

REALIZABILITY OF HYPERGRAPHS AND INTRINSIC LINKING THEORY

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ABSTRACT. In this expository paper we present short simple proofs of Conway-Gordon-Sachs' theorem on intrinsic linking in three-dimensional space, as well as van Kampen-Flores' and Ummel's theorems on intrinsic intersections. The latter are related to nonrealizability of certain hypergraphs in four-dimensional space. The proofs use a reduction to lower dimensions which allows to exhibit relation between these results. We use elementary language which allows to present the main ideas without technicalities. Thus our exposition is accessible to non-specialists in the area, including students who know basic three-dimensional geometry, and who are ready to learn straightforward four-dimensional generalizations.

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‘It’s too difficult.’

‘Write simply.’

‘That’s hardest of all.’

I. Murdoch, The Message to the Planet.

1. INTRODUCTION

1.1. Impossible constructions, intrinsic intersection and intrinsic linking. ‘Impossible constructions’ like the impossible cube, the Penrose triangle, the blivet etc. are well-known, mainly due to pictures by Maurits Cornelis Escher, see Figure 1, [Io], and also [Br68, CKS+, GSS+]. The pictures do not allow the global spatial interpretation because of collision between local spatial interpretations to each other. In geometry, topology and graph theory there are also famous basic examples of ‘impossible constructions’ (of which local parts are ‘possible’).

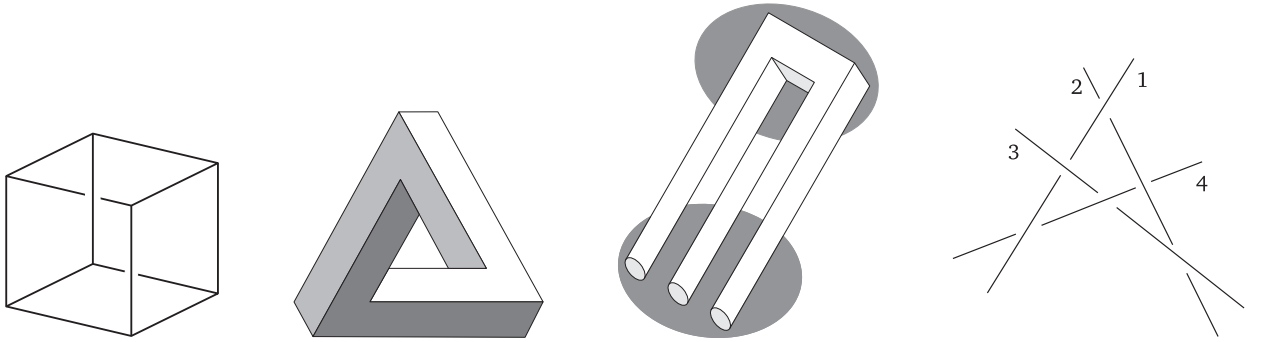


FIGURE 1. The impossible cube, the Penrose triangle, the blivet, an impossible projection

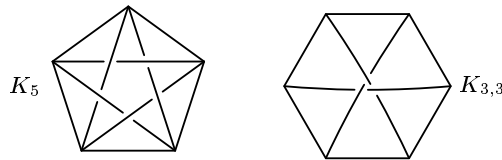


FIGURE 2. Nonplanar graphs K_5 and $K_{3,3}$

The following example of an ‘impossible construction’ or an ‘intrinsic intersection’ is already directly relevant to this paper: *For 5 points in the plane one cannot join each point to each other by a path so that the paths intersect only at their starting points or endpoints.*¹

Proposition 1.1. *For any 5 points in the plane there are two intersecting segments joining these points, and having no common vertices.*

For next results we need some notation. We abbreviate ‘three-dimensional Euclidean space \mathbb{R}^3 ’ to ‘3-space’. Analogous meaning have ‘4-space’ (\mathbb{R}^4) and d -space (\mathbb{R}^d). By a *triangle* we mean the part of the plane bounded by a closed polygonal line of three segments.

¹Also, one cannot take 3 houses and 3 wells in the plane and join each house to each well by a path so that the paths intersect only at their starting points or endpoints. In graph-theoretic terms these assertions mean that the complete graph K_5 on 5 vertices, and the complete bipartite graph $K_{3,3}$ are not planar, see Figure 2. Proposition 1.1 below is a ‘linear’ version of the non-planarity of K_5 .

Take two triangles in 3-space no 4 of whose 6 vertices lie in one plane. The triangles are called **linked**, if the outline of the first triangle intersects the second triangle exactly at one point. E.g. the triangles $A_1A_3A_5$ and $A_2A_4A_6$, Δ and Δ' in Figure 3 are linked.²

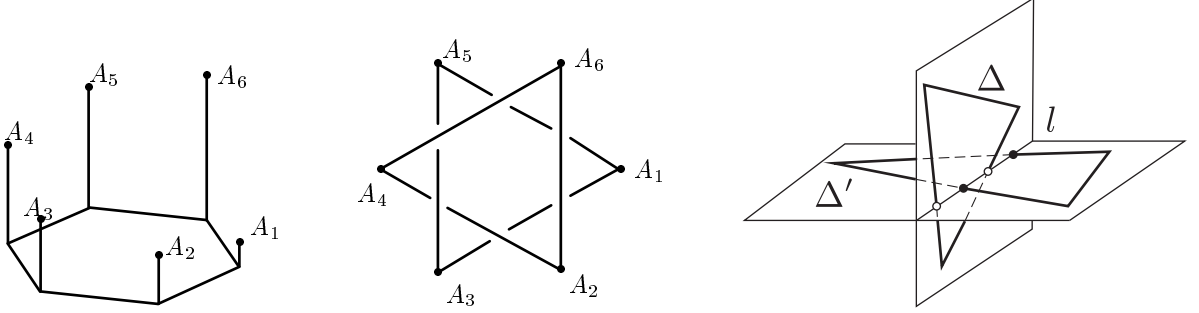


FIGURE 3. Linked triangles

Theorem 1.2 (Linear Conway–Gordon–Sachs Theorem; [Sa81, CG83]). *If no 4 of 6 points in 3-space lie in one plane, then there are two linked triangles with vertices at these 6 points.*

1.2. Why this paper might be interesting? We exhibit striking relations

- of ‘intrinsic intersections’ in the plane (Propositions 1.1 and 2.3) to an ‘intrinsic linking’ result in 3-space (Theorems 1.2 and 2.7 below);
- of the latter results to ‘intrinsic intersections’ in 4-space (Theorems 1.5 and 4.2 below generalizing Proposition 1.1).

