Existence And Uniqueness of Stationary Solutions for 3D Navier-Stokes System with Small Random Forcing via Stochastic Cascades.

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Abstract

We consider 3D Navier–Stokes system in the Fourier space with regular forcing given by a stationary in time stochastic process satisfying a smallness condition. We explicitly build stationary solution to the system and prove the uniqueness theorem for this solution. Moreover we prove the following "One Force — One Solution" principle: the unique stationary solution at time t is presented as a functional of the realization of the forcing in the past up to t. The explicit construction of the solution is based upon the stochastic cascade representation.

1 Introduction. Main result.

The aim of this note is to prove an existence and uniqueness theorem for stationary solutions of randomly forced Navier-Stokes system on 3D-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and $\mathbb{R}^3$ with the help of the stochastic cascade representation of solutions introduced in [JS97] and developed in [BCD$^+$03]. Results on existence and uniqueness of stationary solutions to randomly and stochastically forced Navier-Stokes system in 2D can be found in [CK97], [Fer97], [FM95], [DPZ96], [EMS01], [KS00], [BKL01], [KPS02], [KS02]. See [Mat03] for a recent survey of these results and new ideas. An existence theorem for stationary suitable weak solutions for the Navier-Stokes system in 3D bounded domains is proved in [FR01] and no uniqueness results in 3D are known to the author. Our approach is completely different from that of

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[FR01]. Using the techniques of [JS97] we build an explicit representation of the stationary solution as a functional of the realization of the forcing in the past.

In this section we introduce necessary notation and state an existence-uniqueness result for the Cauchy problem from [BCD$^+$03] as well as our new result for stationary solutions. In Section 2 we describe our main tool, stochastic cascades, and give all the proofs.

We consider first the Cauchy problem for the Navier-Stokes system on domain $G = \mathbb{T}^3$ or $\mathbb{R}^3$:

$$\frac{\partial u(x,t)}{\partial t} + \langle u, \nabla \rangle u(x,t) = \nu \Delta u(x,t) - \nabla p(x,t) + g(x,t), \tag{1}$$

$$\langle \nabla, u \rangle = 0. \tag{2}$$

$$u(x,t_0) = u_0(x). \tag{3}$$

Here $x \in G$, and $u(x,t) \in \mathbb{R}^3$ is a divergence-free velocity field for each time $t \in [t_0, \infty)$. Angular brackets denote the Euclidean inner product, ∇ is the gradient operator, Δ is the Laplacian, $\nu > 0$ is the viscosity parameter, $p : G \to \mathbb{R}$ is the pressure and $g : G \times [t_0, \infty) \to \mathbb{R}^3$ is the external forcing.

We shall assume that the initial data u_0 and the force g are divergence free and zero mean:

$$\int_G u_0(x)dx = 0, \qquad \int_G g(x,t)dx = 0, \quad t \geqslant t_0.$$

Let us rewrite the Navier–Stokes system (1)—(3) in the Fourier space and get rid of the pressure term. Consider the case $G = \mathbb{T}^3$ first:

$$\frac{\partial \widehat{u}(k,t)}{\partial t} = -4\pi^2 \nu |k|^2 \widehat{u}(k,t) - 2\pi i \, \mathrm{P}_{k^\perp} \sum_{l_1+l_2=k} \langle k, \widehat{u}(l_1,t) \rangle \widehat{u}(l_2,t) + \widehat{g}(k,t), \quad k \neq 0, \tag{4}$$

$$\widehat{u}(0,t) = 0.$$

Here $u(x,t) = \sum_{k \in \mathbb{Z}^3} \widehat{u}(k,t) e^{2\pi i \langle k,x \rangle}$, $g(x,t) = \sum_{k \in \mathbb{Z}^3} \widehat{g}(k,t) e^{2\pi i \langle k,x \rangle}$, and $\mathrm{P}_{k^\perp}$ is the orthogonal projection along the vector $e_k = \frac{k}{|k|}$ which corresponds to the projection on the space of divergence free vector fields since the condition of zero divergence is expressed in the Fourier space as $\langle \widehat{u}(k,t), k \rangle = 0$ for all t and k.

