REALIZABILITY OF HYPERGRAPHS AND INTRINSIC LINKING THEORY

A. SKOPENKOV

Abstract. In this expository paper we present short simple proofs of Conway-Gordon-Sachs' theorem on intrinsic linking in three-dimensional space, as well as van Kampen-Flores' and Ummel's theorems on intrinsic intersections. The latter are related to nonrealizability of certain hypergraphs in four-dimensional space. The proofs use a reduction to lower dimensions which allows to exhibit relation between these results. We use elementary language which allows to present the main ideas without technicalities. Thus our exposition is accessible to non-specialists in the area, including students who know basic three-dimensional geometry, and who are ready to learn straightforward four-dimensional generalizations.

Contents

Homepage: https://users.mccme.ru/skopenko/.

This paper is based on the author's lectures at Moscow Institute of Physics and Technology, Independent University of Moscow, Institute of Science and Technology (Austria), and at various summer schools and math circles. I am grateful to M. Skopenkov and A. Zimin for allowing me to use their materials, and to G. Chelnokov, I. Izmestiev, R. Karasev, A. Matushkin, A. Rukhovich, A. Shapovalov, M. Skopenkov, A. Sossinsky, S. Tabachnikov, O. Viro, A. Zimin, J. Zung for useful discussions.

'It's too difficult.'
'Write simply.'
'That's hardest of all.'
I. Murdoch, The Message to the Planet.

1. Introduction

1.1. **Impossible constructions, intrinsic intersection and intrinsic linking.** 'Impossible constructions' like the impossible cube, the Penrose triangle, the blivet etc. (see Figure 1 and [Io]) are well-known, mainly due to pictures by Maurits Cornelis Escher, see also [Br68, CKS+, GSS+]. The pictures do not allow the global spatial interpretation because of collision between local spatial interpretations to each other. In geometry, topology and graph theory there are also famous basic examples of 'impossible constructions' (of which local parts are 'possible').

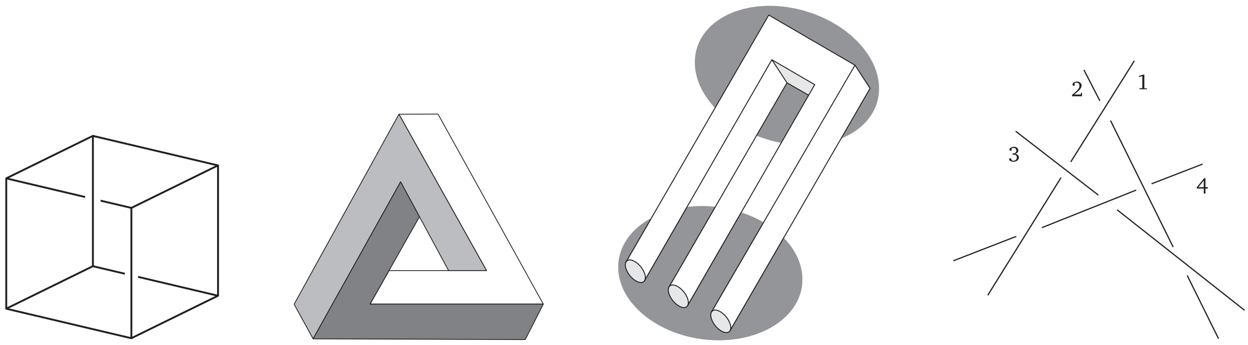

Figure 1. The impossible cube, the Penrose triangle, the blivet, an impossible projection

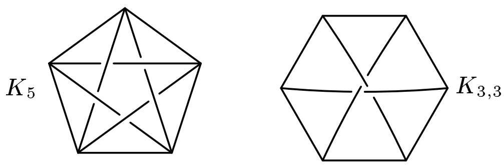

Figure 2. Nonplanar graphs K_5 and $K_{3,3}$

The following example of an 'impossible construction' or an 'intrinsic intersection' is already directly relevant to this paper: *For 5 points in the plane one cannot join each point to each other by a path so that the paths intersect only at their starting points or endpoints.*[1]

Proposition 1.1. *From any* 5 *points in the plane one can choose two disjoint pairs such that the segment joining the first pair intersects the segment joining the second pair.*

For next results we need some notation. We abbreviate 'three-dimensional Euclidean space $\mathbb{R}^3$' to '3-space'. Analogous meaning have '4-space' ($\mathbb{R}^4$) and d-space ($\mathbb{R}^d$). By a *triangle* we mean the part of the plane bounded by the outline of a triangle.

[1]Also, one cannot take 3 houses and 3 wells in the plane and join each house to each well by a path so that the paths intersect only at their starting points or endpoints. In graph-theoretic terms this means that the complete graph K_5 on 5 vertices, and the complete bipartite graph $K_{3,3}$ are not planar, see Figure 2. Proposition 1.1 below is a 'linear' version of the non-planarity of K_5.

Take two triangles in 3-space no 4 of whose 6 vertices lie in the same plane. The triangles are called **linked**, if the outline of the first triangle intersects the second triangle exactly at one point. E.g. the triangles $A_1A_3A_5$ and $A_2A_4A_6$ in Figure 3 are linked.[2]

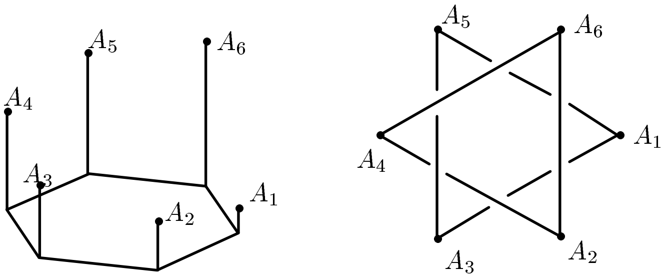

FIGURE 3. Linked triangles

Theorem 1.2 (Linear Conway–Gordon–Sachs Theorem; [Sa81, CG83])**.** *If no 4 of 6 points in 3-space lie in the same plane, then there are two linked triangles with vertices at these 6 points.*

1.2. **Why this paper might be interesting?** We exhibit striking relations

• of 'intrinsic intersections' in the plane (Propositions 1.1 and 2.3) to an 'intrinsic linking' result in three-dimensional space (Theorems 1.2 and 2.10 below);

• of the latter results to 'intrinsic intersections' in four-dimensional space (Theorems 1.5 and 4.3 below generalizing Proposition 1.1).

Remark 1.3 (lowering of dimension)**.** Often it is convenient to reduce a planar result to a linear one (i.e., to a result in a line), and a spatial result to a planar result. Similarly, a tempting approach to a 4-dimensional result is an analogy to, or a reduction to, a spatial result. Some examples are given in Remark 1.4.

Proposition 1.1 on intrinsic intersection in the plane is reduced (in §2.1) to Proposition 2.1 on intrinsic linking in the line. Analogously, Theorem 1.2 on intrinsic linking in 3-space is reduced (in §2.4) to Proposition 1.1. Analogously, Theorem 1.5 below on intrinsic intersection in 4-space is reduced (in §2.5) to Theorem 1.2.

This relation between intrinsic linking and intrinsic intersection in consecutive dimensions generalizes to higher dimensions (Theorem 1.6; for simplicity we mention dimensions higher than 4 only in that theorem). Because of such 'lowering of dimension' the reader not familiar with 4-dimensional space need not be scared.

The results on intrinsic intersections give a natural generalization of non-planarity of graphs: examples of *two-dimensional* analogues of graphs non-realizable in 3- and 4-space. This is explained in Remark 3.1.

We give a simplified exposition accessible to non-specialists in the area. We present examples in terms of certain systems of points. So we do not use the notion of realizability of a hypergraph (we do mention this notion because it is an important *motivation*). For understanding most of the paper it suffices to know basic geometry of 3-space, and to know or learn straightforward 4-dimensional generalizations. We believe that the elementary description of simple applications of topological methods makes these methods more accessible. Comparison with other proofs is discussed in Remark 3.2.

[2]The distance from the point A_j to the projection plane equals j. So the projection in Figure 3, right, is realizable, as opposed to Figure 1, right.
The property of being linked is symmetric (this is not obvious from the definition but does have a simple proof [Sk, Symmetry Lemma 4.1.2]). Introductory results on linked triangles are presented in §2.4 and in [Sk, §4.1 'Linking of triangles in three-dimensional space'].

The striking relation between 'intrinsic intersection' and 'intrinsic linking' not only gives simple proofs of classical results. It also brings a reader to the frontline of research, notably to the solution of the generalized Menger conjecture (explained in §4.1). The exposed intrinsic linking results are the departure point of *intrinsic (Ramsey) linking theory*. See surveys [RA05, PS05, FMM+, Na20] and references therein; for higher-dimensional analogues see [SS92, Sk03, KS20]. The exposed intrinsic intersection results are generalized to *non-realizability of hypergraphs*. See surveys [Sk06, §4, §5], [MTW, §1], [Sk18, §3.2] and references therein; for recent results see [Pa15, Sk18o, AKM, Pa21, Me22]. For analogous problem on embedding dynamical systems see [LT14] and references therein.

The history is exposed in Remark 3.3.

Plan of the paper. The remarks are not formally used later and so could be omitted. The same is true for §1.4 and §1.5. Sections 2, 3 and 4 are independent of each other, so they could be read in any order. Forward references and references to other papers can be ignored for the first reading.