Remark 1.3 (lowering of dimension). Often it is convenient to reduce a planar result to a one-dimensional result (i.e., to a result in a line), and a spatial result to a planar result. Similarly, a tempting approach to a 4-dimensional result is an analogy to, or a reduction to, a spatial result. Some examples are given in Remark 1.4.

Proposition 1.1 on intrinsic intersection in the plane is reduced (in §2.1) to Proposition 2.1 on intrinsic linking in the line. Analogously, Theorem 1.2 on intrinsic linking in 3-space is reduced (in §2.4) to Proposition 1.1 (more precisely, to its quantitative version, Proposition 1.1’). Analogously, Theorem 1.5 below on intrinsic intersection in 4-space is reduced (in §2.5) to Theorem 1.2.

This relation between intrinsic linking and intrinsic intersection in consecutive dimensions generalizes to higher dimensions (Theorem 1.6; for simplicity we mention dimensions higher than 4 only in that theorem). Because of such ‘lowering of dimension’ the reader not familiar with 4-space need not be scared.

The results on intrinsic intersections give a natural generalization of non-planarity of graphs: examples of *two-dimensional* analogues of graphs non-realizable in 3- and 4-space. This is explained in Remark 3.1.

We give a simplified exposition accessible to non-specialists in the area. We state results in terms of systems of points. So we do not use the notion of realizability of a hypergraph (we do mention this notion because it is an important *motivation*). For understanding most of the paper it suffices to know basic geometry of 3-space, and to know or learn straightforward 4-dimensional generalizations. We believe that the elementary description of simple

²The distance from the point A_j to the projection plane equals j . So the projection in Figure 3, middle, is realizable, as opposed to Figure 1, right.

The property of being linked is symmetric (this is not obvious from the definition but does have a simple proof [Sk, Proposition 4.1.2]). Other introductory results on linked triangles are presented in [Sk, §4.1 ‘Linking of triangles in 3-space’].

applications of topological methods makes these methods more accessible. Comparison with other proofs is discussed in Remark 3.2.

The striking relation between ‘intrinsic intersection’ and ‘intrinsic linking’ not only gives simple proofs of classical results. It also brings a reader to the frontline of research, notably to the solution of the generalized Menger conjecture (explained in §4.1). The exposed intrinsic linking results are the departure point of *intrinsic (Ramsey) linking theory*. See surveys [RA05, PS05, FMM+, Na20] and references therein; for higher-dimensional analogues see [SS92, Sk03, KS20]. The exposed intrinsic intersection results are generalized to *non-realizability of hypergraphs*. See surveys [Sk06, §4, §5], [MTW, §1], [Sk18, §3.2] and references therein; for recent results see [Pa15, Sk18o, AKM, Pa21, Me22]. For analogous problem on embedding dynamical systems see [LT14] and references therein.

The history is exposed in Remark 3.3.

Plan of the paper. The remarks are not formally used later and so could be omitted. The same is true for §1.4 and §1.5. Sections 2, 3 and 4 are independent of each other, so they could be read in any order. Forward references and references to other papers can be ignored for the first reading.

1.3. Intrinsic intersection in 4-space.

Remark 1.4 (some intuition on 4-space). (a) ‘Typical’ intersection

- of two segments in the plane is either empty set or a point (here ‘typical’ means that no 3 points among the vertices of segments lie in one line);
- of a segment and a triangle in 3-space is either empty set or a point;
- of two triangles in 4-space is either empty set or a point.

(b) For each two points

- of the plane distinct from a point A in the plane there exists a polygonal line joining these points and not passing through A .
- of 3-space not belonging to a line l in 3-space there exists a polygonal line joining these points and disjoint with l ;
- in 4-space not belonging to a 2-dimensional plane α in 4-space there exists a polygonal line joining these points and disjoint with α .

(c) More intuition on 4-space is not required here, but can be developed by studying e.g. [KRR+, §1], [Sk, §4.10 ‘How to work with four-dimensional space?’].

Theorem 1.5 (Linear Van Kampen-Flores Theorem; [vK32, Fl34]). *From any 7 points in 4-space one can choose two disjoint triples such that the two triangles with vertices at the triples intersect.*

Analogues of Theorem 1.5 are true for 5 points in the plane, and for 6 points in 3-space (Propositions 1.1 and 2.4.b). Analogues of Proposition 1.1 and of Theorem 1.5 are false for 4 points in the plane and for 6 points in 4-space, respectively: in \mathbb{R}^{2k} take the $2k+1$ vertices and an interior point of a $2k$ -simplex, cf. Figure 4.

1.4. Intrinsic intersection and linking in higher dimensions. A subset of \mathbb{R}^d is called *convex*, if for any two points from this subset the segment joining these two points is in this subset. The *convex hull* of $X \subset \mathbb{R}^d$ is the minimal convex set that contains X .

Theorem 1.6. *Take any $d+3$ points in \mathbb{R}^d of which no $d+1$ points lie in one $(d-1)$ -dimensional hyperplane.*

For d even there are two disjoint $(d+2)/2$ -element subsets whose convex hulls intersect.

If d is odd, then there is an unordered pair of $(d+1)/2$ -simplices with vertices at these points which is linked (i.e., the boundary of the first simplex intersects the convex hull of the second simplex exactly at one point).