Consider a function $h : \mathbb{Z}^3 \to \mathbb{R}_+$ such that $h(k) > 0$ for $k \neq 0$ and $h(0) = 0$, denote

$$\chi(k,t) = \frac{\widehat{u}(k,t)}{h(k)}, \quad k \neq 0, \tag{5}$$

$$\chi(0,t) = 0,$$

and rewrite equation (4) in integral form:

$$\begin{aligned}\chi(k,t) &= e^{-4\pi^2\nu(t-t_0)|k|^2}\chi(k,t_0) \\ &+ \frac{1}{2}\int_0^{t-t_0} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 s} m(k)\,\mathrm{P}_{k^\perp}\sum_{l_1+l_2=k}\langle e_k,\chi(l_1,t-s)\rangle\chi(l_2,t-s)H(k,l_1,l_2)ds \\ &+ \frac{1}{2}\int_0^{t-t_0} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 s}\varphi(k,t-s)ds, \quad k\neq 0.\end{aligned} \tag{6}$$

Here

$$m(k) = -\frac{4\pi i h*h(k)}{\nu|k|h(k)}, \quad H(k,l_1,l_2) = \frac{h(l_1)h(l_2)}{h*h(k)}, \quad \varphi(k,t) = \frac{2\widehat{g}(k,t)}{\nu|k|^2 h(k)} \tag{7}$$

and $h*h(k) = \sum_{l_1+l_2=k} h(l_1)h(l_2)$.

In the case $G=\mathbb{R}^3$ an analogous equation for the Fourier transform of u given by

$$\widehat{u}(k) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\langle k,x\rangle}u(x)dx$$

is

$$\begin{aligned}\chi(k,t) &= e^{-\nu(t-t_0)|k|^2}\chi(k,t_0) \\ &+ \frac{1}{2}\int_0^{t-t_0} \nu|k|^2 e^{-\nu|k|^2 s} m(k)\,\mathrm{P}_{k^\perp}\int_{\mathbb{R}^3}\langle e_k,\chi(l,t-s)\rangle\chi(k-l,t-s)H(k,l)dlds \\ &+ \frac{1}{2}\int_0^{t-t_0} \nu|k|^2 e^{-\nu|k|^2 s}\varphi(k,t-s)ds, \quad k\neq 0,\end{aligned} \tag{8}$$

where now

$$m(k) = -\frac{2ih*h(k)}{\nu(2\pi)^{3/2}|k|h(k)}, \quad H(k,l) = \frac{h(l)h(k-l)}{h*h(k)}, \quad \varphi(k,t) = \frac{2\widehat{g}(k,t)}{\nu|k|^2 h(k)}$$

and $h*h(k) = \int_{\mathbb{R}^3} h(l)h(k-l)dl$ for a function $h:\mathbb{R}^3\to\mathbb{R}_+$ such that $h(k)>0$ for $k\neq 0$ and $h(0)=0$. The scaled solution χ is given by (5).

The following result was proved in [BCD+03] and [JS97]:

Theorem 1 *Let $G=\mathbb{T}^3$ (respectively, $\mathbb{R}^3$). Suppose $|m(k)|\leqslant 1$, $|\chi(k,t_0)|\leqslant 1$ for all k and $|\varphi(k,t)|\leqslant 1$ for all k and $t\in[t_0,t_1]$. Then there exists a solution of* (6) *(respectively,* (8)*) on $[t_0,t_1]$ with initial data $\chi(\cdot,t_0)$. This solution χ satisfies $|\chi(k,t)|\leqslant 1$ for all k and $t\in[t_0,t_1]$. It is unique in the space of functions bounded by* 1 *and defined on $\mathbb{Z}^3\times[t_0,t_1]$ (respectively, $\mathbb{R}^3\times[t_0,t_1]$).*

The main new result of this paper is concerned with the case of the external forcing given by a stochastic process. Now $\varphi(k,t) = \varphi(k,t,\omega)$ where ω is an element of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 2 *Let $G = \mathbb{T}^3$ (respectively, $\mathbb{R}^3$). Suppose $|m(k)| \leqslant 1$ for all $k \neq 0$, and the external forcing is a stationary process taking values in the space of functions defined on $\mathbb{Z}^3$ (respectively, $\mathbb{R}^3$) and bounded by 1: $|\varphi(k,t)| \leqslant 1$ for all k and t. Then there exist a solution of* (6) *(respectively,* (8)*) defined for $t \in \mathbb{R}$ which is a stationary process. This stationary solution χ satisfies $|\chi(k,t)| \leqslant 1$ for all k and t. It is the only solution of* (6) *defined for all $t \in \mathbb{R}$ with this property.*

The following "one force — one solution" principle holds: There exists a functional Ψ of realizations of the force in the past up to time t such that the unique stationary solution χ is given by $\chi(\cdot,t) = \Psi(\varphi(-\infty,t])$.