1.3. Intrinsic intersection in 4-space.

Remark 1.4 (some intuition on 4-space)**.** (a) 'Typical' intersection of two segments in the plane is either empty set or a point. Here 'typical' means that no 3 points among the vertices of segments lie in the same line. Analogously, 'typical' intersection

• of a segment and a triangle in 3-space is either empty set or a point;

• of two triangles in 4-space is either empty set or a point.

(b) For each two points distinct from a point A in the plane there exists a polygonal line joining these points and not passing through A. Analogously,

• for each two points not belonging to a line l in 3-space there exists a polygonal line joining these points and disjoint with l;

• for each two points not belonging to a 2-dimensional plane α in 4-space there exists a polygonal line joining these points and disjoint with α.

(c) More intuition on 4-space is not required here, but can be developed by studying e.g. [KRR+, §1].

Theorem 1.5 (Linear Van Kampen-Flores Theorem; [vK32, Fl34])**.** *From any 7 points in 4-space one can choose two disjoint triples such that the two triangles with vertices at the triples intersect.*

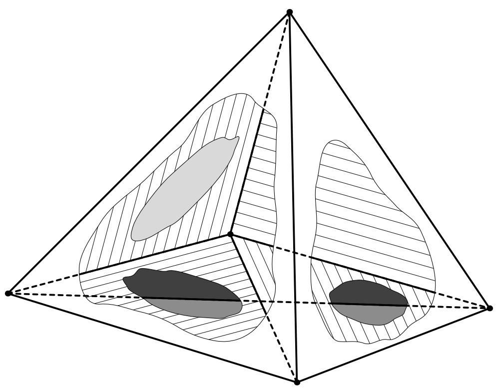

FIGURE 4. Five points in $\mathbb{R}^3$ (realization of the complete 3-homogeneous hypergraph on 5 vertices)

Analogues of Theorem 1.5 are true for 5 points in the plane, and for 6 points in 3-space (Propositions 1.1 and 2.4.b). Analogues of Proposition 1.1 and of Theorem 1.5 are false for

4 points in the plane and for 6 points in 4-space, respectively: in $\mathbb{R}^{2k}$ take the $2k+1$ vertices and an interior point of a $2k$-simplex, cf. Figure 4.

1.4. **Intrinsic intersection and linking in higher dimensions.** A subset of $\mathbb{R}^d$ is called *convex*, if for any two points from this subset the segment joining these two points is in this subset. The *convex hull* of $X \subset \mathbb{R}^d$ is the minimal convex set that contains X.

Theorem 1.6. *Take any $d+3$ points in $\mathbb{R}^d$ of which no $d+1$ points lie in the same $(d-1)$-dimensional hyperplane.*

For d even there are two disjoint $(d+2)/2$-element subsets whose convex hulls intersect.

If d is odd, then there is an unordered pair of $(d+1)/2$-simplices with vertices at these points which is linked (i.e., the boundary of the first simplex intersects the convex hull of the second simplex exactly at one point).

This is Proposition 1.1 for $d = 2$, Theorem 1.2 for $d = 3$, and Theorem 1.5 for $d = 4$. Theorem 1.6 is due to van Kampen-Flores for even $d > 2$ [vK32, Fl34], and to Segal-Spież-Lovász-Schrijver-Taniyama for odd $d > 3$ [LS98, Corollary 1.1], [Ta00] (the index argument of [SS92, §1] has a simple generalization to Theorem 1.6; thus Theorem 1.6 for odd $d > 3$ is implicit in [SS92]).

Theorem 1.6 is proved by induction on d. The base is $d = 1$ and is trivial. The inductive step is proved in §2 for $d = 2, 3, 4$; the proof for the general case is analogous.

The analogue of Theorem 1.6 for $d + 2$ points

- does not make sense for d odd because $(d + 1)/2$-simplex has $(d + 3)/2$ vertices;
- is false for d even, analogously to Theorem 1.5.

For d odd there is Proposition 2.4.b on intrinsic intersection and its higher-dimensional analogue. They are weaker than the corresponding Theorems 1.2, 1.6 on intrinsic linking. More results are presented in [KRR+, §3], [Sk16, §4].

1.5. **Multiple intersection and linking.** Let us formulate the analogues of the above results for r-fold intrinsic intersections.

Theorem 1.7 ([Sa91g])**.** *From any 11 points in 3-space one can choose 3 triangles having pairwise disjoint vertices but having a common point.*

It is surprising that the only known proof of such an elementary result involves algebraic topology. It would be interesting to obtain an elementary proof.

Example 1.8. *(a) In $\mathbb{R}^3$ take the vertices of a 3-dimensional simplex and its center, see Figure 4. Either take every of these 5 points with multiplicity two, or for every point take two close points. We obtain 10 points for which the analogue of Theorem 1.7 is false.*

(b) In $\mathbb{R}^{kr}$ take the vertices of a kr-dimensional simplex and its center. Either take every of these $kr + 2$ points with multiplicity $r - 1$, or for every point take close $r - 1$ points in general position. We obtain $(r-1)(kr+2)$ points in $\mathbb{R}^{kr}$ such that for any r pairwise disjoint $(k(r-1)+1)$-tuples all the r convex hulls of the tuples do not have a common point.

Conjecture 1.9 (Linear r-fold van Kampen–Flores conjecture)**.** *For every integers $k, r > 0$ from any $(r-1)(kr+2)+1$ points in $\mathbb{R}^{kr}$ one can choose r pairwise disjoint $(k(r-1)+1)$-tuples whose r convex hulls have a common point.*

This generalizes Theorems 1.5 (take $k = r = 2$) and 1.7 (take $r = 3 = 3k$). This was proved for a prime r by Sarkaria [Sa91g], for a prime power r by Volovikov [Vo96v, Corollary in §1], and is an open problem for other r [Fr17, beginning of §2]. The 'topological version' of Conjecture 1.9 played in important role in resolution of *the topological Tverberg conjecture*, see surveys [Sk16], [Sk18, §3.3] and references therein.

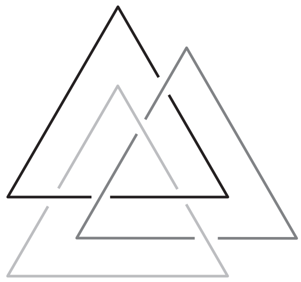

FIGURE 5. Borromean triangles

Let us formulate the analogues of the above results for intrinsic triple linking.

There are three triangles in 3-space which are pairwise unlinked but linked together (Figure 5), i.e., each two are isotopic to triangles which are contained in disjoint cubes, but all the three of them are not. (One can check that the projection in Figure 5 is realizable, as opposed to Figure 1, right.) They are called *Borromean triangles*, cf. [Val].

Theorem 1.10 (Negami [Ne91])**.** *There is N such that if no 4 of N points in 3-space lie in the same plane, then there are three Borromean triangles with the vertices at these points.*

See also [PS05, FNP]. It would be interesting to obtain an analogue of Theorem 1.10 with specific N. By Example 1.8 one cannot take $N = 10$. Can one take $N = 11$ (as in Theorem 1.7)? One can make computer experiments to solve this problem using equivalent definitions of Borromean triangles [Ko19]. It would be interesting to obtain a higher-dimensional higher-multiplicity analogues of Theorem 1.10, cf. [BL, FFN+].

2. Proofs

By *k points in* $\mathbb{R}^d$ (in this paper mostly $d \le 4$) we mean a k-element subset of $\mathbb{R}^d$; so these k points are assumed to be pairwise distinct.

2.1. Intersections in the plane: proof of Proposition 1.1.

Proposition 1.1 is easily proved by analyzing the convex hull of the 5 points. In order to illustrate the 'lowering of dimension' idea (see Remark 1.3) in the simplest situation, we deduce Proposition 1.1 from the following obvious 1-dimensional result.

Proposition 2.1. *Every 4 points in a line can be colored in two red and two blue so that they alternate: red-blue-red-blue or blue-red-blue-red (one says 'the red pair is linked with the blue pair').*

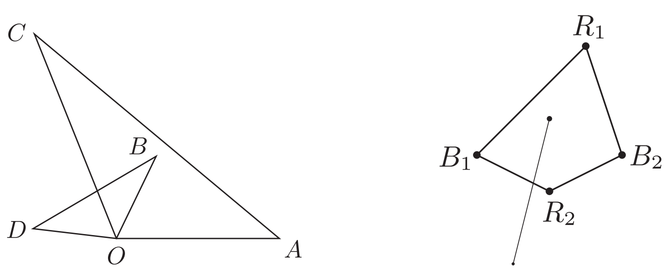

FIGURE 6. Left: To the proof of Proposition 1.1. Right: To Proposition 2.3.b.

Proof of Proposition 1.1. We may assume that there is a unique 'lowest' point O among given ones. Consider a 'horizontal' line slightly above the point O. Denote by A, B, C, D the remaining points.

If for some two points $X, Y \in \{A, B, C, D\}$ the point X belongs to the segment OY, then we are done. Otherwise we can assume that the points A, B, C, D are seen from O in this order, see Figure 6. Then the outlines of the triangles OAC and OBD have an intersection point different from O (this is Lemma 2.2 below). Hence some two sides of the triangles have disjoint vertices and intersect. $\square$

Lemma 2.2 (see Figure 6, left). *Assume that two triangles Δ, Δ' in the plane have a common vertex O, and no 3 of their vertices lie in the same line. Then the outlines $\partial\Delta, \partial\Delta'$ of the triangles intersect at an even number of points if and only if the intersection $\partial\Delta \cap \Delta'$ contains precisely one segment with vertex O.*

This lemma is trivial. It is explicitly stated in order to conveniently use it (here and in §2.5), and to illustrate its generalization to higher-dimensions (Lemma 4.8).