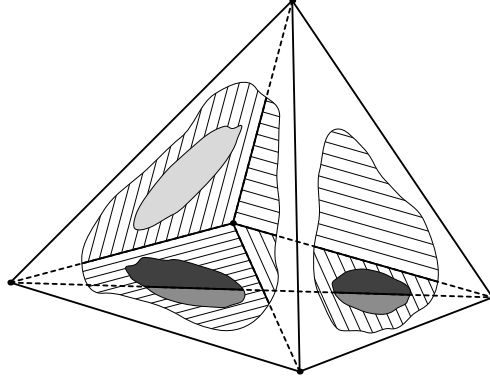


FIGURE 4. Five points in \mathbb{R}^3 (realization of the complete 3-homogeneous hypergraph on 5 vertices)

This is Proposition 1.1 for $d = 2$, Theorem 1.2 for $d = 3$, and Theorem 1.5 for $d = 4$. Theorem 1.6 is due to van Kampen-Flores for even $d > 2$ [vK32, F134], and to Segal-Spież-Lovász-Schrijver-Taniyama for odd $d > 3$ [LS98, Corollary 1.1], [Ta00] (the index argument of [SS92, §1] has a simple generalization to Theorem 1.6; thus Theorem 1.6 for odd $d > 3$ is implicit in [SS92]).

Theorem 1.6 is proved by induction on d . The base is $d = 1$ and is trivial. The inductive step is proved in §2 for $d = 2, 3, 4$; the proof for the general case is analogous.

The analogue of Theorem 1.6 for $d + 2$ points

- does not make sense for d odd because $(d + 1)/2$ -simplex has $(d + 3)/2$ vertices;
- is false for d even, analogously to the corresponding counterexample to Theorem 1.5.

For d odd there is Proposition 2.4.b on intrinsic intersection and its higher-dimensional analogue. They are weaker than the corresponding Theorems 1.2, 1.6 on intrinsic linking. More results are presented in [KRR+, §3], [Sk16, §4].

1.5. Multiple intersection and linking. Let us formulate the analogues of the above results for r -fold intrinsic intersections.

Theorem 1.7 ([Sa91g]). *From any 11 points in 3-space one can choose 3 triangles having pairwise disjoint vertices but having a common point.*

It is surprising that known proof of such an elementary result involves algebraic topology. It would be interesting to obtain an elementary proof.

Example 1.8. *In 3-space take the vertices of a 3-dimensional simplex and its center, see Figure 4. For every of these 5 points either take it with multiplicity two or take a close point. We obtain 10 points for which the analogue of Theorem 1.7 is false.*

For a higher-dimensional higher-multiplicity analogue of Theorems 1.6 and 1.7 see [Sk16, Theorem 1.6], [Sk18, Conjecture 3.1.4 and the text below].

Let us formulate the analogues of the above results for intrinsic triple linking.

There are three triangles in 3-space which are pairwise unlinked but linked together (Figure 5; one can check that this projection is realizable, as opposed to Figure 1, right.) Such a triple of triangles is called *Borromean*, cf. [Val] and [Sk, §4.6].

Theorem 1.9 (Negami [Ne91]). *There is N such that if no 4 of N points in 3-space lie in one plane, then there is a Borromean triple of triangles with the vertices at these points.*

See also [PS05, FNP]. It would be interesting to obtain an analogue of Theorem 1.9 with specific N . By Example 1.8 one cannot take $N = 10$. Can one take $N = 11$ (as



FIGURE 5. Borromean triple of triangles

in Theorem 1.7)? One can make computer experiments to solve this problem using an equivalent definitions of Borromean triple [Ko19]. It would be interesting to obtain a higher-dimensional higher-multiplicity analogues of Theorem 1.9, cf. [BL, FFN+].

2. PROOFS

By k points in \mathbb{R}^d (in this paper mostly $d \leq 4$) we mean k pairwise distinct points. Denote $[n] := \{1, 2, \dots, n\}$.

2.1. Intersections in the plane: proof of Proposition 1.1. Proposition 1.1 is easily proved by analyzing the convex hull of the 5 points. In order to illustrate the ‘lowering of dimension’ idea (see Remark 1.3) in the simplest situation, we also deduce Proposition 1.1 from the following obvious 1-dimensional result.

Proposition 2.1. *Every 4 points in a line can be colored in two red and two blue so that they alternate: red-blue-red-blue or blue-red-blue-red (one says ‘the red pair is linked with the blue pair’).*

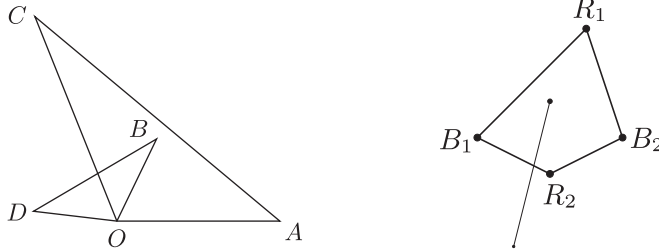


FIGURE 6. Left: To the proof of Proposition 1.1. Right: To Proposition 2.3.b.

Proof of Proposition 1.1. There is a line l such that one of the given points, say O , lies on one side of l , and the other points, say A, B, C, D , lie on the other side of l . If for some two points $X, Y \in \{A, B, C, D\}$ the point X belongs to the segment OY , then we are done. Otherwise we can assume that the points A, B, C, D are seen from O in this order, see Figure 6. Then the required assertion follows by Lemma 2.2 below. \square

Lemma 2.2 (lowering of dimension; see Figure 6, left). *Two triangles in the plane have a common vertex. A line l splits this point from the bases of the triangles. The intersections of l with the triangles alternate along l . Then some two sides of the triangles intersect but do not have common vertices.*

This lemma is trivial. It is explicitly stated in order to conveniently use it (here and in §2.5), and to illustrate its generalization to higher-dimensions (Lemma 4.6).

The following propositions are proved analogously to Proposition 1.1. They are used for some 3-dimensional results (Propositions 4.1.ab and Theorem 2.7).