Remark 1 The condition on $|m(\cdot)|$ is fulfilled iff $4\pi h * h(k) \leqslant \nu|k|h(k)$ for $\mathbb{T}^3$ and $2h * h(k) \leqslant \nu(2\pi)^{3/2}|k|h(k)$ for $\mathbb{R}^3$. A possible choice of $h(\cdot)$ for both cases is $h(k) = c|k|^{-2}$ with sufficiently small constant $c > 0$, see [JS97], [BCD$^+$03].

Remark 2 In particular this result is applicable if the force is constant with respect to time. Thus, one obtains an existence-uniqueness theorem for steady state of the Navier-Stokes system.

Remark 3 Our method of constructing stationary solutions is actually applicable for any PDE which admits a stochastic cascade representation of solutions to the Cauchy problem, such as linear parabolic and fractional diffusion equations, some quasilinear equations, e.g. the Burgers equation.

Since the construction of the functional Ψ will be based on the notion of stochastic cascade which plays the central role in the proof of Theorem 1, we give proofs of both Theorems 1 and 2 in the next section.

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2 Stochastic cascades and proofs of Theorems 1 and 2.

The proofs will be given only for the case $G = \mathbb{T}^3$. The case $G = \mathbb{R}^3$ is treated in the same way.

The basic notion of this section is that of stochastic cascade. To introduce it we need some notation. Let

$$V = \bigcup_{j=0}^{\infty} \{1,2\}^j = \{\theta, (1), (2), (1,1), \ldots\}$$

be a complete binary tree rooted at θ, where $\{1,2\}^0 = \{\theta\}$.

For $v = (v_1, v_2, \ldots, v_m) \in V$ and $1 \leqslant n \leqslant m$ denote $v|n = (v_1, v_2, \ldots, v_n)$ and $v|0 = \theta$.

For a finite subtree W of V rooted at θ we define ∂W to be the set of all leaves of W, where a leave is a vertice of W with no children in W.

Informally, the stochastic cascade we need can be described as a branching random walk. A single particle corresponding to the root θ of V is placed at a point $(k,t) \in \mathbb{Z}^3 \times \mathbb{R}$ and then the process takes place in the reverse time. The particle waits an exponentially distributed length of time S_θ with parameter $4\pi^2\nu|k|^2$ and then, at time $t - S_\theta$ an independent coin κ_θ is tossed and either with probability 1/2 the event $\{\kappa_\theta = 0\}$ occurs and the particle dies, or with probability 1/2 one has $\{\kappa_\theta = 1\}$ and the particle branches into two particles which are placed at $(l_1, t - S_\theta)$ and $(l_2, t - S_\theta)$ where the positions l_1 and l_2 are chosen according to the probability distribution $H(k, l_1, l_2)$ defined in (7). This procedure is repeated independently for these new particles which correspond to vertices (1) and (2) of a complete binary tree.

More exactly, for a given $k \in Q = \mathbb{Z} \setminus \{0\}$ we need a stochastic process $(k_v, \kappa_v, S_v)_{v \in V}$ indexed by vertices of a binary tree (or, equivalently, a corresponding probability measure $\mathbb{P}$) with the following properties.

1. $\mathbb{P}\{k_v \in Q, \kappa_v \in \{0,1\}, S_v \in \mathbb{R}_+\} = 1$ for all $v \in V$.

2. Random variables k_θ, κ_θ and S_θ are independent with $\mathbb{P}\{k_\theta = k\} = 1$. $\mathbb{P}\{\kappa_\theta = i\} = \frac{1}{2}$ for $i = 0,1$ and $\mathbb{P}\{S_\theta > s\} = e^{-4\pi^2\nu|k|^2 s}$ for $s \geqslant 0$.

3. Let the concatenation (v, b) of $v \in V$ and $b \in \{1,2\}$ denote a child vertex of v. Suppose W is a subtree of V rooted at θ and $v \in \partial W$. Then,

$$\begin{aligned} &\mathbb{P}\{k_{(v,1)} = l_1, \kappa_{(v,1)} = i_1, S_{(v,1)} \geqslant s_1, k_{(v,2)} = l_2, \kappa_{(v,2)} = i_2, S_{(v,2)} \geqslant s_2 \mid \mathcal{F}_W\} \\ &\quad = \frac{1}{4} H(k_v, l_1, l_2) e^{-4\pi^2\nu|l_1|^2 s_1} e^{-4\pi^2\nu|l_2|^2 s_2}, \quad l_1, l_2 \in Q, i_1, i_2 \in \{0,1\}, s_1, s_2 \geqslant 0, \end{aligned}$$

where $\mathcal{F}_W = \sigma\{(k_w, \kappa_w, S_w)_{w \in W}\}$.