The following propositions are proved analogously to Proposition 1.1. They are used for some 3-dimensional results (Proposition 4.1 and Theorems 2.10, 4.2).

Proposition 2.3. *(a) (See Figure 2, right) Two triples of points are given in the plane. Then there exist two intersecting segments without common vertices and such that each segment joins the points from distinct triples.*

(b) (See Figure 6, right) Suppose that there are 4 red and 2 blue points B_1, B_2 in the plane. Suppose further that any two segments joining points of different colors either are disjoint or intersect at their common vertex. Then there are 2 red points R_1, R_2 such that the quadrilateral $R_1B_1R_2B_2$ does not have self-intersections and the remaining 2 red points lie on different sides of the quadrilateral. (I.e., a general position polygonal line joining the remaining 2 red points intersects the outline of the quadrilateral at an odd number of points.)

2.2. **Weaker versions of Theorem 1.2.** First we illustrate the 'lowering of the dimension' idea (see Remark 1.3) by proving the following weaker versions of Theorem 1.2.

Proposition 2.4. *(a) From any 6 points in 3-space one can choose 5 points O, A, B, C, D such that the triangles OAB and OCD have a common point other than O.*

(b) From any 6 points in 3-space one can choose disjoint pair and triple such that the segment joining points of the pair intersects the triangle spanned by the triple.

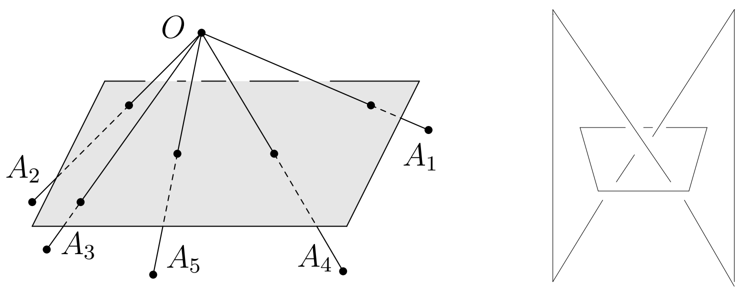

FIGURE 7. Left: To the proofs of Proposition 2.4.a and Theorem 2.8. A plane in $\mathbb{R}^3$ intersects the segments $OA_1, \ldots, OA_5$ by points $A_1', \ldots, A_5'$.
Right: Whitehead link formed by space quadrilaterals

Proof of (a). We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane slightly below the point O. Consider the intersection of this plane with the union of triangles OAB for all pairs A, B of given points. Now the assertion follows by Proposition 1.1. □

Part (b) follows from (a). Part (b) is an improvement of (a) and is a spatial analogue of Proposition 1.1.

Figure 4 shows that the analogue of (a) for 5 points is false.

2.3. **'Quantitative' versions.** The deduction of Theorem 1.2 from Proposition 1.1 has a technical detail (footnote 3) which is hard to generalize to higher dimensions. So instead we prove the following stronger 'quantitative' (algebraic modulo 2) version of these results.

Proposition 2.5 (obvious). *For any 4 points in the line there is a unique linked unordered splitting of the 4 points into two pairs.*

Proposition 2.6 (cf. Proposition 1.1). *If no 3 of 5 points in the plane lie in the same line, then the number of intersection points of interiors of segments joining the 5 points is odd.*

This is easily proved by analyzing the convex hull of the 5 points. Alternatively, this is reduced to Proposition 2.5 by a simple additional counting analogous to the proof of Theorem 2.8 in §2.4.

Remark 2.7. Proposition 2.6 is indeed stronger than Proposition 1.1 because it suffices to prove Proposition 1.1 under assumption that no 3 of the 5 points lie in the same line,

(a) first, since otherwise Proposition 1.1 is obvious: if points A, B, C among given 5 points lie in the same line, B between A and C, and D is any other given point, then segments AC and BD intersect.

(b) second, since we can make a small shift so that no 3 of 5 shifted points lie in the same line, and no intersection points of segments with disjoint vertices are added.

Analogously to (b), Theorems 2.8 and 2.9 below are stronger than Theorems 1.2 and 1.5.

Theorem 2.8 (cf. Theorem 1.2; [Sa81, CG83]). *If no 4 of 6 points in 3-space lie in the same plane, then the number of linked unordered pairs of triangles with vertices at these 6 points is odd.*

Theorem 2.9 (cf. Theorem 1.5; [vK32, Fl34]). *If no 5 of 7 points in 4-space lie in the same 3-dimensional hyperplane, then the number of intersection points of triangles with vertices at these points is odd.*

Theorem 1.6 has an analogous quantitative version. For counterexamples to 'integer versions' see [KS20, Proposition 1.2 and Theorem 1.4].

2.4. **Linking in 3-space: proof of Theorem 2.8.** We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane α slightly below the point O. Denote by $A'_1, \dots, A'_5$ the intersection points of α and segments joining O to other given points. See Figure 7. Clearly, no 3 of the obtained 5 points lie in the same line.

In 3-space *a segment p is below a segment q* looking from point O, if there exists a half-line OX with the endpoint O that intersects the segment p at a point $P := p \cap OX$, the segment q at a point $Q := q \cap OX$, $P \neq Q$, so that Q is in the segment OP. So in the plane α we obtain a picture analogous to Figure 3, right. Since no 4 of the given 6 points $O, A_1, \dots, A_5$ lie in the same plane, the number of those sides of the triangle $A_3A_4A_5$ that are higher than A_1A_2 equals to the number of intersection points of the outline of the triangle $A_3A_4A_5$ with the triangle OA_1A_2. Also, a segment cannot intersect a triangle by more than 2 points. Hence *the triangles OA_1A_2 and $A_3A_4A_5$ are linked if and only if A_1A_2 is below an odd number of sides of the triangle $A_3A_4A_5$.*

Then the following numbers have the same parity:

• the number of linked unordered pairs of triangles formed by given 6 points;
• the number of segments A_iA_j that are below an odd number of sides of their 'complementary' triangles $A_kA_lA_m$, $\{i,j,k,l,m\} = \{1,2,3,4,5\}$;
• the number of ordered pairs (A_iA_j, A_kA_l) of segments of which the first is below the second;
• the number of intersection points of interiors of segments whose vertices are $A'_1, \dots, A'_5$.

By Proposition 2.6 the latter number is odd.[3] □

The following version of Theorem 1.2 is analogously reduced to Proposition 2.3.b [Zi13]. This version is used for some 4-dimensional result (Theorem 4.3) in §4.6.

In 3-space take two quadrilaterals (i.e., closed quadrangular polygonal lines) $ABCD$ and $A'B'C'D'$, no 4 whose 8 vertices lie in the same plane. The quadrilaterals are called *linked modulo 2* if the number of intersection points of the quadrilateral $ABCD$ with the union of the triangles $A'B'C'$ and $A'D'C'$ is odd. (As opposed to triangles, there are space quadrilaterals *linked* but not linked modulo 2, see Figure 1.2, right. This notion of linking is illustrated by Proposition 4.7.)

Theorem 2.10 (Linear Sachs Theorem; [Sa81])**.** *Suppose that there are 8 points in 3-space, of which no 4 lie in the same plane, 4 points are red and 4 are blue. Then there are two linked space quadrilaterals consisting of segments joining points of different colors.*

2.5. **Intersection in 4-space: proof of Theorem 1.5.** We may assume that no 5 of the given 7 points $O, A_1, \dots, A_6$ lie in the same 3-dimensional hyperplane (analogously to Remark 2.7.b). We may also assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' 3-dimensional hyperplane α slightly below the point O. Take the 6 intersection points $A'_1, \dots, A'_6$ of α with the segments $OA_1, \dots, OA_6$. See Figure 8. Clearly, no 4 of the obtained 6 points lie in the same plane.

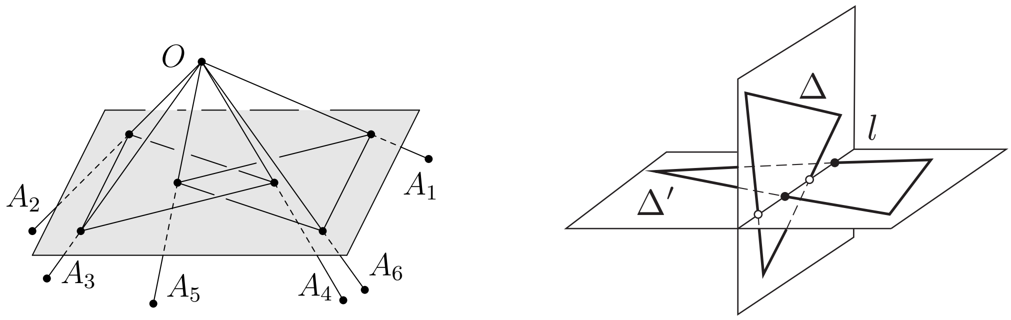

FIGURE 8. Left: Hyperplane α in $\mathbb{R}^4$ (shown as a plane in $\mathbb{R}^3$) intersects the segments $OA_1, \dots, OA_6$ at 6 points $A'_1, \dots, A'_6$ which are vertices of two linked triangles.
Right: Section by the 3-hyperplane α; the planes are the planes of the linked triangles $\Delta = \alpha \cap O\Delta_1$ and $\Delta' = \alpha \cap O\Delta_2$; $l = \gamma \cap \alpha$.