Proposition 2.3. (a) (See Figure 2, right) Two triples of points are given in the plane. Then there exist two intersecting segments without common vertices and such that each segment joins the points from distinct triples.

(b) (See Figure 6, right) Four red and two blue points B_1, B_2 are given in the plane. Suppose that any two segments joining points of different colors either are disjoint or intersect at their common vertex. Then there are two red points R_1, R_2 such that the quadrilateral $R_1 B_1 R_2 B_2$ does not have self-intersections, and the remaining two red points lie in different sides w.r.t. the quadrilateral. ('In different sides' means that a general position polygonal line joining the remaining two red points intersects the outline of the quadrilateral at an odd number of points.)

2.2. Weaker versions of Theorem 1.2. First we illustrate the 'lowering of the dimension' idea (see Remark 1.3) by proving the following weaker versions of Theorem 1.2.

Proposition 2.4. From any 6 points in 3-space one can choose

(a) 5 points O, A, B, C, D such that the triangles OAB and OCD have a common point other than O .

(b) disjoint pair and triple such that the segment joining points of the pair intersects the triangle spanned by the triple.

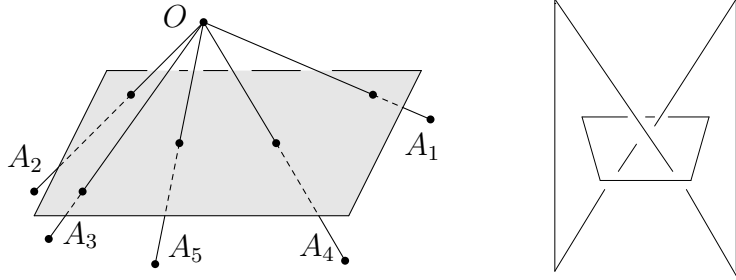


FIGURE 7. Left: To the proofs of Proposition 2.4.a and Theorem 1.2'. A plane in \mathbb{R}^3 intersects the segments OA_1, \dots, OA_5 by points A'_1, \dots, A'_5 .

Right: Whitehead link formed by space quadrilaterals

Proof of (a). There is a plane α such that one of the given points, say O , lies on one side of α , and the other 5 points lie on the other side of α (see Figure 7). Consider the intersection of α with the union of triangles OAB for all pairs A, B of given points. Now part (a) follows by Proposition 1.1. \square

Part (b) follows from (a) (and vice versa).

Part (b) is a spatial analogue of Proposition 1.1.

Figure 4 shows that the analogue of (a) for 5 points is false.

2.3. 'Quantitative' versions. We prove the following stronger 'quantitative' (i.e., algebraic modulo 2) version of the results from §1. (The deduction of Theorem 1.2 from Proposition 1.1, not from its quantitative version, has a technical detail which is hard to generalize to higher dimensions.)

Proposition 2.1' (obvious). Any 4 points in the line have a unique unordered splitting into two linked pairs.

Proposition 1.1'. *No 3 of given 5 points in the plane lie in one line. Then the number of intersection points of interiors of segments joining the 5 points is odd.*

This is easily proved by analyzing the convex hull of the 5 points, or follows by Proposition 2.1' and the following lemma.

Lemma 2.2'. (a) *Two triangles in the plane have a common vertex O . No 3 of their 5 vertices lie in one line. A line l splits O from the bases of the triangles. Then the outlines of the triangles intersect at an even number of points if and only if the intersections of l with the outlines alternate along l (i.e., if the intersection of one triangle with the outline of the other contains exactly one segment with vertex O).*

(b) *No 3 of 5 points O, A_1, A_2, A_3, A_4 in the plane lie in one line. Then the number from Proposition 1.1' equals to the sum of the numbers of intersection points of the interiors of sides of triangles OPQ and ORS , over all unordered splittings of points A_1, A_2, A_3, A_4 into two unordered pairs P, Q and R, S .*

(c) [DGN+, Proposition 7.5.a] Denote by $X = \binom{[5]}{2}$ the set of all unordered pairs of 2-element subsets of $[5]$. For any partition $\sigma \sqcup \tau = [4]$ into 2-element sets denote

$$T_{\{\sigma, \tau\}} := \{ \{\alpha, \beta\} \in X : \alpha \subset \sigma \sqcup \{5\}, \beta \subset \tau \sqcup \{5\} \}.$$

Then the sum modulo 2 (i. e., the symmetric difference) of the sets $T_{\{\sigma, \tau\}}$ over all the 3 non-ordered partitions $\sigma \sqcup \tau = [4]$ into 2-element sets equals $\{ \{\alpha, \beta\} \in X : \alpha \cap \beta = \emptyset \}$.

Part (a) is analogous to lemma 2.2 (and so is trivial). Part (b) follows from (c). A simple proof of (c) is left to a reader.

Remark 2.5. Proposition 1.1' is indeed stronger than Proposition 1.1 because it suffices to prove Proposition 1.1 under the assumption that no 3 of the 5 points lie in one line,

(a) first, since otherwise Proposition 1.1 is obvious: if points A, B, C among given 5 points lie in one line, B between A and C , and D is any other given point, then segments AC and BD intersect.

(b) second, since we can make a small shift so that no 3 of the 5 shifted points lie in one line, and no intersection points of segments with disjoint vertices are added.

Analogously to (b), Theorems 1.2' and 1.5' below are stronger than Theorems 1.2 and 1.5.

Triangles differing only by permutation of vertices are considered to be the same.

Theorem 1.2' ([Sa81, CG83]). *No 4 of given 6 points in 3-space lie in one plane. Then the number of linked unordered pairs of triangles with vertices at these 6 points is odd.*

Theorem 1.5' ([vK32, Fl34]). *If no 5 of 7 points in 4-space lie in one 3-dimensional hyperplane, then the number of intersection points of triangles with vertices at these points is odd.*

Theorem 1.6 has an analogous quantitative version.

Another 'quantitative versions' are presented in §2.6. For counterexamples to 'integer versions' see [KS20, Proposition 1.2 and Theorem 1.4].

