

REALIZABILITY OF HYPERGRAPHS AND RAMSEY LINK THEORY

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Аннотация

We present simple proofs of Conway-Gordon-Sachs' theorem on graphs in 3-dimensional space and van Kampen-Flores' theorem on nonrealizability of hypergraphs in 4-dimensional space. The proofs use a reduction to lower dimensions. Our exposition is simplified and so accessible to students who know basic geometry of 3-dimensional space and who are ready to learn straightforward 4-dimensional generalizations. We use elementary language which allows to present the main ideas without technicalities.

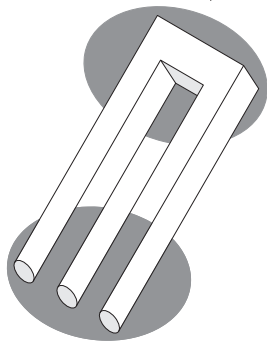
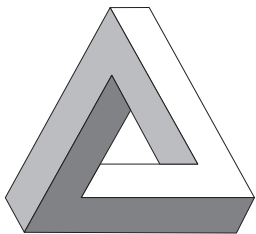
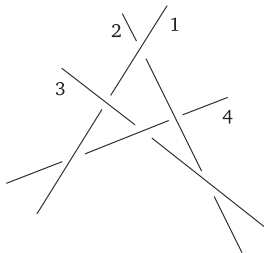
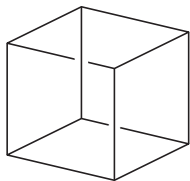


Рис.: The impossible cube, an impossible projection, the Penrose triangle, the blivet

Impossible constructions and intrinsic linking

'Impossible constructions' like the impossible cube, the Penrose triangle, the blivet etc are well-known, mainly due to pictures by Maurits Cornelis Escher. The pictures do not allow the global spatial interpretation because of collision between local spatial interpretations to each other. In geometry, topology and graph theory there are also famous basic examples of 'impossible constructions' (of which local parts are 'possible'). See also

[B26] *P. Bruegel. Die Elster auf dem Galgen, 1526,*

https://de.wikipedia.org/wiki/Die_Elster_auf_dem_Galgen

[GSS+] *Projections of skew lines, presented by A. Gaifullin, A.*

Shapovalov, A. Skopenkov and M. Skopenkov,

<http://www.turgor.ru/lktg/2001/index.php>

[CKS+] *New ways of weaving baskets, presented by G. Chelnokov, Yu.*

Kudryashov, A. Skopenkov and A. Sossinsky,

<http://www.turgor.ru/lktg/2004/lines.en/index.htm>

We exhibit a striking relation of 'impossible constructions' in four-dimensional space to 'intrinsic linking' results in three-dimensional space. Such a relation was found by M. Skopenkov in 2003. He used it to obtain a short proof of the Menger 1929 conjecture (first proved by B. Ummel in 1978 using complicated calculations) and its generalizations.

Let us state the beautiful result on 'intrinsic linking'. We abbreviate 'three-dimensional space \mathbb{R}^3 ' to '3-space'. Analogous meaning has '4-space'.

Linked triangles

Take two triangles in 3-space no 4 of whose 6 vertices lie in the same plane. The triangles are called **linked**, if the outline of the first triangle intersects the part of the plane bounded by the outline of the second triangle exactly at one point. (It is not obvious from the definition that the property of being linked is symmetric.) E.g. the triangles $A_1A_3A_5$ and $A_2A_4A_6$ are linked. (The projection is realizable.)

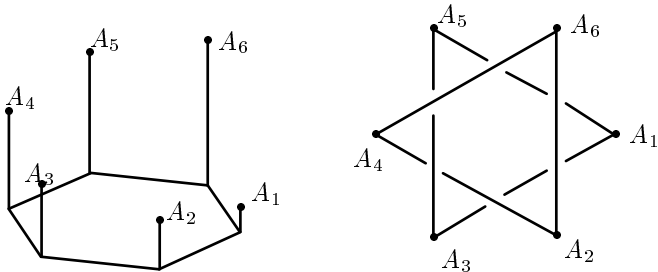


Рис.: Linked triangles

Theorem (Linear Conway–Gordon–Sachs Theorem; 1981-1983)

If no 4 of 6 points in 3-space lie in the same plane, then there are two linked triangles with vertices at these 6 points.

We shall reduce this result to impossibility of the following construction in the plane:

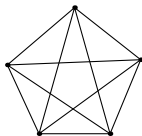


Рис.: Nonplanar graph K_5

Lemma

From any 5 points in the plane one can choose two disjoint pairs such that the segment joining the first pair intersects the segment joining the second pair. (This is a 'linear' version of nonplanarity of K_5 .)