Hence by Theorem 1.2 there are two linked triangles with vertices at these 6 points. Denote by Δ_1 and Δ_2 those triangles formed by given 7 points for which the linked triangles are the

[3]For the proof of Theorem 1.2 the latter paragraph can be replaced by the following. By Proposition 2.4.b some triangle, say ABC, with vertices at given points intersects some segment, say, DE, with vertices at given points. Denote by F the remaining point. If triangles ABC and DEF are linked, then we are done. If not, $|ABC \cap \partial(DEF)| = 2$. W.l.o.g. DF intersects ABC and EF does not. Denote by E' and F' the intersection points of DE and DF with ABC. W.l.o.g. $E' \in ABF'$. Then the triangles ABF' and CDE are linked. But F' is not given point; the triangles ABF and CDE need not be linked.

intersections $\alpha\cap O\Delta_1$ and $\alpha\cap O\Delta_2$ of the hyperplane α with tetrahedra $O\Delta_1$ and $O\Delta_2$ (e.g. $\Delta_1 = A_2A_3A_4$ and $\Delta_2 = A_1A_5A_6$).

Denote by γ the plane containing O and the intersection line of the planes of the linked triangles. Then $\gamma\cap\alpha$ is this line and $\Delta_j^\gamma := \gamma\cap O\Delta_j$ is a triangle (for each $j = 1, 2$). The side of Δ_j^γ not containing O is $\gamma\cap\Delta_j$. The two sides of Δ_j^γ containing O form the intersection of γ and the lateral surface of the tetrahedron $O\Delta_j$ (whose base is Δ_j). We may assume that Figure 6 (left) represents the section by the plane γ: $\Delta_1^\gamma = OAC$ and $\Delta_2^\gamma = OBD$.

Since the triangles $\alpha\cap O\Delta_1$ and $\alpha\cap O\Delta_2$ are linked, the intersection points of the line $\gamma\cap\alpha$ and the outlines of Δ_1^γ and Δ_2^γ alternate along the line. Hence the outlines have a common point distinct from O (this is Lemma 2.2). This point is the intersection of either

- the sides $\gamma\cap\Delta_1$ and $\gamma\cap\Delta_2$ or, without loss of generality,
- the side $\gamma\cap\Delta_1$ and the union of the two sides of Δ_2^γ containing O.

In the first case Δ_1 intersects Δ_2. In the second case Δ_1 intersects the lateral surface of the tetrahedron $O\Delta_2$; this lateral surface consists of triangles having disjoint vertices with the vertices of Δ_1. □

Theorem 2.9 is reduced to Theorem 2.8 by a simple additional counting (analogous to the proof of Theorem 2.8 in §2.4).

The following higher-dimensional version of Proposition 2.3.a is related (analogously to Theorem 1.5) to some 3-dimensional linking result [DS22, Remark 2.5].

Theorem 2.11 ([Fl34])**.** *Three triples of points in 4-space are given. Then there exist two intersecting triangles without common vertices such that the vertices of each triangle belong to distinct triples.*

3. Some important remarks

Remark 3.1 (Relation to hypergraphs)**.** (a) Two-dimensional analogues of graphs are *3-homogeneous*, or *2-dimensional hypergraphs* defined as collections of 3-element subsets of a finite set.[4] For brevity, we omit '3-homogeneous' and '2-dimensional'. For instance, *a complete hypergraph* on k vertices is the collection of all 3-element subsets of a k-element set. *Realizability* (also called embeddability) of a hypergraph in $\mathbb{R}^d$ is defined similarly to the realizability of a graph in the plane: one 'draws' a triangle for every three-element subset (see Figures 10 and 9, where a subdivision of quadrilaterals analogous to Figure 9, left, is not shown). See rigorous definitions e.g. in [Sk18, §3.2].

Hypergraphs (and simplicial complexes) play an important role in mathematics. One cannot imagine topology and combinatorics without them. They are also used in computer science and in bioinformatics, see e.g. [PS11].

A 'small shift' (or 'general position') argument shows that every graph is realizable in 3-space. A straightforward generalization shows that every hypergraph is realizable in 5-space.

The complete hypergraph on 6 vertices contains 'the cone over K_5' and hence is non-realizable in 3-space (Proposition 2.4.a). Already in the early history of topology (1920s) mathematicians tried to construct hypergraphs non-realizable in 4-space. Egbert van Kampen and A. Flores in 1932-34 proved that the complete hypergraph on 7 vertices is not realizable in 4-space (Theorem 1.5). This is both an early application of *combinatorial topology* (nowadays called algebraic topology) and one of the first results of *topological combinatorics* (also an area of ongoing active research).

(b) Realizations (=embeddings) are maps without self-intersections. For topological combinatorics and discrete geometry it is interesting to study maps whose self-intersections are

[4]In topology such objects are called *pure, or dimensionally homogeneous*, 2-dimensional *simplicial complexes*. The term 'hypergraph' is more convenient to generic mathematician or computer scientist.

non-empty (like for embeddings), but 'not too complicated'. An important particular case is studying maps *without triple intersections* and, more generally, maps *without r-tuple intersections*, see §1.5 and surveys [Sk16], [Sk18, §3.3].

(c) We present *linear* versions of the results. *PL (piecewise-linear) and topological* realizations (=embeddings) of hypergraphs are defined and discussed e.g. in [Sk18, §3.2], [Sk, §5]. The proofs we expose are interesting because they easily generalize to the PL (and quantitative) versions [Sk03, Zi13], as opposed to the proofs of [BM15, So12]. The paper [BM15] presents short algebraic proof of Theorem 1.6 (and so of its particular cases). That proof is in the spirit of the algebraic proof of the *Radon theorem*[5]. That proof is presumably a direct (i.e., without use of the Gale transform) version of the proof of [So12, Theorem 5] for $k = 1$, which is Theorem 1.6 for d even.

PL versions of 'quantitative' results (see §2.3) imply the PL versions for *almost-embeddings* (see the PL case of [Sk18, Theorem 1.4.1 and 3.1.6]). The latter imply the *topological* versions (see explanation in [Sk18, the paragraph after Theorem 1.4.1]).

Remark 3.2 (Comparison with other proofs)**.** Theorem 1.6 for d even (and so its particular cases, Proposition 1.1 and Theorem 1.5) has an alternative simple proof using *the van Kampen number*, see e.g. [Sk18, §1.4], [Sk, §1.4, §5]. (That proof works for PL, quantitative, and topological versions, see Remark 3.1.c. For such a proof of a quantitative version of Theorem 2.11 see [DS22, Lemma 2.2].) That proof and the proof sketched in this paper, are presumably the simplest known proofs ('proofs from the Book').

Usually (the topological version of) Theorem 1.6 for d even (and Theorem 2.11) is proved using the Borsuk-Ulam theorem [Sk20, §8], [Ma03, §5]. As opposed to this paper (and to the alternative simple proof using the van Kampen number), this requires some knowledge of algebraic topology. This knowledge does not make things simpler: known proofs of the Borsuk-Ulam theorem (see [Ma03] and the references therein) are not easier than the above-discussed direct proofs of Theorem 1.6 for d even. (The Borsuk-Ulam theorem is proved using *the degree* analogously to the direct proof of Theorem 1.6 for d even using *the van Kampen number*.)

Remark 3.3 (history)**.** General 'lowering of dimension' or 'the link of a vertex' ideas are simple and well-known (see Remark 1.3). For proofs of the Radon theorem based on this idea see [Pe72, Ko18, RRS]. For a recent application in computer science see [DE94, proof of 2.3.i]. Also well-known is relation between linking and intersection.[6] An elaboration of this idea to a relation between intrinsic linking and non-realizability is non-trivial (cf. the difference between Proposition 2.4.a and Theorem 1.2). Proofs that discover and use that relation seem to have not been published

• before [RST, RST'], Alexander Shapovalov's 2003 solution of an olympic problem, [RSS+, Zi13], for a proof of the Conway–Gordon–Sachs Theorem 1.2 by reducing intrinsic linking to intrinsic intersection in lower dimension,

• before [Sk03, Example 2, Lemmas 2 and 1'], [RSS+], for proofs of Theorem 1.5 and of the Menger conjecture (see §4.1.a) by reducing intrinsic intersection to intrinsic linking in lower dimension.

4. Realizability of products and the Menger conjecture

[5]See e.g. [Sk16, §1] for the statement of the Radon theorem. See [Sk16, §4] for relations between the Radon, the van Kampen-Flores and the Conway–Gordon–Sachs theorems.

[6]E.g. the linking number of two disjoint closed polygonal lines in 3-dimensional sphere ∂D^4 equals to the intersection number of two *general position* 2-dimensional disks in 4-dimensional ball D^4 spanning the two polygonal lines. For an inductive proof involving assertion on linking in odd dimensions and assertion on intersection in even dimensions see [RS72, Whitney Lemma 5.12 and Theorem 5.16 for $p = q$].