2.4. Linking in 3-space: proof of Theorem 1.2'. In 3-space a segment p is below a segment q (looking from point O), if there exists a point $X \in p$ such that the segments OX and q intersect.

Lemma 2.6 (lowering of dimension). *No 4 of 6 points O, A_1, \dots, A_5 lie in one plane. Then the triangles OA_1A_2 and $A_3A_4A_5$ are linked if and only if A_1A_2 is below exactly one side of the triangle $A_3A_4A_5$.*

The lemma follows because the number of those sides of the triangle $A_3A_4A_5$ that are higher than A_1A_2 equals to the number of intersection points of the outline of the triangle $A_3A_4A_5$ with the triangle OA_1A_2 .

Proof of Theorem 1.2. There is a plane α such that one of the given points, say O , lies on one side of α , and the other points, say A_1, \dots, A_5 , lie on the other side of α (Figure 7). Take the intersection points of α and segments OA_1, \dots, OA_5 . Since no 4 of the 6 points lie in one plane, no 3 of the taken 5 points lie in one line. So in the plane α we obtain a picture analogous to Figure 3, middle. Then the following numbers have the same parity:

- the number P of linked unordered pairs of triangles formed by given 6 points;
- the number Q of segments A_iA_j that are below an odd number of sides of their ‘complementary’ triangles $A_kA_lA_m$, $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$;
- the number of ‘undercrossings’, i.e., of ordered pairs (A_iA_j, A_kA_l) of segments in which the first segment is below the second one;
- the number of intersection points of interiors of segments whose vertices are the 5 taken points in α .

Here the numbers P and Q have the same parity by the Lowering of Dimension Lemma 2.6: a segment cannot intersect a triangle by more than two points, so in the lemma we can replace ‘exactly one side’ by ‘an odd number of sides’.

By Proposition 1.1’ the latter number is odd. Hence P is also odd. \square

The following version of Theorem 1.2 is analogously reduced to Proposition 2.3.b [Zi13]. This version is used for some 4-dimensional result (Theorem 4.2) in §4.5.

In 3-space take two quadrilaterals (i.e., closed quadrangular polygonal lines) $ABCD$ and $A'B'C'D'$, no 4 whose 8 vertices lie in one plane. The quadrilaterals are called *linked modulo 2* if the number of intersection points of the quadrilateral $ABCD$ with the union of the triangles $A'B'C'$ and $A'D'C'$ is odd. (As opposed to triangles, there are space quadrilaterals *linked* but not *linked modulo 2*, see Figure 1.2, right. Cf. definition at the beginning of §4.5.)

Theorem 2.7 ([Sa81]). *There are 4 points red and 4 blue points in 3-space. No 4 of these 8 points lie in one plane. Then there are two linked modulo 2 space quadrilaterals consisting of segments joining points of different colors.*

2.5. Intersection in 4-space: proof of Theorem 1.5.

Proof of Theorem 1.5. There is a 3-dimensional hyperplane α such that one of the given points, say O , lies on one side of α , and the other 6 points lie on the other side of α (Figure 8). Take the 6 intersection points of α with the segments joining O to the other 6 points. We may assume that no 5 of the given 7 points lie in one 3-dimensional hyperplane (analogously to Remark 2.5.b). Hence no 4 of the 6 points in α lie in one plane. Then by Theorem 1.2 there are two linked triangles with vertices at the taken 6 points. So we are done by Lemma 2.8. \square

Lemma 2.8 (lowering of the dimension; Figure 8). *Two tetrahedra τ and τ' in 4-space have a common vertex. A 3-dimensional hyperplane α splits this vertex from the bases of the tetrahedra. The outlines of the triangles $\alpha \cap \tau$ and $\alpha \cap \tau'$ are disjoint and linked in α . Then some two faces of the tetrahedra intersect but do not have common vertices.*

This lemma is not as obvious as its low-dimensional analogues (Lemma 2.2 and analogous result for a triangle and a tetrahedron in 3-space) because *the surface of a tetrahedron in 4-space does not split 4-space* (cf. Remark 1.4.b).

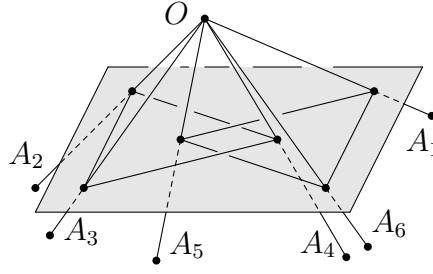


FIGURE 8. To the proof of Theorem 1.5; the hyperplane α in 4-space is shown as a plane in 3-space

Proof of Lemma 2.8. Denote by γ the intersection plane of the 3-dimensional hyperplanes spanned by the tetrahedra. Then $\alpha \cap \gamma$ is the intersection line of the planes of the linked triangles $\alpha \cap \tau$ and $\alpha \cap \tau'$. Hence $\tau \cap \gamma$ and $\tau' \cap \gamma$ are triangles with a common vertex O , which is a common vertex of the tetrahedra (Figure 6, left). Since α splits O from the bases of the tetrahedra, we have that $\alpha \cap \gamma$ splits O from the bases of the triangles. Since the triangles $\alpha \cap \tau$ and $\alpha \cap \tau'$ are linked, the intersection points of the line $\alpha \cap \gamma$ and the outlines of the triangles $\tau \cap \gamma$ and $\tau' \cap \gamma$ alternate along the line [Sk, Proposition 4.1.3.b] (cf. Figure 3, right). Hence by Lemma 2.2 two sides of the triangles $\tau \cap \gamma$ and $\tau' \cap \gamma$ intersect but do not have common vertices. At most one of these sides contains O . Hence the two sides are contained in two faces of the tetrahedra intersect but do not have common vertices. \square

Theorem 1.5' follows from Theorem 1.2' and the following Lemma 2.8'.

