u. If (l, h) is a solution to (9) *on $\mathbb{R}_-$ then $h(0) = \Psi(l)$.*

Proof: Note that condition (14) implies (12) since $\delta_1 < \delta_0$. It also implies that $\gamma < \delta_0/2$. Due to linearity of the system one can apply inequality (13) with $h = h_n - h_{n+1}$ on the interval $[-n, 0]$ because both of h_1 and h_2 are subject to one and the same low-mode trajectory l on this interval. Together with inequality (4) it gives for $\delta \in (2\gamma, \delta_0)$

$$|h_{n+1}(0) - h_n(0)| \leq |h_{n+1}(-n)| \exp\left\{-\frac{\delta}{2}n\right\}$$

for $n > T(\delta)$. Thus to prove existence of the limit in (15) it is sufficient to show that

$$\sum_{n\geq T} |h_{n+1}(-n)| \exp\left\{-\frac{\delta}{2}n\right\} < \infty.$$

To do that let's estimate $|h_{n+1}(-n)|$ with the help of the Gronwall inequality.

First take the inner product of (11) with h to obtain

$$\begin{aligned}\frac{1}{2}\frac{d|h|^2}{dt} &= -\nu|\Lambda h|^2 - \langle B_h(h,u),h\rangle - \langle B_h(l,u),h\rangle - \langle B_h(u,l),h\rangle \\ &\leq -\nu|\Lambda h|^2 + C_1|h||\Lambda h||\Lambda u| + C_2|\Lambda u||\Lambda^2 l||h| \\ &\leq \left[-\frac{\nu}{2}N^2 + \frac{C_1^2}{2\nu}|\Lambda u(s)|^2 + \frac{1}{4}\right]|h|^2 + C_2^2|\Lambda u|^2|\Lambda^2 l|^2\end{aligned}$$

Since the trajectory u is typical and $l \in \mathcal{X}_{\gamma,l}$ one can write

$$|\Lambda u(s)| \leq \|\!|\Lambda u|\!\|_\alpha(1+|s|^\alpha),\ \ \alpha\in(0,1)$$

(here $\|\!|f|\!\|_\alpha = \sup_{s\leq 0}\frac{|f(s)|}{1+|s|^\alpha}$) and

$$|\Lambda^2 l(s)| \leq \|l\|_\gamma N^2 e^{\gamma|s|}.$$

These estimates and the Gronwall inequality imply that

$$\begin{aligned}|h_{n+1}(-n)|^2 \leq & C_2^2 N^4 \|\!|\Lambda u|\!\|_\alpha^2 \|l\|_\gamma^2 e^{2\gamma(n+1)}(1+(n+1)^{2\alpha})\times \\ & \times \int_{-n-1}^{n} \exp\left\{\int_t^n \left[-\nu N^2 + \frac{C_1^2}{\nu}|\Lambda u(s)|^2 + \frac{1}{2}\right]ds\right\}dt \\ \leq & C_2^2 \|\!|\Lambda u|\!\|_\alpha^2 \|l\|_\gamma^2 e^{2\gamma|n+1|}(1+|n+1|^{2\alpha})\times \\ & \times \exp\left\{\frac{C_1^2}{\nu}\sup_{s\in[-n-1,n]}|\Lambda u(s)|^2 + \frac{1}{2}\right\} \\ \leq & C_2^2 N^4 \|\!|\Lambda u|\!\|_\alpha^2 \|l\|_\gamma^2 e^{2\gamma(n+1)}(1+(n+1)^{2\alpha})\times \\ & \times \exp\left\{\frac{C_1^2}{\nu}\|\!|\Lambda u|\!\|_\alpha(1+(n+1)^\alpha) + \frac{1}{2}\right\}\end{aligned}$$

So,

$$|h_{n+1}(-n)| \leq C_2 \|\!|\Lambda u|\!\|_\alpha \|l\|_\gamma e^{\gamma(n+1) + \frac{C_1^2}{2\nu}\|\!|\Lambda u|\!\|_\alpha(1+(n+1)^\alpha) + \frac{1}{4}}(1+(n+1)^\alpha)$$

and inequality $\gamma < \delta/2$ implies now that

$$\sum_{n\geq T}|h_{n+1}(-n)|\exp\left\{-\frac{\delta}{2}n\right\} \leq C_3(\|\!|u|\!\|_\alpha)\|\!|\Lambda u|\!\|_\alpha\|l\|_\gamma$$

where C_3 is a locally bounded function. This proves the existence of the limit $\Psi(l)$ and continuity of $\Psi(\ \cdot\ )$ with respect to $\|\ \cdot\ \|_\gamma$. Analogous estimates imply the last assertion of the theorem. □

This theorem shows that equation (9) can be written as equation with memory. Indeed, substitute $h(t) = \Psi(\pi_t l)$ into (10) to obtain

$$\frac{dl}{dt} = \nu\Delta l - B_l(u,l) - B_l(l,u) - B_l(u,\Psi(\pi_t l)) - B_l(\Psi(\pi_t l),u). \quad (16)$$

Moreover this equation satisfies all the assumptions of Theorem 2 with one-to-one correspondence between trajectories of ergodic process u and corresponding operator-valued trajectories A_t. Thus we obtain the following result.

Theorem 6 *Under the conditions of Theorem 5 the linear cocycle defined by the corresponding linear SDEM* (16) *satisfies the conclusions of Theorem 2. In particular the space of low mode trajectories for which Lyapunov exponent is negative has finite codimension. Lyapunov exponents λ_i are constant a.s.*

5 Acknowledgments

The author would like to thank Jonathan Mattingly and Yakov Sinai for their input and comments. The author would also like to thank the hospitality and support of the Institute for Advanced Study in Princeton and California Institute of Technology in Pasadena where this work was undertaken. Support from the NSF under grant DMS-9729992 is gratefully acknowledged.

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