4.1. **The Menger conjecture.** The *(Cartesian) product* $F \times F'$ of two figures F, F' in $\mathbb{R}^3$ is the set of all points $(x, y, z, x', y', z') \in \mathbb{R}^6$ such that $(x, y, z) \in F$ and $(x', y', z') \in F'$.

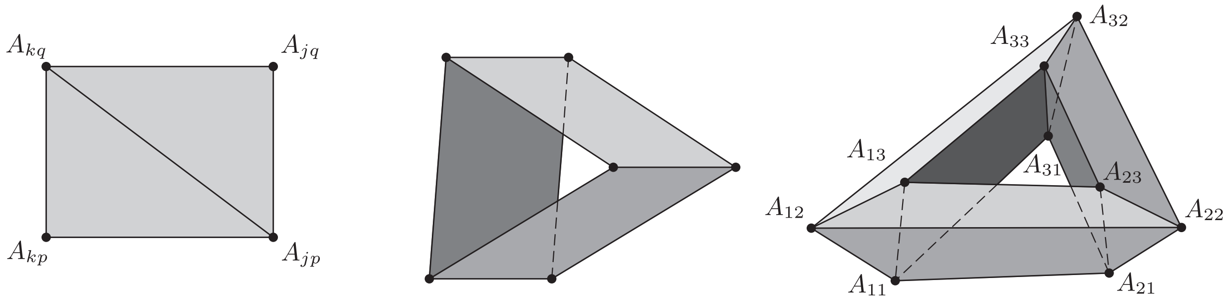

FIGURE 9. Realizations of the products: $K_2 \times K_2$ in $\mathbb{R}^2$ (left), $K_2 \times K_3$ (middle), $K_3 \times K_3$ (right)

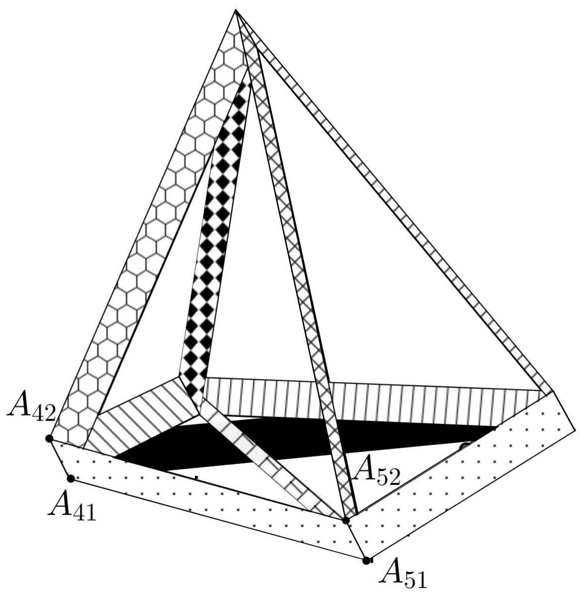

FIGURE 10. Realization in $\mathbb{R}^3$ of the product $K_5 \times K_2$

Examples of realization of products are given in Figures 9 and 10. For definition of realization see e.g. [Sk18, §3.2]. Karl Menger conjectured in 1929 that the square of a nonplanar graph is not realizable in $\mathbb{R}^4$ [Me29] (cf. Theorem 4.3). This was proved only in 1978 by Brian Ummel [Um78] using *spectral sequences*. A simple proof was obtained in 2003 by Mikhail Skopenkov [Sk03] using *lowering of dimension*, see exposition below. This argument proves the generalized Menger conjecture (the k-th power of a nonplanar graph is not realizable in $\mathbb{R}^{2k}$), and even gives a short formula for the minimal number d such that given product of several graphs is realizable in $\mathbb{R}^d$ [Sk03].

A combinatorial version of the product is *product* of two graphs (not necessarily planar). This product can be considered (although not canonically) as a hypergraph; see Figure 9, left. In Figure 9, middle and right, splitting of quadrilaterals into triangles is not shown.

Proposition 4.1 and Theorem 4.2 below imply that neither $K_4 \times K_4$ nor $K_5 \times K_3$ are linearly realizable in $\mathbb{R}^3$. Proof of Proposition 4.1 shows that even $K_{3,1} \times K_{3,1}$ is not linearly realizable in $\mathbb{R}^3$. Analogous improvements of Theorems 4.2 and 4.3 are false.

Remark. Proofs of the Menger conjecture using the van Kampen number or the Borsuk-Ulam theorem (see Remark 3.2) are unknown. The proof of the Menger conjecture in [Um78] works for the topological version but is complicated. The simpler proof in [Sk03] uses for the topological version the non-trivial *Bryant approximation theorem.* A simpler proof of the topological version can be obtained by inventing a quantitative PL version of the Menger conjecture (i.e., by improving the PL version of Theorem 4.3 analogously to §2.3):

Find a subset

$$M \subset \left\{\{(X,Y),(X',Y')\}\ :\ X,Y,X',Y' \in \binom{[5]}{2},\ \textit{either } X\cap X'=\emptyset \textit{ or } Y\cap Y'=\emptyset\right\}$$

such that

(linear version) for any 25 general position points $A_{jp}\in\mathbb{R}^4$, $j,p\in[5]$, there is an odd number of pairs $\{(X,Y),(X',Y')\}\in M$ for which $|(X\times Y)\cap(X'\times Y')|\equiv 1 \mod 2$, where $\times$ is defined at the beginning of §4.2.

(PL version) for any PL map $f: K_5\times K_5\to\mathbb{R}^4$, there is an odd number of pairs $\{(X,Y),(X',Y')\}\in M$ for which $|f(X\times Y)\cap f(X'\times Y')|\equiv 1 \mod 2$, where by X,Y,X',Y' we understand edges of K_5. [7]

This is related to the following algebraic Menger problem [Pa20, Conjecture 2]: *Complexes K,L have non-trivial van Kampen obstructions to embeddability in $\mathbb{R}^m$ and in $\mathbb{R}^n$, respectively (see definition e.g. in [Sk18, §1.5]). Does the cartesian product $K\times L$ of K and L have non-trivial van Kampen obstruction to embeddability in $\mathbb{R}^{m+n}$?*

4.2. **Realizability of products.** Suppose that we have mn points A_{jp}, where $j\in[m]:=\{1,2,\dots,m\}$ and $p\in[n]$, in 3- or 4-space. For two-element subsets $\{j,k\}\subset[m]$, $j<k$, and $\{p,q\}\subset[n]$, $p<q$, denote by $jk\times pq$ the collection, or the union, of two triangles $A_{jp}A_{kq}A_{jq}$ and $A_{jp}A_{kq}A_{kp}$ having a common side (see Figure 9, left). This union could be, but need not be, a plane quadrilateral. An (m,n)**-product** is a collection of triangles from

$$jk\times pq,\quad \text{where}\quad 1\le j<k\le m,\quad 1\le p<q\le n.$$

(There are $mn(m-1)(n-1)/2$ such triangles.) The union of triangles of (m,n)-product is a polyhedral and possibly self-intersecting

- square, if $m=n=2$ (Figure 9, left);
- lateral surface of a cylinder, if $m=3$ and $n=2$ (Figure 9, middle);
- torus, if $m=n=3$ (Figure 9, right).

A typical example is the Cartesian product of m points in the plane and n points in the line (or in the plane).

Proposition 4.1. *Any $(4,4)$-product in 3-space has two triangles which have disjoint vertices but intersect.*

Proposition 4.1 is reduced to Proposition 2.3.a in §4.4.

Theorem 4.2 (Product; [Sk03])**.** *Any $(5,3)$-product in 3-space has two triangles which have disjoint vertices but intersect.*

The Product Theorem 4.2 is reduced to Proposition 2.3.b in §4.4.

Theorem 4.3 (Square; [Um78, Sk03])**.** *Any $(5,5)$-product in 4-space has two triangles which have disjoint vertices but intersect.*

[7]For M the whole set the number of pairs has apparently the same parity as the number of linked pairs of cycles in $K_{4,4}$ embedded in $\mathbb{R}^3$, which is even. So either M should be different, or one should use integers (or residues modulo 4) instead of residues modulo 4.

The Square Theorem 4.3 is reduced to Theorem 2.10 in §4.6.

Example 4.4. *The analogues of Theorems 4.2 and 4.3 are false for*

(a) $(2,n)$*-products in 3-space for every* n *(for* $n\le 4$ *this is obvious; for* $n=5$ *see Figure 10: the vertices of the parallelograms are the required* 10 *points; for* $n\ge 6$ *the construction is analogous, see §4.3; cf.* [RSS', Theorem 1.5]*);*

(b) $(3,n)$*-products in 3-space for every* $n\le 4$ *(for* $n\le 3$ *this is obvious, see Figure 9, right; for* $n=4$ *the construction is analogous, see §4.3);*

(c) $(4,n)$*-products in 4-space for every* n *(see §4.3).*

4.3. **Realization of products in 3- and 4-space.**

Proof of Example 4.4.a. Let $(0,0,0),V,A_{11},\dots A_{1n}$ be points in $\mathbb{R}^3$ of which no 4 lie in the same plane. For every $p\in[n]$ denote $A_{2p}:=V+A_{1p}$. If V is close enough to $(0,0,0)$, then the points A_{jp}, $j\in\{1,2\}$, $p\in[n]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect.