Lemma 2.8'. (a) *Two tetrahedra τ and τ' in 4-space have a common vertex O . No 5 of their 7 vertices lie in one 3-dimensional hyperplane. A 3-dimensional hyperplane α splits O from the bases of the tetrahedra. Then the surfaces of the tetrahedra τ, τ' intersect at an even number of points if and only if the triangles $\alpha \cap \tau$ and $\alpha \cap \tau'$ are linked in α (i.e., if the intersection of one tetrahedron with the surface of the other contains exactly one segment with the end O).*

(b) *No 5 of 7 points $O, A_1, \dots, A_6 \in \mathbb{R}^4$ lie in one 3-dimensional hyperplane. Then the number from Theorem 1.5' equals to the sum of the numbers of intersection points of the interiors of faces of tetrahedra $O\Delta$ and $O\Delta'$, over all unordered splittings of points A_1, \dots, A_6 into two unordered triples Δ and Δ' . (The interior of a triangle is its complement to the outline.)*

(c) *Denote by $X = \binom{[7]}{3}$ the set of all unordered pairs of 3-element subsets of $[7]$. For any partition $\sigma \sqcup \tau = [6]$ into 3-element sets denote*

$$T_{\{\sigma, \tau\}} := \{ \{\alpha, \beta\} \in X : \alpha \subset \sigma \sqcup \{7\}, \beta \subset \tau \sqcup \{7\} \}.$$

Then the sum modulo 2 of the sets $T_{\{\sigma, \tau\}}$ over all the ten non-ordered partitions $\sigma \sqcup \tau = [6]$ into 3-element sets equals $\{ \{\alpha, \beta\} \in X : \alpha \cap \beta = \emptyset \}$.

Part (a) is analogous to lemma 2.8. Part (b) follows from (c). A simple proof of (c) is left to a reader.

The following higher-dimensional version of Proposition 2.3.a is related (analogously to Theorem 1.5) to some 3-dimensional intrinsic linking result [DS22, Remark 2.5].

Theorem 2.9 ([vK32, FI34]). *Three triples of points in 4-space are given. Then there exist two intersecting triangles without common vertices such that the vertices of each triangle belong to distinct triples.*

2.6. Unlinking properties. Here we present quantitative versions asserting that the number of intersections (or linkings) is even, cf. §2.3.

Proposition 2.10. (2) *There are 5 points in the plane such that no 3 of them lie in a line, and every segment joining two of the points intersects the outline of the triangle formed by the remaining three points at an even number of points. (I.e., every pair of points is ‘unlinked’ with the triangle formed by the remaining three points.)*

(2') *No 3 of 5 given points in the plane lie in a line. Then the number of those segments joining two of the points that intersect the outline of the triangle formed by the remaining three points exactly at one point, is even.*

Proofs are easy and are left to the reader.

In 3-space instead of unlinking properties 2.10.2,2' there is a linking property (Theorem 1.2) and the following unlinking properties.

Proposition 2.11. (3) *There are 6 points in 3-space such that no 4 of them lie in a plane, and every segment joining two of them intersects the surface of the tetrahedron formed by the remaining four points at an even number of points. (I.e., every pair of points is ‘unlinked’ with the tetrahedron formed by the remaining four points.)*

(3') *No 4 of given 6 points in 3-space lie in one plane. Then the number of intersection points of segments joining two of the points, and surfaces of tetrahedra formed by the remaining four points, is even.*

Sketch of a proof. (3) Take points close to the vertices of regular octahedron, points close to the vertices of a triangular prism, or points on the moment curve.

(3') Any two triangles spanned by two disjoint triples of given points either are disjoint or intersect by a segment (non-degenerate to a point). There is an even number of ends of such segments. The ends of such segments are exactly intersection points of segments joining pairs of points, and surfaces of ‘complementary’ tetrahedra.

Alternatively, we have

$$\sum_{\{A,B\}} |AB \cap \partial T_{AB}|_2 = \sum_{\{A,B\}} (|A \cap T_{AB}|_2 + |B \cap T_{AB}|_2) = \sum_A \sum_{B \neq A} |A \cap T_{AB}|_2 = 0.$$

Here

- $|S|_2$ is the parity of the number of elements in a finite set S ,
- T_{AB} is the tetrahedron formed by the four given points distinct from A, B , and
- the last equality holds because for every A the set $\{T_{AB}\}_{B \neq A}$ is a 3-cycle, cf. [Sk18, Remark 1.3.5b]. □

Propositions 1.1, 1.1', 2.4.b and 2.11.3' show that under transition from dimension 2 to dimension 3 the property of the existence of intersection is preserved, while the parity of the number of intersections change. The 3-dimensional versions of Propositions 1.1, 1.1' have a stronger form: Theorems 1.2 and 1.2'.

Proposition 2.12. (4-3) *There are 7 points in 4-space such that no 5 of them lie in a 3-dimensional hyperplane, and every triangle formed by 3 of them intersects the surface of the tetrahedron formed by the 4 remaining points at an even number of points. (I.e., every triangle formed by three of the points is ‘unlinked’ with the tetrahedron formed by the remaining four points.)*

(4'-3) *No 5 of 7 given points in 4-space lie in a 3-dimensional hyperplane. Then the number of those triangles spanned by three of the points that intersect exactly at one point the surface of the tetrahedron formed by the remaining four points, is even.*

(4-2) (conjecture) *There are 7 points in 4-space such that no 5 of them lie in a 3-dimensional hyperplane, and every segment joining two of them intersects the 3-dimensional surface of the 4-simplex formed by the remaining five points at an even number of points. (I.e., every pair of points is ‘unlinked’ with the 4-simplex formed by the remaining five points.)*

(4'-2) *No 5 of 7 given points in 4-space lie in a 3-dimensional hyperplane. Then the number of intersection points of segments joining them and 3-dimensional surfaces of 4-dimensional simplices formed by the remaining five points, is even.*

Sketch of a proof. (4-3) Take points on the moment curve. See details in [St].

(4'-3) Analogously to Proposition 2.11.3'.