Hypergraphs

We present a natural generalization of nonplanarity of K_5 : beautiful and nontrivial examples of *two-dimensional* analogues of graphs non-realizable in 3- and 4-space.

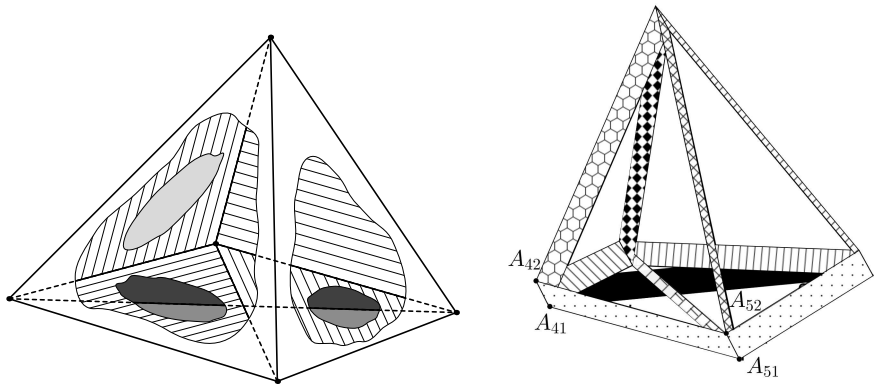


Рис.: Realizations in 3-space of the complete 3-homogeneous hypergraph on 5 vertices and of the *product* of the complete graphs on 5 and on 2 vertices.

Such analogues are *3-homogeneous*, or *2-dimensional hypergraphs* defined as collections of 3-element subsets of a finite set.

Hypergraphs play an important role in mathematics. One cannot imagine topology and combinatorics without them. They are also used in computer science and bioinformatics.

For brevity, we omit '3-homogeneous, or 2-dimensional'. For instance, a *complete hypergraph* on k vertices is the collection of all 3-element subsets of a k -element set.

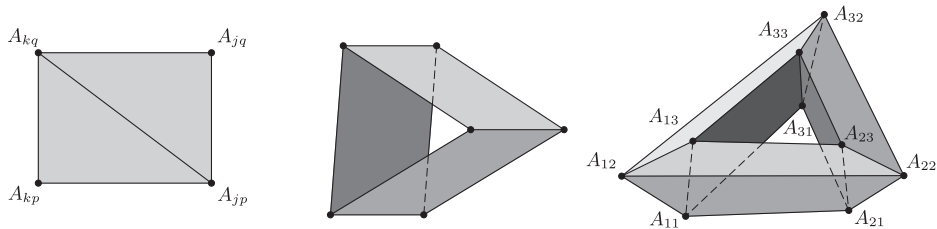


Рис.: Realizations in 3-space of the *square* of the complete graph on 2 vertices, of the *product* of the complete graphs on 2 and on 3 vertices and of the *square* of the complete graph on 3 vertices. A subdivision of quadrilaterals analogous to the leftmost figure is not shown.

Realizability of hypergraphs

Realizability of a hypergraph in d -dimensional Euclidean space is defined similarly to the realizability of a graph in the plane: one 'draws' a triangle for every three-element subset.

A 'small shift' (or 'general position') argument shows that every graph is realizable in 3-space. A straightforward generalization shows that every hypergraph is realizable in 5-space. It is easy to see (and we shall prove below) that the complete hypergraph on 6 vertices is non-realizable in 3-space. Already in the early history of topology (1920s) mathematicians tried to construct hypergraphs non-realizable in 4-space. E. van Kampen and A. Flores in 1932-34 proved that the complete hypergraph on 7 vertices is not realizable in 4-space. This is one of the first results of both *combinatorial (algebraic) topology* and *topological combinatorics*, areas of ongoing active research.

We shall state this result in terms of systems of points, like the above lemma for nonplanarity of K_5 . So we do not use the notions of a hypergraph and its realizability. However, we did mention hypergraphs in order to *motivate* the results.

What is 4-space?

One can define

- the *line* as the set of all real numbers;
- the *plane* as the set of all ordered pairs (x, y) of real numbers;
- *3-space* as the set of all ordered triples (x, y, z) of real numbers;
- *4-space* as the set of all ordered quadruples (x, y, z, t) of real numbers.

Then one can 'analytically' define lines in a plane, lines and planes in 3-space, lines, planes and (3-dimensional) hyperplanes in 4-space.