Indeed, $12\times pq$ is a parallelogram for every $p\ne q$. Since no 4 of the points $(0,0,0)$, V, $A_{11},\dots A_{1n}$ lie in the same plane, for any distinct p,q,r,s the segments $A_{1p}A_{1q}$ and $A_{1r}A_{1s}$ are disjoint. Since V is close enough to $(0,0,0)$, the same holds for 1 replaced by 2. Then any two (convex hulls of) parallelograms $12\times pq$ and $12\times rs$ that have no common side are disjoint. Therefore the points A_{jp} are as required. $\square$

Proof of Example 4.4.b. Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be the rotation through $\frac{2\pi}{3}$ w.r.t. x-axis. Let

$$(A_{11},A_{12},A_{13},A_{14})=((1,0,1),(-1,0,1),(0,0,2),(0,0,3)).$$

Let $A_{2p}=f(A_{1p})$ and $A_{3p}=f(f(A_{1p}))$ for every $p\in[4]$. Cf. Figure 9, right. Then the points A_{jp}, $j\in[3]$, $p\in[4]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect.

Indeed, $jk\times pq$ is a parallelogram for every $j\ne k, p\ne q$. Since every two segments joining points A_{1p} either are disjoint or intersect at a common vertex, any two of such parallelograms that have no common side are disjoint. Therefore the points A_{jp} are as required. $\square$

Sketch of the proof of a weaker version of Example 4.4.c: $(3,5)$*-product in 4-space.* Take a 3-dimensional hyperplane in $\mathbb{R}^4$ (shown in Figure 11, left, as a plane in 3-space). In this hyperplane take 10 vertices A_{jp}, where $j\in[5]$, $p\in\{1,2\}$, shown in Figure 10. Take a vector v not parallel to the hyperplane. Set $A_{j3}:=A_{j1}+v$. (In Figure 11, left, we see the lateral surface of the prismoid $A_{41}A_{42}A_{43}A_{51}A_{52}A_{53}$.) Then the points A_{jp}, $j\in[5]$, $p\in[3]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect. $\square$

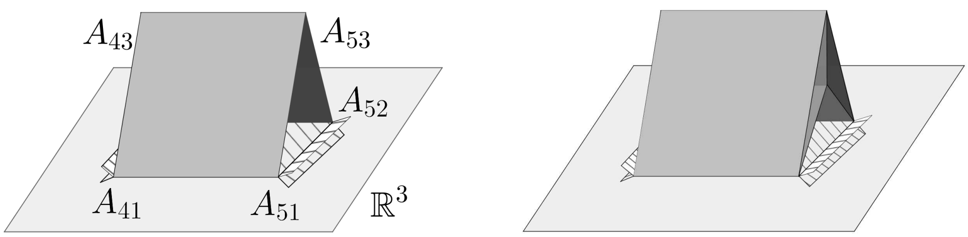

FIGURE 11. Left: to realization in $\mathbb{R}^4$ of the product $K_5\times K_3$.
Right: to realization in $\mathbb{R}^4$ of the product $K_5\times K_4$.

Sketch of the proof of Example 4.4.c. See Figure 11, right. Take points $A_{jp} \in \mathbb{R}^3 \subset \mathbb{R}^4$, $j \in \{1,2\}$, $p \in [n]$ from the proof of Example 4.4.a. Then $\overrightarrow{A_{1p}A_{1q}} = \overrightarrow{A_{2p}A_{2q}}$ for every $p \neq q$. Take vectors $v_3, v_4 \in \mathbb{R}^4$ not parallel to the hyperplane $\mathbb{R}^3 \subset \mathbb{R}^4$. Denote $A_{jp} := A_{1p} + v_j$, $j \in \{3,4\}$. We can take v_3, v_4 so that A_{14} is an interior point of the triangle $A_{11}A_{12}A_{13}$. Then the points A_{jp}, $j \in [4]$, $p \in [n]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect. □

4.4. **Non-realizability of products in 3-space.**

Proof of Proposition 4.1. (The proof is analogous to Proposition 2.4.) Take a small tetrahedron containing A_{11} in its interior. For every $j = 2,3,4$ color in red the intersection point of the surface S of the tetrahedron with the segment $A_{11}A_{j1}$, see Figure 12, left. For every $k = 2,3,4$ color in blue the intersection point of S with the segment $A_{11}A_{1k}$. (The intersection of S with the union of the triangles of the $(4,4)$-product is the image of a *piecewise linear map* of the graph $K_{3,3}$ to S.) Then by an analogue of Proposition 2.3.a (cf. [Sk18, Remark 1.5.1.d]) there are $2 \le j < k \le 4$ and $2 \le p < q \le 4$ such that the triangles $A_{11}A_{1p}A_{j1}$ and $A_{11}A_{1q}A_{k1}$ have a common point other than A_{11}. Hence without loss of generality the segment $A_{1p}A_{j1}$ intersects the triangle $A_{11}A_{1q}A_{k1}$. So the triangles $A_{jp}A_{1p}A_{j1}$ and $A_{11}A_{1q}A_{k1}$ have disjoint vertices but intersect. [8] □

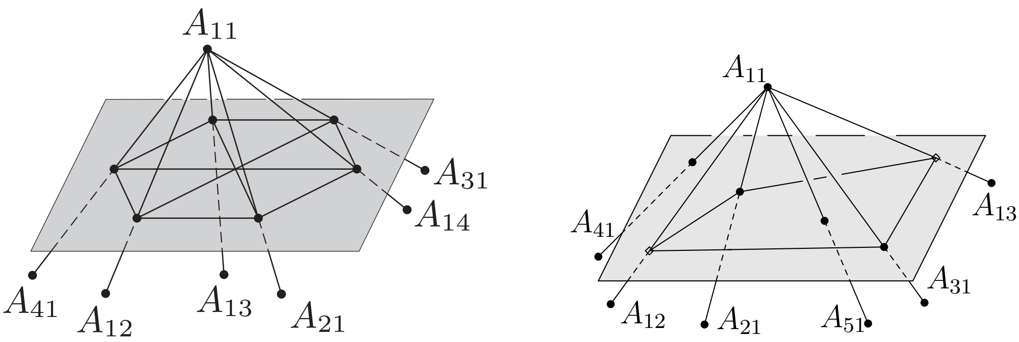

FIGURE 12. To the proofs of Proposition 4.1 (left) and the Product Theorem 4.2 (right)

Given 9 points A_{jk}, $j,k \in \{u,v,w\}$, in 3- or 4-space denote by T_{uvw} the *body* of the corresponding $(3,3)$-product, i.e., the union of products $jk \times pq$ (defined at the beginning of §4.2) taken for every two-element subsets $\{j,k\}, \{p,q\} \subset \{u,v,w\}$. See Figure 9, right. (As opposed to the figure, T_{uvw} can have self-intersections.) We abbreviate 'the body of a $(3,3)$-product' to 'a $(3,3)$-product'.

Proof of the Product Theorem 4.2. Take a small tetrahedron containing A_{11} in its interior. For every $j = 2,3,4,5$ color in red the intersection point of the surface S of the tetrahedron with the segment $A_{11}A_{j1}$, see Figure 12, right. For every $k = 2,3$ color in blue the intersection point of S with the segment $A_{11}A_{1p}$. (The intersection of S with the union of the triangles of the $(5,3)$-product is the image of a piecewise linear map of the graph $K_{4,2}$ to S.)

Denote the blue points by B_1, B_2. The intersection of a triangle $A_{11}A_{j1}A_{1p}$ with S is called an *arc*. Analogously to the last sentences from the proof of Proposition 4.1 either some two triangles of the $(5,3)$-product have disjoint vertices and intersect, or any two arcs joining points of different colors can only intersect at their common vertex.

[8]It is here that we use a specific triangulation of $K_4 \times K_4$. Thus the point A_{11} is not interchangeable with other A_{jp}. So we have to consider a tetrahedron instead of a (hyper)plane as in Theorems 1.2, 1.5 and 4.3. Analogous remark applies for the proof of the Product Theorem 4.2 below.

In the second case by an analogue of Proposition 2.3.b there are 2 red points R_1, R_2 such that the polygonal line $R_1B_1R_2B_2$ formed by arcs does not have self-intersections and the remaining two red points R_3, R_4 lie in S on different sides of the polygonal line. Without loss of generality, R_1, B_1, R_2, B_2 belong to the segments joining A_{11} to A_{21}, A_{12}, A_{31}, A_{13}, respectively, and R_3, R_4 belong to the segments joining A_{11} to A_{41}, A_{51}, respectively. Then the points R_3 and R_4 are intersection points of S and the outline of the triangle $A_{11}A_{41}A_{51}$. The intersection of $S \cap A_{11}A_{41}A_{51}$ is a polygonal line joining R_3 and R_4. The polygonal line $R_1B_1R_2B_2$ is the intersection of S and the $(3,3)$-subproduct T_{123}. Since R_3, R_4 lie in S on different sides of the polygonal line, $(S \cap A_{11}A_{41}A_{51}) \cap (S \cap T_{123}) \neq \emptyset$. Thus $A_{11}A_{41}A_{51} \cap T_{123} \neq \emptyset$. Hence one of the two triangles $A_{11}A_{41}A_{51}$, $A_{45}A_{41}A_{51}$ and some triangle from T_{123} have disjoint vertices but intersect. $\square$

4.5. **Parity Lemmas.** For the proof of the Square Theorem 4.3 we need Lemma 4.8 whose simpler analogues were already used above (see Lemma 2.2 and an argument on a triangle and a $(3,3)$-product in 3-space from the proof of the Product Theorem 4.2). Proof of Lemma 4.8 allows to exhibit a basic idea of homology theory (i.e., of Poincaré Lemma on the homology of Euclidean space) in an elementary language accessible to non-specialists. See a similar alternative proof in [Zu] and more on parity lemmas in [Sk18, §1.3], [Sk, §4].