(4-2) Take points on the moment curve?

(4'-2) Analogously to the alternative proof of Proposition 2.11.3'. □

Conjecture 2.13. *(d-k) For any $d \neq 2k - 1$ there are $d + 3$ points in \mathbb{R}^d , of which no $d + 1$ lie in one $(d - 1)$ -hyperplane, and such that any k -simplex spanned by $k + 1$ of them, intersects the surface of $(d + 1 - k)$ -spanned by the remaining $d + 2 - k$ points, at an even number of points.*

Hint. Take points on the moment curve?

(d'-k) *No $d + 1$ of $d + 3$ points in \mathbb{R}^d lie in the same $(d - 1)$ -hyperplane. Then the number of intersection points of k -simplices, spanned by $k + 1$ of them, with the surfaces of $(d + 1 - k)$ -simplices, spanned by the remaining $d + 2 - k$ points, is even.*

Hint. Induction on k ; analogously to the alternative proof of Proposition 2.11.3'.

3. SOME IMPORTANT REMARKS

Remark 3.1 (Relation to hypergraphs). (a) Two-dimensional analogues of graphs are *3-homogeneous*, or *2-dimensional hypergraphs* defined as collections of 3-element subsets of a finite set.³ For brevity, we omit ‘3-homogeneous’ and ‘2-dimensional’. For instance, a *complete hypergraph* on k vertices is the collection of all 3-element subsets of a k -element set. *Realizability* (also called *embeddability*) of a hypergraph in \mathbb{R}^d is defined similarly to the realizability of a graph in the plane: one ‘draws’ a triangle for every three-element subset. See Figures 4, 9 and 10; on the last figures a subdivision of quadrilaterals analogous to Figure 9, left, is not shown. See rigorous definitions e.g. in [Sk18, §3.2].

Hypergraphs (and simplicial complexes) play an important role in mathematics. One cannot imagine topology and combinatorics without them. They are also used in computer science and in bioinformatics, see e.g. [PS11].

A ‘small shift’ (or ‘general position’) argument shows that every graph is realizable in 3-space. A straightforward generalization shows that every hypergraph is realizable in 5-space.

The complete hypergraph on 6 vertices contains ‘the cone over K_5 ’ and hence is not realizable in 3-space (Proposition 2.4.a). Already in the early history of topology (1920s) mathematicians tried to construct hypergraphs non-realizable in 4-space. Egbert van Kampen and A. Flores in 1932-34 proved that the complete hypergraph on 7 vertices is not realizable in 4-space (Theorem 1.5). This is both an early application of *combinatorial topology* (nowadays called *algebraic topology*) and one of the first results of *topological combinatorics* (also an area of ongoing active research).

(b) Realizations (=embeddings) are maps without self-intersections. For topological combinatorics and discrete geometry it is interesting to study maps whose self-intersections are non-empty (like for embeddings), but ‘not too complicated’. An important particular case

³In topology such objects are called *pure*, or *dimensionally homogeneous*, 2-dimensional *simplicial complexes*. The term ‘hypergraph’ is more convenient to generic mathematician or computer scientist.

is studying maps *without triple intersections* and, more generally, maps *without r -tuple intersections*, see §1.5 and surveys [Sk16], [Sk18, §3.3].

(c) We present *linear* versions of the results. *PL* (*piecewise-linear*) and *topological* realizations (=embeddings) of hypergraphs are defined and discussed e.g. in [Sk18, §3.2], [Sk, §5]. The proofs we expose are interesting because they easily generalize to the PL versions [Sk03, Zi13], as opposed to the proofs of [BM15, So12].

PL versions of ‘quantitative’ results (see §2.3) imply the PL versions for *almost-embeddings* (see the PL case of [Sk18, Theorems 1.4.1 and 3.1.6]). The latter imply the *topological* versions (see explanation in [Sk18, the paragraph after Theorem 1.4.1]).

Remark 3.2 (Comparison with other proofs). Theorem 1.6 for d even (and so its particular cases, Proposition 1.1 and Theorem 1.5) has an alternative simple proof using *the van Kampen number*, see e.g. [Sk18, §1.4], [Sk, §1.4, §5]. (That proof works for quantitative, PL, and topological versions, see Remark 3.1.c.) That proof and the proof sketched in this paper, are presumably the simplest known proofs (‘proofs from the Book’).

Usually Theorem 1.6 for d even (more precisely, the topological version of Theorems 1.6 and 2.9) is proved using the Borsuk-Ulam theorem [Sk20, §8], [Ma03, §5]. As opposed to this paper (and to the alternative proof using the van Kampen number), this requires some knowledge of algebraic topology. This knowledge does not make things simpler: known proofs of the Borsuk-Ulam theorem (see [Ma03] and the references therein) are not easier than the above-discussed direct proofs of Theorem 1.6 for d even. (The Borsuk-Ulam theorem is proved using *the degree* analogously to the direct proof of Theorem 1.6 for d even using *the van Kampen number*.)

The paper [BM15] presents short algebraic proof of Theorem 1.6 (and so of its particular cases). That proof is in the spirit of the algebraic proof of the *Radon theorem*⁴.

Remark 3.3 (history). General ‘lowering of dimension’ or ‘the link of a vertex’ ideas are simple and well-known (see Remark 1.3). For proofs of the Radon theorem based on this idea see [Pe72, Ko18, RRS]. For an application in computer science see [DE94, proof of 2.3.i]. Also well-known is relation between linking and intersection.⁵ An elaboration of this idea to a relation between intrinsic linking and non-realizability is non-trivial (cf. the difference between Proposition 2.4.a and Theorem 1.2). Proofs that discover and use that relation seem to have not been published

- before [RST, RST’, Sh03, RSS+, Zi13], for a proof of the Conway–Gordon–Sachs Theorem 1.2 by reducing intrinsic linking to intrinsic intersection in lower dimension,
- before [Sk03, Example 2, Lemmas 2 and 1’], [RSS+], for proofs of Theorem 1.5 and of the Menger conjecture (see §4.1) by reducing intrinsic intersection to intrinsic linking in lower dimension.