We will call the construction described above stochastic cascade emitted from point k.

We will need two easily verified propositions concerning stochastic cascades. Each of them is a form of the Markov property. Let $K = (k_v, \kappa_v, S_v)_{v \in V}$ be a stochastic cascade emitted from a point k, and $u \in V$. Then one can define "shifted" stochastic cascade $K(u) = (\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v)_{v \in V}$ via

$$(\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v) = (k_{(u,v)}, \kappa_{(u,v)}, S_{(u,v)}), \quad v \in V,$$

where (u, v) means concatenation. Define also $\mathcal{F}^*_W = \sigma\{(k_w, \kappa_w, S_w)_{w \in W \setminus \partial W}, (k_w)_{w \in \partial W}\}$.

Proposition 1 (First Markov Property) *Suppose W is a subtree of V rooted at θ. Then stochastic cascades $\{K(u), u \in \partial W\}$ are conditionally independent given $\mathcal{F}^*_W$ and the conditional distribution of $K(u)$ is a.s. equal to the distribution of stochastic cascade emitted from k_u.*

Let $K = (k_v, \kappa_v, S_v)_{v \in V}$ is a stochastic cascade emitted from a point k. Consider $t > 0$ and a point $l \in Q$. Suppose there exists such $u \in V$ that $k_u = l$ and $A_u \leqslant t < B_u$ where $A_u = \sum_{i=0}^{|u|-1} S_{u|i}$ and $B_u = \sum_{i=0}^{|u|} S_{u|i}$ for $u \in V$. Then one can define a new shifted cascade $K(u, t) = (\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v)_{v \in V}$:

$$(\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v) = \begin{cases} (k_{(u,v)}, \kappa_{(u,v)}, S_{(u,v)}), & v \neq \theta, \\ (l, \kappa_u, B_u - t), & v = \theta. \end{cases}$$

Proposition 2 (Second Markov Property) *The distribution of the stochastic cascade $K(u, t)$ conditioned on the event that $\{A_u \leqslant t < B_u, k_u = l\}$ for $u \in V$, coincides a.s. with the distribution of the stochastic cascade emitted from l.*

Let us now introduce a deterministic functional $X(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \chi_0(\cdot), \varphi(\cdot, \cdot), t_0)$ of a realization of stochastic cascade $(k_v, \kappa_v, S_v)_{v \in V}$ emitted from $k_\theta = k$, initial data χ_0, and external forcing φ. Define for $v \in V$

$$X_v = \begin{cases} \chi_0(k_v), & T_v \leqslant t_0 \\ \varphi(k_v, T_v), & T_v > t_0, \kappa_v = 0 \\ m(k_v)\, \mathrm{P}_{k_v^\perp} \langle e_{k_v}, X_{(v,1)} \rangle X_{(v,2)}, & T_v > t_0, \kappa_v = 1 \end{cases} \tag{9}$$

where $T_v = t - B_v$ for $v \in V$ and let

$$X(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \chi_0(\cdot), \varphi(\cdot, \cdot), t_0) = X_\theta.$$

Define $\tau(k, t, t_0)$ to be the maximal subtree of V rooted at θ such that $\kappa_v = 1$ and $t - A_v \geqslant t_0$ for all $v \in \tau(k, t, t_0)$. Since the tree $\tau(k, t, t_0)$ can be viewed as a truncated representation of a critical branching process it is a.s. finite and one can evaluate X_θ recursively starting with the leaves of $\tau(k, t, t_0)$.

We are now ready to give a proof of the existence part of Theorem 1. Since all the terms in the definition of X_θ are bounded by 1, $|X_\theta| \leqslant 1$ a.s. We claim that $\chi(k,t) = \mathsf{E}\, X(k,t,(k_v,\kappa_v,S_v)_{v\in V},\chi_0(\cdot),\varphi(\cdot,\cdot),t_0)$ is a solution of (6). Recall that the expectation is taken with respect to the stochastic cascade emitted from point $k_\theta = k$.