In order to illustrate the idea in the simplest situation, we start with a planar Parity Lemma 4.5, then proceed 3-dimensional Parity Lemma 4.6, then present Proposition 4.7 on linking in 3-space. All of them are required for Lemma 4.8.

Some points in the plane **are in general position**, if no three of them lie in the same line and no three segments joining them have a common interior point.

Lemma 4.5 (Parity; [Sk18, Parity Lemma 1.3.2])**.** *Any two closed polygonal lines in the plane whose vertices are in general position intersect at an even number of points.*

Now we generalize the following evident fact: *if no 4 of the vertices of a closed polygonal line and of a tetrahedron in 3-space lie in the same plane, then the polygonal line and the surface of the tetrahedron intersect at an even number of points.*

Some points in 3-space **are in general position**, if no 4 of them lie in the same plane, and for every pairwise disjoint pair, triple and triple among them the line passing through the pair, the plane passing through the first triple and the plane passing through the second triple, have no common points.

E.g. in general position are the 6 points in Figure 3 ('helix curve'), and the points with *Cartesian coordinates* $(t; t^2; t^3)$, where $t \in (0,1)$ ('moment curve').

A **2-cycle** is a collection of (different) triangles such that every segment is the side of an even number (possibly, zero) of triangles from the collection. *The vertices* of a 2-cycle are the vertices of its triangles. *The body* of a 2-cycle is the union of its triangles.

An example of a 2-cycle is the surface of a tetrahedron (possibly, degenerate). Also, (the body of) the $(3,3)$-product T_{uvw} defined in §4.4 is the body of a 2-cycle.

Lemma 4.6 (Parity)**.** *If the vertices of a closed polygonal line and a 2-cycle in 3-space are in general position, then the polygonal line intersects the body of the 2-cycle at an even number of points.*

Sketch of the proof. The lemma follows by its particular case when the polygonal line is a triangle (analogously to [Sk18, §1.3, proof of the Parity Lemma 1.3.2]). This particular case is reduced to (the case when one polygonal line is a triangle of) the Parity Lemma 4.5 by proving that the intersection of the 2-cycle and the plane containing the triangle is the union of closed polygonal lines. $\square$

Proposition 4.7. *Let $ABCD$ and $A'B'C'D'$ be two closed quadrangular polygonal lines in 3-space no 4 of whose 8 vertices lie in the same plane.*

(a) The polygonal lines are linked if and only if an odd number among the following pairs of triangles are linked pairs:

$$(ABC, A'B'C'),\quad (ABC, A'D'C'),\quad (ADC, A'B'C'),\quad (ADC, A'D'C').$$

(b) Assume that $\Delta_1, \dots, \Delta_k$ are triangles in 3-space such that $\Delta_1, \dots, \Delta_k, ABC, ADC$ is a 2-cycle and the union of their vertices is in general position. (Such a collection of triangles is called a coboundary of $ABCD$.) Assume that $\Delta'_1, \dots, \Delta'_{k'}$ is an analogous collection of triangles for $A'B'C'D'$. The polygonal lines are linked if and only if an odd number among the kk' pairs $(\Delta_j, \Delta'_{j'})$ of triangles are linked pairs.

Proof. Part (a) is a particular case of (b) for $k = k' = 2$, $\Delta_1 = ABC$, $\Delta_2 = ADC$, $\Delta'_1 = A'B'C'$, $\Delta'_2 = A'D'C'$.

Denote by $\partial\Delta$ the outline of a triangle or a quadrilateral Δ. For a finite set S denote by $|S|$ the number of elements in S. By $\underset{2}{\equiv}$ denote congruence modulo 2. Part (b) follows because

$$|ABCD \cap (A'B'C' \cup A'D'C')| \underset{2}{\equiv} \sum_{j'=1}^{k'} |ABCD \cap \Delta'_{j'}| \underset{2}{\equiv} \sum_{j=1,\ j'=1}^{k,k'} |(\partial\Delta_j) \cap \Delta'_{j'}|.$$

Here the first congruence follows by the Parity Lemma 4.6. $\square$

Lemma 4.8. *Assume that two $(3,3)$-products T_{123} and T_{145} in 4-space intersect at a unique point A_{11}, which is their common vertex, no 5 of their vertices lie in the same 3-dimensional hyperplane, and the triangles of $(3,3)$-products having disjoint vertices are disjoint. Consider the intersection of the union of triangle of T_{123} containing A_{11} and the union of (the convex hulls of) tetrahedra $A_{11}A_{14}A_{41}A_{15}$ and $A_{11}A_{14}A_{41}A_{51}$. Then this intersection contains an even number of segments with vertex A_{11}.*

Proof. The conclusion of the lemma is equivalent to the following: a small 3-dimensional sphere containing O in its interior intersects T_{123} and T_{145} by two quadrangular polygonal lines which are *linked* in the sphere.

Denote by $\Delta_1, \dots, \Delta_9$ ($\Delta'_1, \dots, \Delta'_9$) those triangles of T (of T') that do not contain O. Let $OX = \mathrm{conv}\{\{O\} \cup X\}$ be the cone over X with the center O. Then $(T \cap T') - \{O\} = \emptyset$ consists of an even number of points. Hence there is an even number of pairs $(j, j') \in [9]^2$ such that the surfaces of tetrahedra $O\Delta_j$ and $O\Delta'_{j'}$ intersect at an odd number of points. By (a spherical analogue of) the argument in the proof of Theorem 1.5 the latter number has the same parity as the number of pairs $(j, j') \in [9]^2$ such that the triangles $\pi \cap O\Delta_j$ and $\pi \cap O\Delta'_{j'}$ are linked. So the lemma follows by (a spherical analogue of) Proposition 4.7.b. $\square$

4.6. **Non-realizability of products in 4-space.**

Proof of the Square Theorem 4.3. We may assume that no 5 of the given 25 points A_{jp} lie in the same 3-dimensional hyperplane (analogously to Remark 2.7.b). We may also assume that there is a unique 'highest' point A_{11} among the given ones. Consider a 'horizontal' 3-dimensional hyperplane α slightly below the point A_{11}.

For every $j = 2, 3, 4, 5$ color in red the intersection point of α with the segment $A_{11}A_{1j}$; see Figure 13. For every $p = 2, 3, 4, 5$ color in blue the intersection point of α with the segment $A_{11}A_{p1}$. Clearly, no 5 of the 8 colored points in α lie in the same plane. Hence by Theorem 2.10 there are two linked closed quadrangular polygonal lines whose vertices are the colored points and whose edges have endpoints of different colors. Without loss

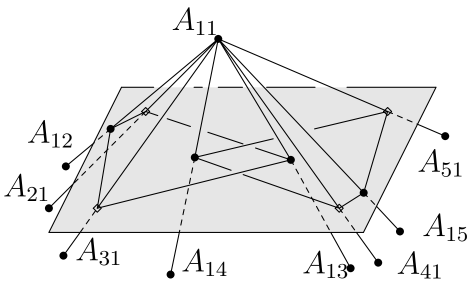

FIGURE 13. To the proof of the Square Theorem 4.3

of generality, the vertices of the first polygonal line belong to the segments joining A_{11} to $A_{12}, A_{21}, A_{13}, A_{31}$, and the vertices of the second polygonal line belong to the segments joining A_{11} to $A_{14}, A_{41}, A_{15}, A_{51}$. Then the polygonal lines are the intersections with the hyperplane of the $(3,3)$-products T_{123} and T_{145}. By Lemma 4.8 T_{123} and T_{145} have an intersection point distinct from A_{11}. Hence analogously to the end of the proof of Theorem 1.5, some two triangles of T_{123} and T_{145} have disjoint vertices but intersect. $\square$

REFERENCES

[AKM] *M. Abrahamsen, L. Kleist and T. Miltzow.* Geometric Embeddability of Complexes is $\exists\mathbb{R}$-complete, arXiv:2108.02585.

[BL] * https://en.wikipedia.org/wiki/Brunnian_link

[BM15] *I. Bogdanov and A. Matushkin.* Algebraic proofs of linear versions of the Conway–Gordon–Sachs theorem and the van Kampen–Flores theorem, arXiv:1508.03185.

[Br68] *P. Bruegel,* 1568, https://en.wikipedia.org/wiki/The_Magpie_on_the_Gallows.

[CG83] *J. H. Conway and C. M. A. Gordon*, Knots and links in spatial graphs, J. Graph Theory 7 (1983), 445–453.

[CKS+] * New ways of weaving baskets, presented by G. Chelnokov, Yu. Kudryashov, A.Skopenkov and A. Sossinsky, http://www.turgor.ru/lktg/2004/lines.en/index.htm.

[DE94] *T.K. Dey and H. Edelsbrunner.* Counting triangle crossings and halving planes, Discrete Comput. Geom. 12 (1994), 281–289.

[DS22] *S. Dzhenzher and A. Skopenkov,* To the Kühnel conjecture on embeddability of k-complexes into $2k$-manifolds, arXiv:2208.04188.