Usually only the simplest properties are deduced from the analytic definition (or just accepted as axioms). More complicated properties can be deduced from the simplest ones 'synthetically' (i.e., as in school geometry, without using the analytic definition).

Main result for 4-space

A *triangle* is the part of the plane bounded by the outline of a triangle.

Theorem (Linear Van Kampen-Flores Theorem; 1932-1934)

From any 7 points in 4-space one can choose two disjoint triples such that the two triangles with vertices at the triples intersect.

Observe that 'typical' intersection of two segments in the plane is either empty set or a point. Analogously, 'typical' intersection

- of a segment and a triangle in 4-space is empty.
- of two triangles in 4-space is either empty set or a point.

An analogue of the above theorem

- is true for 5 points in the plane or 6 points in 3-space.
- is false for 4 points in the plane, 5 points in 3-space or 6 points in 4-space (in \mathbb{R}^d take the $d + 1$ vertices and an interior point of a d -simplex).

Lowering of dimension: simple examples

Often it is convenient to reduce a planar problem to a linear one (i.e., to a problem in a line), and a spatial problem to a planar one. Similarly, the best approach to some 4-dimensional problems is an analogy to, or a reduction to, spatial ones.

Observe that

- for each two points not belonging to a line in 3-space there exists a broken line which joins these points and does not intersect the line.
- for each plane in 3-space there exist two points not in this plane such that any broken line joining them intersects the plane.

Analogously,

- for each two points not belonging to a 2-dimensional plane in 4-space (e.g. to the plane $x = y = 0$) there exists a broken line which joins these points and does not intersect the plane.
- for each hyperplane in 4-space (e.g. for the hyperplane $x = 0$) there exist two points not in this hyperplane such that any broken line joining them intersects the hyperplane.

Lowering of dimension: proofs of main results

A striking idea is to reduce the 4-dimensional Linear Van Kampen - Flores Theorem to 3-dimensional Linear Conway - Gordon - Sachs Theorem. Because of such 'lowering of dimension' a reader unfamiliar with 4-space need not be scared. Before such a reduction we reduce the 3-dimensional theorem to the lemma on 5 points in the plane.

Lemma (5PP)

From any 5 points in the plane one can choose two disjoint pairs such that the segment joining the first pair intersects the segment joining the second pair.

This is easily proved by analyzing the convex hull of the points. In order to illustrate the 'lowering of dimension' argument in the simplest situation we present another proof, by reduction to the following trivial 1-dimensional statement.

Take 4 points on a line, 2 red and 2 blue. The red and the blue pairs of points are called **linked** if they alternate: red-blue-red-blue or blue-red-blue-red. *Any 4 points in a line can be colored in 2 red and 2 blue so that the red pair is linked with the blue pair.*

Proof of the 5PP Lemma.

Denote the points by O, A, B, C, D . If for some two points $X, Y \in \{A, B, C, D\}$ the point X belongs to the segment OY , then we are done. Otherwise we can assume that the points A, B, C, D are seen from O in this order. Then the outlines of the triangles OAC and OBD have an intersection point different from O . Hence some two sides of the triangles have disjoint vertices and intersect. \square

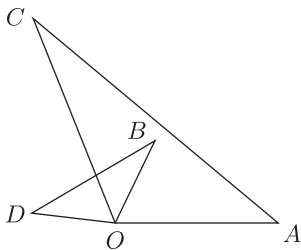


Рис.: To the proof of the 5PP Lemma

A weaker version of the Linear Conway–Gordon–Sachs Theorem

We illustrate the ‘lowering of the dimension’ idea of proof of the LCGS Theorem by a spatial analogue of the lemma on 5 points in the plane.

Lemma

From any 6 points in 3-space one can choose disjoint pair and triple such that the segment joining points of the pair intersects the triangle spanned by the triple.

We have seen that the analogue of this lemma for 5 points is false.

Доказательство.

We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane slightly below the point O . Take the intersection of this plane with the segment OA_j , for every given point A_j . Then by the lemma on 5 points in the plane there are 4 given points A, B, C, D such that the triangles OAB and OCD have a common point other than O . Now the lemma follows. \square

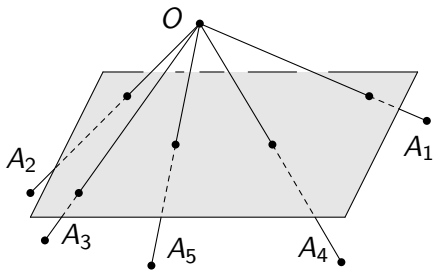


Рис.: A plane in 3-space intersects the segments OA_j by points A'_j .