To verify our claim let us write

$$\mathsf{E}\, X_\theta = \mathsf{E}\, X_\theta \mathbf{1}\{S_\theta > t-t_0\} + \mathsf{E}\, X_\theta \mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 1\} + \mathsf{E}\, X_\theta \mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 0\}. \quad (10)$$

Clearly, the first and the third terms of this decomposition exactly coincide with those of (6). To show that the second terms of these equations also coincide, use the First Markov Property:

$$\begin{aligned} &\mathsf{E}\left[m(k_\theta)\, \mathrm{P}_{k_\theta^\perp}\langle e_{k_\theta}, X_{(1)}\rangle X_{(2)} \mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 1\}\right] \\ &\quad = m(k)\, \mathsf{E}\left[\mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 1\}\, \mathrm{P}_{k_\theta^\perp}\, \mathsf{E}\left[\langle e_{k_\theta}, X_{(1)}\rangle X_{(2)} \mid k_{(1)}, k_{(2)}, S_\theta, \kappa_\theta\right]\right] \\ &\qquad = m(k)\, \mathsf{E}\left[\mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 1\}\, \mathrm{P}_{k_\theta^\perp}\langle e_{k_\theta}, \chi(k_{(1)}, t-S_\theta)\rangle \chi(k_{(2)}, t-S_\theta)\right] \end{aligned}$$

which is obviously equal to the second term in (6). The existence part is proved and details of a proof of the uniqueness part can be found in [BCD$^+$03].

To prove Theorem 2, for each realization of the random forcing φ we introduce a stochastic cascade $(k_v,\kappa_v,S_v)_{v\in V}$ emitted from point k described above and a functional Z analogous to (9):

$$Z_v = \begin{cases} \varphi(k_v, T_v), & \kappa_v = 0 \\ m(k_v)\, \mathrm{P}_{k_v^\perp}\langle e_{k_v}, Z_{(v,1)}\rangle Z_{(v,2)}, & \kappa_v = 1. \end{cases} \quad (11)$$

and

$$Z(k,t,(k_v,\kappa_v,S_v)_{v\in V},\varphi(\cdot,\cdot)) = Z_\theta.$$

Define $\tau(k)$ as the maximal subtree of V such that $\kappa_v = 1$ for all $v \in \tau(k)$. The tree $\tau(k)$ is a.s. finite and one can evaluate Z_θ recursively starting with the leaves of the tree $\tau(k)$.

As well as for the Cauchy problem case $|Z_\theta| < 1$ samplewize since all the multipliers in (11) are bounded by 1. Let us show that $\chi(k,t) = \mathsf{E}_\varphi\, Z(k,t,(k_v,\kappa_v,S_v)_{v\in V},\varphi(\cdot,\cdot))$ is a solution of (6) on any time interval $[t_0,\infty)$, where E_φ means the expectation with respect to the stochastic cascade with fixed realization of the forcing φ. To that end fix a time $t > t_0$ and consider the following decomposition:

$$\mathsf{E}_\varphi\, Z_\theta = \mathsf{E}_\varphi\, Z_\theta \mathbf{1}\{S_\theta > t-t_0\} + \mathsf{E}_\varphi\, Z_\theta \mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 1\} + \mathsf{E}_\varphi\, Z_\theta \mathbf{1}\{S_\theta \leqslant t-t_0, \kappa_\theta = 0\}.$$

The Second Markov Property implies

$$\mathsf{E}_\varphi\, Z_\theta \mathbf{1}\{S_\theta > t-t_0\} = e^{-4\pi^2\nu|k|^2(t-t_0)}\, \mathsf{E}_\varphi\, Z(k,t_0,(k_v,\kappa_v,S_v)_{v\in V},\varphi(\cdot,\cdot)) = e^{-4\pi^2\nu|k|^2(t-t_0)}\chi(k,t_0),$$

and the two other terms are treated in the same way as in representaion (10). Note that the resulting solution χ at time t is a functional of the realization of the forcing term φ in the past up to time t.

Suppose now that there is another solution $\gamma(k,t)$ of (6) which is bounded by 1 and defined for all $t \in \mathbb{R}$. Pick a $t_0 \in \mathbb{R}$ and consider $\gamma(k,t), t \geqslant t_0$ as the solution to the Cauchy problem for (6) with initial data $\gamma(k,t_0)$. Consider the stochastic cascade representation of χ and γ. Since $Z_\theta = X_\theta$ if $d(\tau(k)) < t - t_0$ where $d(\tau(k)) = \sup\{B_v : v \in \tau(k)\}$, we obtain

$$|\chi(k,t) - \gamma(k,t)| \leqslant 2\mathbb{P}\{d(\tau(k)) \geqslant t - t_0\} \to 0 \quad \text{as } t_0 \to -\infty.$$

This implies that $\chi(k,t) = \gamma(k,t)$ and the theorem is proved.

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