Convex chains for Schubert varieties

Valentina Kiritchenko

(joint work with Evgeny Smirnov and Vladlen Timorin)

In [4], we constructed generalized Newton polytopes for Schubert subvarieties in the variety of complete flags in $\mathbb{C}^n$. Every such "polytope" is a union of faces of a Gelfand–Zetlin polytope (the latter is a well-known Newton–Okounkov body for the flag variety). These unions of faces are responsible for Demazure characters of Schubert varieties and were originally used for Schubert calculus.

The methods of [4] lead to an extension of Demazure (or divided difference) operators from representation theory and topology to the setting of convex geometry. Below I define divided difference operators acting on convex polytopes and outline some applications such as a simple inductive construction of Gelfand-Zetlin polytopes and their generalizations.

The definition is based on the following observation. Let $\Pi(\mu,\nu)$ where μ, $\nu \in \mathbb{Z}^m$ denote the integer *coordinate parallelepiped* $\{(x_1,\ldots,x_m)|\mu_i \le x_i \le \nu_i\} \subset \mathbb{R}^m$, and let $\sigma(x)$ for $x \in \mathbb{R}^m$ denote the sum of coordinates $\sum_{i=1}^m x_i$. Given a parallelepiped $\Gamma = \Pi(\mu,\nu) \subset \mathbb{R}^m$ of dimension $m-1$ (assume that $\mu_m = \nu_m$) and an integer C, there is a unique parallelepiped $\Pi = \Pi(\mu,\nu') \subset \mathbb{R}^m$ such that $\Gamma = \Pi \cap \{x_m = \mu_m\}$ (that is, $\nu'_i = \nu_i$ for $i < m$) and

$$\sum_{x\in\Pi\cap\mathbb{Z}^d} t^{\sigma(x)} = D_C\Big(\sum_{x\in\Gamma\cap\mathbb{Z}^d} t^{\sigma(x)}\Big), \qquad (*)$$

where D_C is a Demazure-type operator on the ring $\mathbb{Z}[t,t^{-1}]$ of Laurent polynomials in t:

$$D_C(f) := \frac{f - tf^*}{1-t}, \quad f^* := t^C f(t^{-1}).$$

Indeed, an easy calculation (using the formula for the sum of a geometric progression) shows that $\sum_{i=1}^m(\mu_i + \nu'_i) = C$ which yields the value of ν'_m. Note that Γ is a facet of Π unless $\Pi = \Gamma$.

We now use this observation in a more general context. A *root space* of rank n is a coordinate space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n}$$

and a collection of linear functions $l_1, \ldots, l_n \in (\mathbb{R}^d)^*$ such that l_i vanishes on $\mathbb{R}^{d_i}$. We always assume that the summands are coordinate subspaces so that $\mathbb{R}^{d_1}$ is spanned by the first d_1 basis vectors etc.

Let $P \subset \mathbb{R}^d$ be a convex polytope in the root space. It is called a *parapolytope* if for all $i = 1,\ldots, n$, the intersection of P with any parallel translate of $\mathbb{R}^{d_i}$ is a coordinate parallelepiped. For instance, if $d = n$, that is, $d_1 = \ldots = d_n = 1$, then every polytope is a parapolytope. For each $i = 1,\ldots, n$, we now define a *divided difference operator* A_i on parapolytopes. In general, the operator A_i takes values in *convex chains* in $\mathbb{R}^d$ (see [3] for a definition) but in many cases of interest (see examples below) these convex chains will just be single convex parapolytopes.

First, consider the case where $P \subset (c+\mathbb{R}^{d_i})$ for some $c \in \mathbb{R}^d$, i.e. $P = P(\mu,\nu)$ is a coordinate parallelepiped. Here $\mu = (\mu_1,\ldots,\mu_d)$, $\nu = (\nu_1,\ldots,\nu_d)$. Put $N_i := d_1+\ldots+d_i$ and $N_0 = 0$. Assume that $\dim(P) < d_i$. Choose the smallest $j \in [N_{i-1}+1, N_i]$ such that $\mu_j = \nu_j$. Define $A_i(P)$ to be the coordinate parallelepiped $\Pi(\mu,\nu')$, where $\nu'_k = \nu_k$ for all $k \neq j$ and ν'_j is chosen so that

$$\sum_{k=N_{i-1}+1}^{N_i} (\mu_k+\nu'_k) = l_i(c), \tag{$**$}$$

that is, an analog of formula $(*)$ holds for $\Gamma = P$, $\Pi = A_i(P)$ and $C = l_i(c)$. The definition yields a non-virtual coordinate parallelepiped if $l_i(c)$ is sufficiently large and can be extended to other values of $l_i(c)$ by linearity.

For an arbitrary parapolytope $P \subset \mathbb{R}^d$ define $A_i(P)$ as the union of $A_i(P\cap(c+\mathbb{R}^{d_i}))$ over all $c \in \mathbb{R}^d$:

$$A_i(P) = \bigcup_{c\in\mathbb{R}^d} \{A_i(P\cap(c+\mathbb{R}^{d_i}))\}$$

(assuming that $\dim(P\cap(c+\mathbb{R}^{d_i})) < d_i$ for all $c \in \mathbb{R}^d$). In other words, we first slice P by subspaces parallel to $\mathbb{R}^{d_i}$ and then replace each slice with another parallelepiped according to $(**)$. Note that P is a facet of $A_i(P)$ unless $A_i(P) = P$. It is easy to check that $A_i^2 = A_i$ (the same identity as for the classical Demazure operators).