[FFN+] *E. Flapan, J. Foisy, R. Naimi, J. Pommersheim,* Intrinsically n-linked graphs, J. Knot Theory Ramifications, 10 (8) (2001) 1143-1154.

[Fl34] *A. Flores*, Über n-dimensionale Komplexe die im E^{2n+1} absolut selbstverschlungen sind, Ergeb. Math. Koll. 6 (1934) 4–7.

[FMM+] * *E. Flapan, T. Mattman, B. Mellor, R. Naimi, R. Nikkuni,* Recent Developments in Spatial Graph Theory, Contemp. Math., 689 (2017), AMS. arXiv:1602.08122.

[FNP] *E. Flapan, R. Naimi, J. Pommersheim,* Intrinsically triple linked complete graphs, Topology Appl. 115 (2) (2001) 239-246.

[Fr17] *F. Frick*, On affine tverberg-type results without continuous generalization, arXiv:1702.05466

[GSS+] * Projections of skew lines, presented by A. Gaifullin, A. Shapovalov, A. Skopenkov and M. Skopenkov, http://www.turgor.ru/lktg/2001/index.php.

[Io] * https://en.wikipedia.org/wiki/Category:Impossible_objects

[Ko19] *E. Kogan.* Linking of three triangles in 3-space, arXiv:1908.03865.

[KRR+] * Towards higher-dimensional combinatorial geometry, presented by E. Kogan, V. Retinskiy, E. Riabov and A. Skopenkov, https://www.mccme.ru/circles/oim/multicomb.pdf .

[Ko18] * *E. Kolpakov.* A proof of Radon Theorem via lowering of dimension, Mat. Prosveschenie, 23 (2018), arXiv:1903.11055.

[KS20] *R. Karasev and A. Skopenkov.* Some 'converses' to intrinsic linking theorems, Discr. Comp. Geom., to appear, arXiv:2008.02523.

[LS98] *L. Lovasz and A. Schrijver,* A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, Proc. Amer. Math. Soc. 126:5 (1998), 1275-1285.

[LT14] *E. Lindenstrauss and M. Tsukamoto,* Mean dimension and an embedding problem: an example, Israel J. Math. 199 (2014).

[Ma03] * *J. Matoušek.* Using the Borsuk-Ulam theorem: Lectures on topological methods in combinatorics and geometry. Springer Verlag, 2008.

[Me29] *K. Menger.* Über plättbare Dreiergraphen und Potenzen nicht plättbarer Graphen, Ergebnisse Math. Kolloq., 2 (1929) 30–31.

[Me22] *S. A. Melikhov,* Embeddability of joins and products of polyhedra, Topol. Methods in Nonlinear Analysis, 60:1 (2022), 185-201.

[MTW] *J. Matoušek, M. Tancer, U. Wagner.* Hardness of embedding simplicial complexes in $\mathbb{R}^d$, J. Eur. Math. Soc. 13:2 (2011), 259–295. arXiv:0807.0336.

[Na20] * *R. Naimi,* A brief survey on intrinsically knotted and linked graphs, arXiv:2006.07342.

[Ne91] *S. Negami.* Ramsey theorems for knots, links and spatial graphs, Trans. Amer. Math. Soc., 324 (1991), 527–541.

[Pa15] *S. Parsa,* On links of vertices in simplicial d-complexes embeddable in the euclidean $2d$-space, Discrete Comput. Geom. 59:3 (2018), 663–679. This is arXiv:1512.05164v4 up to numbering of sections, theorems etc; we refer to numbering in arxiv version. Correction: Discrete Comput. Geom. 64:3 (2020) 227–228.

[Pa20] *S. Parsa,* On the Smith classes, the van Kampen obstruction and embeddability of $[3] * K$, arXiv:2001.06478.

[Pa21] *S. Parsa,* Instability of the Smith index under joins and applications to embeddability, Trans. Amer. Math. Soc. 375 (2022), 7149–7185, arXiv:2103.02563.

[Pe72] * *B. B. Peterson.* The Geometry of Radon's Theorem, Amer. Math. Monthly 79 (1972), 949-963.

[PS05] * *V. V. Prasolov and M.B. Skopenkov.* Ramsey link theory, Mat, Prosvescheniye, 9 (2005), 108–115.

[PS11] *Y. Ponty and C. Saule.* A combinatorial framework for designing (pseudoknotted) RNA algorithms, Proc. of the 11th Intern. Workshop on Algorithms in Bioinformatics, WABI'11, 250–269.

[RA05] * *J. L. Ramírez Alfonsín.* Knots and links in spatial graphs: a survey. Discrete Math., 302 (2005), 225–242.

[RRS] * *V. Retinskiy, A. Ryabichev and A. Skopenkov.* Motivated exposition of the proof of the Tverberg Theorem (in Russian). Mat. Prosveschenie, 27 (2021), 166–169. arXiv:2008.08361.

[RS72] * *C. P. Rourke and B. J. Sanderson,* Introduction to Piecewise-Linear Topology, *Ergebn. der Math.* 69, Springer-Verlag, Berlin, 1972.

[RSS'] *D. Repovš, A. B. Skopenkov and E. V Ščepin.* On embeddability of $X \times I$ into Euclidean space, Houston J. Math. 1995. 21. P. 199-204.

[RSS+] * *A. Rukhovich, A. Skopenkov, M. Skopenkov, A. Zimin*, Realizability of hypergraphs, https://www.turgor.ru/lktg/2013/1/1-1en.pdf .

[RST] * *N. Robertson, P. Seymour and R. Thomas*, A survey of linkless embeddings, Graph Structure Theory (Seattle, WA, 1991), Contemp. Math. 147, (1993) 125–136.

[RST'] *N. Robertson, P. Seymour and R. Thomas*, Linkless embeddings of graphs in 3-space, Bull. of the Amer. Math. Soc., 21 (1993) 84–89.

[Sa81] *H. Sachs.* On spatial representation of finite graphs, in: Finite and infinite sets (Eger, 1981), 649–662, Colloq. Math. Soc. Janos Bolyai, 37, North-Holland, Amsterdam, 1984.

[Sa91g] *K. S. Sarkaria.* A generalized Van Kampen-Flores theorem, Proc. Amer. Math. Soc. 111 (1991), 559–565.

[Sk] * *A. Skopenkov.* Algebraic Topology From Algorithmic Standpoint, draft of a book, mostly in Russian, http://www.mccme.ru/circles/oim/algor.pdf.

[Sk03] *M. Skopenkov,* Embedding products of graphs into Euclidean spaces, Fund. Math. 179 (2003), 191–198, arXiv:0808.1199.

[Sk06] * *A. Skopenkov,* Embedding and knotting of manifolds in Euclidean spaces, London Math. Soc. Lect. Notes, 347 (2008) 248–342. arXiv:math/0604045.

[Sk16] * *A. Skopenkov,* A user's guide to the topological Tverberg Conjecture, arXiv:1605.05141v4. Abridged earlier published version: Russian Math. Surveys, 73:2 (2018), 323–353.

[Sk18] * *A. Skopenkov.* Invariants of graph drawings in the plane. Arnold Math. J., 6 (2020) 21–55; full version: arXiv:1805.10237.

[Sk18o] * *A. Skopenkov.* A short exposition of S. Parsa's theorems on intrinsic linking and non-realizability. Discr. Comp. Geom. 65:2 (2021), 584–585; full version: arXiv:1808.08363.

[Sk20] * *A. Skopenkov,* Algebraic Topology From Geometric Viewpoint (in Russian), MCCME, Moscow, 2020 (2nd edition). Part of the book: http://www.mccme.ru/circles/oim/obstruct.pdf . Part of the English translation: https://www.mccme.ru/circles/oim/obstructeng.pdf.

[So12] *P. Soberon,* Equal coefficients and tolerance in coloured Tverberg partitions, Combinatorica, 35:2 (2015) 235–252. arXiv:1204.1202. We refer to numbering in arxiv version.

[SS92] *J. Segal and S. Spież.* Quasi embeddings and embeddings of polyhedra in $\mathbb{R}^m$, Topol. Appl., 45 (1992) 275–282.

[Ta00] *K. Taniyama,* Higher dimensional links in a simplicial complex embedded in a sphere, Pacific Jour. of Math. 194:2 (2000), 465-467.

[Um78] *B. Ummel.* The product of nonplanar complexes does not imbed in 4-space, Trans. Amer. Math. Soc., 242 (1978) 319–328.

[Val] * https://en.wikipedia.org/wiki/Valknut

[vK32] *E. R. van Kampen*, Komplexe in euklidischen Räumen, Abh. Math. Sem. Hamburg, 9 (1933) 72–78; Berichtigung dazu, 152–153. English translation by Tu Tâm Nguễn-Phan: https://sites.google.com/site/tutamnguyenphan/van_Kampen.pdf

[Vo96v] *A. Yu. Volovikov,* On the van Kampen-Flores Theorem. Math. Notes 59:5 (1996), 477–481.

[Zi13] *A. Zimin.* Alternative proofs of the Conway-Gordon-Sachs Theorems, arXiv:1311.2882.

[Zu] *J. Zung.* A non-general-position Parity Lemma, http://www.turgor.ru/lktg/2013/1/parity.pdf.

Books, surveys and expository papers in this list are marked by the stars.