Proof of the LCGS Theorem.

We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane α slightly below the point O . Denote by A'_1, \dots, A'_5 the intersection points of α and segments joining O to other given points.

In 3-space a segment p is below a segment q (looking from point O), if there exists a half-line OX with the endpoint O that intersects the segment p at a point $P := p \cap OX$, the segment q at a point $Q := q \cap OX$, $P \neq Q$, so that Q is in the segment OP . So in the plane α we obtain a picture analogous to the following:

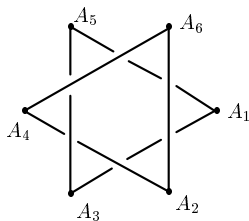


Рис.: A picture in the 'horizontal' plane slightly below the point O

Since no 4 of the given points O, A_1, \dots, A_5 lie in the same plane, the number of those sides of the triangle $A_3A_4A_5$ that are higher than A_1A_2 equals to the number of intersection points of the outline of the triangle $A_3A_4A_5$ with the triangle OA_1A_2 . Also, a segment cannot intersect a triangle by more than 2 points. All this implies that *the triangles OA_1A_2 and $A_3A_4A_5$ are linked if and only if A_1A_2 is below an odd number of sides of the triangle $A_3A_4A_5$.*

Remark. Now it suffices to prove that *if no 3 of 5 points in the plane lie in the same line and the intersection points (different from vertices) of segments joining these points are marked so as to show that one segment 'passes below the other', then there is a segment that is below exactly one side of its 'complementary' triangle.* This can be proved by considering all possible cases. Such an argument hard to generalize to higher dimensions. So instead of giving details, let us present a counting argument that gives the 'quantitative' version of the LCGS Theorem.

'Quantitative' (algebraic modulo 2) versions

The 'quantitative' versions of the above results are as follows.

Lemma (quantitative 5PP)

If no 3 of 5 points in the plane lie in the same line, then the number of intersection points of interiors of segments joining the 5 points is odd.

This is easily proved by analyzing the convex hull of the points, or by reduction to the corresponding trivial 'quantitative' version of the previous 1-dimensional statement.

Theorem (quantitative LCGS)

If no 4 of 6 points in 3-space lie in the same plane, then the number of linked unordered pairs of triangles with vertices at these 6 points is odd.

Theorem (quantitative LVKF)

If no 5 of 7 points in 4-space lie in the same 3-dimensional hyperplane, then the number of intersection points of triangles with vertices at these 7 points is odd.

Proof of the quantitative LCGS Theorem.

The beginning is in the above proof of the LCGS Theorem. Then the following numbers have the same parity:

- the number of linked unordered pairs of triangles formed by given points;
- the number of segments A_iA_j that are below an odd number of sides of their 'complementary' triangles $A_kA_lA_m$,
 $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$;
- the number of ordered pairs (A_iA_j, A_kA_l) of segments of which the first is below the second;
- the number of intersection points of segments whose vertices are A'_1, \dots, A'_5 .

By the quantitative 5PP lemma the latter number is odd. □

Proof of the LVKF Theorem

We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' 3-dimensional hyperplane α slightly below the point O . Denote by A'_1, \dots, A'_6 the intersection points of α and segments joining O to other given points A_1, \dots, A_6 . Clearly, no 4 of the obtained 6 points lie in the same plane. Hence by the LCGS Theorem there are two linked triangles with vertices at these points.

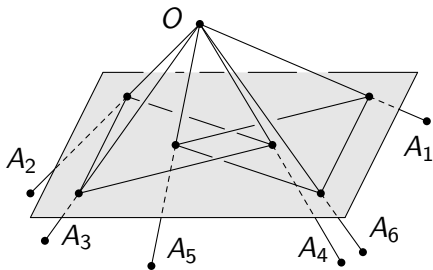


Рис.: To the proof of the quantitative LVKF Theorem. A hyperplane in 4-space (shown as a plane in 3-space) intersects the segments OA_1, \dots, OA_6 at 6 points A'_1, \dots, A'_6 which are vertices of two linked triangles.