Examples: (1) The simplest meaningful example is $\mathbb{R}^2 = \mathbb{R}\oplus\mathbb{R} = \{(x,y)\}$ with the functions $l_1 = y$ and $l_2 = x$. If $P = (a,b)$ is a point, then $A_1(P)$ and $A_2(P)$ are segments:

$$A_1(P) = [(a,b),(b-a,b)], \quad A_2(P) = [(a,b),(a,a-b)],$$

assuming that $\frac{1}{2}b \geq a \geq 2b$. If $b < 2a$, then $A_1(P)$ is a virtual segment. If $2b > a$, then $A_2(P)$ is virtual.

If $P = [(a,b),(a',b)]$ is a horizontal segment, then $A_2(P)$ is the trapezoid (or a skew trapezoid) with the vertices (a,b), (a',b), $(a,a-b)$, $(a',a'-b)$.

(2) A more interesting example is $\mathbb{R}^3 = \mathbb{R}^2\oplus\mathbb{R} = \{(x,y,z)\}$ with the functions $l_1 = z$ and $l_2 = x+y$. If $P = [(a,b,c),(a',b,c)]$ is a segment in $\mathbb{R}^2$, then $A_1(P)$ is the rectangle with the vertices (a,b,c), (a',b,c), $(a,c-a-a'-b,c)$, $(a',c-a-a'-b,c)$. Using this calculation and those in (1), it is easy to show that if $P = (b,c,c)$ is a point and $-b-c > b > c$, then $A_1A_2A_1(P)$ is the 3-dimensional Gelfand–Zetlin polytope given by the inequalities $a \geq x \geq b$, $b \geq y \geq c$ and $x \geq z \geq y$, where $a+b+c = 0$.

(3) Generalizing the last example we now construct Gelfand–Zetlin polytopes for arbitrary n via divided difference operators. For $n \in \mathbb{N}$, put $d = \frac{n(n-1)}{2}$. Consider the root space $\mathbb{R}^d = \mathbb{R}^{n-1}\oplus\mathbb{R}^{n-2}\oplus\ldots\oplus\mathbb{R}^1$ of rank $(n-1)$ with the functions l_i given by the formula: $l_i(x) = \sigma_{i-1}(x)+\sigma_{i+1}(x)$. Here $\sigma_i(x)$ denotes the sum of those coordinates of $x \in \mathbb{R}^d$ that correspond to the subspace $\mathbb{R}^{d_i}$ (put $\sigma_0 = \sigma_n = 0$).

For every strictly dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ (that is, $\lambda_1 > \ldots > \lambda_n$) of GL_n such that $\lambda_1 + \ldots + \lambda_n = 0$, the Gelfand–Zetlin polytope Q_λ coincides with the polytope

$$[(A_1 \ldots A_{n-1})(A_1 \ldots A_{n-2}) \ldots (A_1)]\,(p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_2, \ldots, \lambda_n; \lambda_3, \ldots, \lambda_n; \ldots; \lambda_n)$.

Similarly, divided difference operators for suitable root spaces allow one to construct the classical Gelfand–Zetlin polytopes for symplectic and orthogonal groups. They also yield an elementary description of more general *string polytopes* defined in [5] and might help to extend the results of [4] to arbitrary semisimple groups.

As outlined below, these convex geometric operators are well suited for inductive constructions of Newton–Okounkov polytopes for line bundles on Bott towers and on Bott-Samelson varieties (for natural choice of a geometric valuation). The former polytopes were described in [2] and the latter are currently being computed by Dave Anderson.

Bott towers. Consider a root space with $d = n$, that is, $d_1 = \ldots = d_n = 1$. We have the decomposition

$$\mathbb{R}^n = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{n}; \quad y = (y_1, \ldots, y_n)$$

into coordinate lines. Assume that the linear function l_i for $i < n$ does not depend on $y_1, \ldots, y_i$, and $l_n = y_1$. I can show that the polytope $P := A_1 \ldots A_n(p)$ (for a point $p \in \mathbb{R}^n$) coincides with the Newton–Okounkov body for a *Bott tower* (that depends on $l_1, \ldots, l_n$) together with a line bundle (that depends on p). For $n = 2$, a Bott tower is a Hirzebruch surface and P is a trapezoid (or a skew trapezoid) constructed similarly to the one in example (1). In general, a Bott tower is a toric variety obtained by successive projectivizations of rank two split vector bundles, and P is a multidimensional version of a trapezoid.

Bott–Samelson resolutions. Let $X = R(i_1, \ldots, i_l)$ be the *Bott-Samelson variety* corresponding to any sequence $(\alpha_{i_1}, \ldots, \alpha_{i_l})$ of roots of the group GL_n. It can be obtained by successive projectivizations of rank two (usually non-split) vector bundles. Consider the root space $\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2} \oplus \ldots \oplus \mathbb{R}^{d_{n-1}}$ with the functions l_i given by the formula $l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x)$, where d_i is the number of times the root α_i occurs in the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_l})$. Denote by T_v the parallel translation in the root space by a vector $v \in \mathbb{R}^d$. Consider the polytope

$$P = \left[A_{i_1} T_{v_1} A_{i_2} \ldots T_{v_{l-1}} A_{i_l}\right](p).$$

In his talk, Dave Anderson described an algorithm for computing the Newton–Okounkov body of a line bundle on X with respect to the valuation given by the flag of subvarieties $\{\ldots \supset R(i_{l-1}, i_l) \supset R(i_l)\}$. Based on his computations for $l = 3$ [1], I conjecture that this Newton–Okounkov body coincides with P for suitable choice of a point $p \in \mathbb{R}^d$ and vectors $v_j \in \mathbb{R}^{d_{i_j}}$ for $j = 1, \ldots, l-1$.

References

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