Denote by Δ_1 and Δ_2 the triangles formed by given points so that the linked triangles are the intersections $\alpha \cap O\Delta_1$ and $\alpha \cap O\Delta_2$ of the hyperplane α with tetrahedra $O\Delta_1$ and $O\Delta_2$ (e.g. $\Delta_1 = A_2A_3A_4$ and $\Delta_2 = A_1A_5A_6$). Denote by γ the plane containing O and the intersection line of the planes of the linked triangles. Then $\gamma \cap \alpha$ is a line and $\Delta_j^\gamma := \gamma \cap O\Delta_j$ is a triangle ($j = 1, 2$). The side of Δ_j^γ not containing O is $\gamma \cap \Delta_j$. The two sides of Δ_j^γ containing O form the intersection of γ and the lateral surface of the tetrahedron $O\Delta_j$ (whose base is Δ_j).

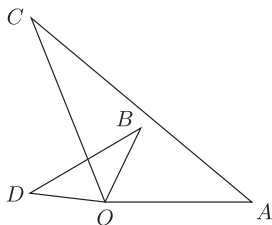


Рис.: Section by the plane γ : $\Delta_1^\gamma = OAC$, $\Delta_2^\gamma = OBD$.

Since the triangles $\alpha \cap O\Delta_1$ and $\alpha \cap O\Delta_2$ are linked, the intersection points of the line $\gamma \cap \alpha$ and the outlines of Δ_1^γ and Δ_2^γ alternate along the line. Hence the outlines have a common point distinct from O .

This point is either the intersection of the sides $\gamma \cap \Delta_1$ and $\gamma \cap \Delta_2$ or, without loss of generality, of the side $\gamma \cap \Delta_1$ and the union of the two sides of Δ_2^γ containing O . In the first case Δ_1 intersects Δ_2 . In the second case Δ_1 intersects the lateral surface of the tetrahedron $O\Delta_2$.

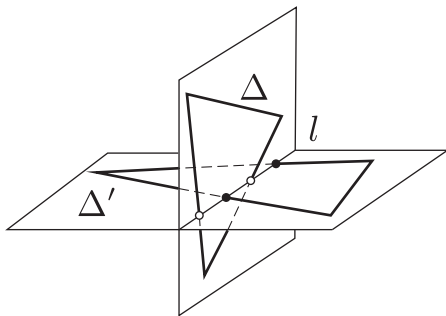


Рис.: Linked triangles and alternating pairs of points

Important remarks

(1) The quantitative LVKF Theorem follows by a simple additional counting (analogous to the proof of the quantitative LCGS Theorem) using the quantitative LCGS Theorem.

(2) We present elementary statements and simple proofs of the *linear* versions of classical results. Our proofs are easily generalized to the *piecewise linear (PL)* and *topological* versions.

(3) *Comparison with other expositions.* The quantitative (linear, PL and topological) CGS and VKF theorems have alternative simple proofs based on showing that the parity in the statement is independent of the set of given points. That proof and the proof sketched here are presumably the simplest known proofs ('proofs from the Book'). Usually the VKF theorem is proved using the Borsuk-Ulam theorem; such a proof requires some knowledge of algebraic topology.

Short algebraic proofs of the linear versions (in the spirit of the 'standard' proof of the Radon theorem) are given by Soberón and Bogdanov-Matushkin. However, those proofs do not generalize to PL (or topological) versions.

Multiple intersection and linking

Realizations (=embeddings) are maps without self-intersections. For topological combinatorics and discrete geometry it is interesting to study of maps whose self-intersections are 'not too complicated'. An important particular case is studying maps *without triple intersections* and, more generally, maps *without r -tuple intersections*. Let us formulate the triple analogues of the above-discussed results.

Theorem (Linear Sarkaria Theorem; 1991)

From any 11 points in 3-space one can choose 3 pairwise disjoint triples whose 3 convex hulls have a common point.

It is surprising that proof of such an elementary result involves algebraic topology. It would be interesting to obtain an elementary proof.

Let us formulate the analogue of this result for *triple linking*.

There are three *Borromean triangles* in 3-space: they are pairwise unlinked but linked together.

Theorem (Linear Negami Theorem; 1991)

There is N such that if no 4 of N points in 3-space lie in the same plane, then there are three Borromean triangles with the vertices at these points.

It would be interesting to obtain an analogue of this result with specific N . Show that one cannot take $N = 10$. Can one take $N = 11$? One can make computer experiments to solve this problem using equivalent definitions of Borromean triangles (E. Kogan, arXiv:1908.03865).

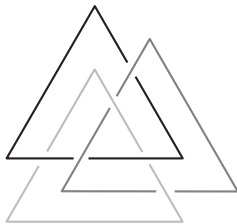


Рис.: Borromean triangles; cf. <https://en.wikipedia.org/wiki/Valknut>