4. REALIZABILITY OF PRODUCTS AND THE MENDER CONJECTURE

4.1. The Menger conjecture. The (*Cartesian*) *product* $F \times F'$ of two figures F, F' in \mathbb{R}^3 is the set of all points $(x, y, z, x', y', z') \in \mathbb{R}^6$ such that $(x, y, z) \in F$ and $(x', y', z') \in F'$.

Examples of realization of products are given in Figures 9 and 10. For definition of realization see e.g. [Sk18, §3.2], [Sk, §5]. Karl Menger conjectured in 1929 that the square of

⁴See e.g. [Sk16, §1] for the statement of the Radon theorem. See [Sk16, §4] for relations between the Radon theorem and Theorems 1.2, 1.5, 1.6. The proof of [BM15] is presumably a direct (i.e., without use of the Gale transform) version of the proof of [So12, Theorem 5] for $k = 1$, which is Theorem 1.6 for d even.

⁵E.g. the linking number of two disjoint closed polygonal lines in 3-dimensional sphere ∂D^4 equals to the algebraic intersection number of two *general position* 2-dimensional disks in 4-dimensional ball D^4 spanning the two polygonal lines. For a certain inductive argument involving assertion on linking in odd dimensions and assertion on intersection in even dimensions see [RS72, Whitney Lemma 5.12 and Theorem 5.16].

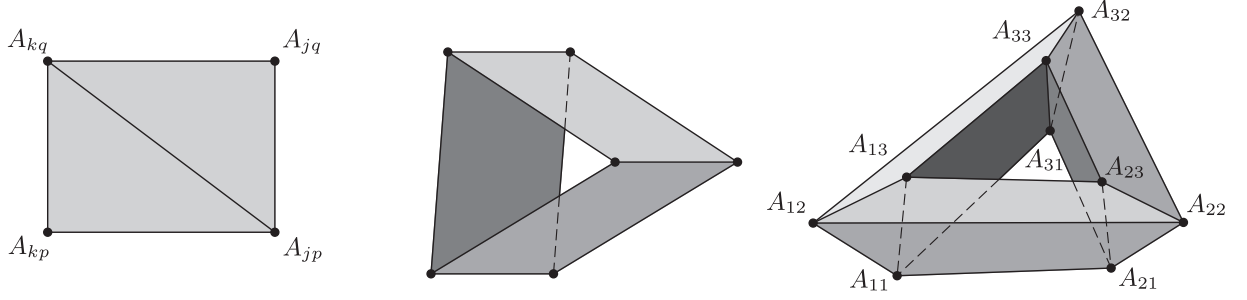


FIGURE 9. Realizations of the products: $K_2 \times K_2$ (left), $K_2 \times K_3$ (middle), $K_3 \times K_3$ (right)

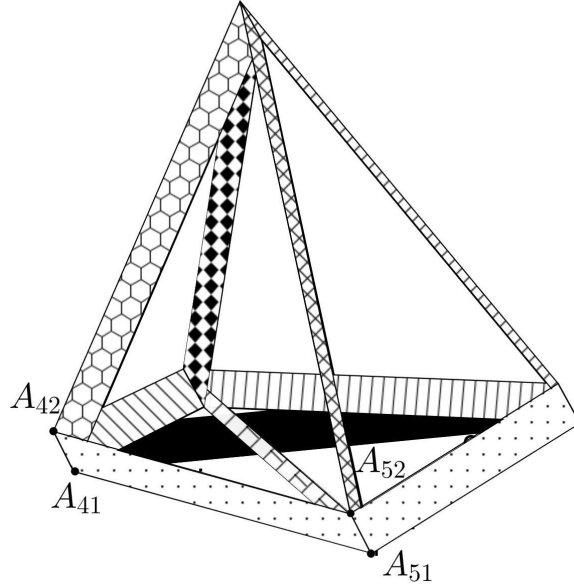


FIGURE 10. Realization of the product $K_5 \times K_2$

a nonplanar graph is not realizable in \mathbb{R}^4 [Me29] (cf. Theorem 4.2). This was proved only in 1978 by Brian Ummel [Um78] using advanced algebraic topology. A simple proof was obtained in 2003 by Mikhail Skopenkov [Sk03] using *lowering of dimension*, see exposition below. His argument proves the generalized Menger conjecture (‘the k -th power of a nonplanar graph is not realizable in \mathbb{R}^{2k} ’), and even gives a short formula for the minimal number d such that given product of several graphs is realizable in \mathbb{R}^d [Sk03].

A combinatorial version of the product is *product of two graphs* (not necessarily planar). This product can be considered (although not canonically) as a hypergraph. See Remark 3.1.a; cf. [ADN+, §2, §5].

Propositions 4.1.ab below imply that neither $K_4 \times K_4$ nor $K_5 \times K_3$ are (linearly) realizable in \mathbb{R}^3 . Proof of Proposition 4.1.a shows that $K_{3,1} \times K_{3,1}$ is not realizable in \mathbb{R}^3 (but this product is realizable in \mathbb{R}^4).

Remark. Proofs of the Menger conjecture using the van Kampen number or the Borsuk-Ulam theorem (see Remark 3.2) are unknown. The proof of the Menger conjecture in [Um78] works for the topological version but is complicated. The simpler proof in [Sk03] uses for the topological version the non-trivial *Bryant approximation theorem*. A simpler proof of the topological version can be obtained by inventing a quantitative PL version of the Menger

conjecture (i.e., by improving the PL version of Theorem 4.2 analogously to §2.3, see Remark 4.4).

4.2. Realizability of products. Let us formalize the idea of $K_m \times K_n$ drawn in 3- or 4-space. Suppose that A_{jp} , where $j \in [m]$ and $p \in [n]$, are mn points in 3- or 4-space. For numbers $j, k \in [m]$, $j < k$, and $p, q \in [n]$, $p < q$, denote

$$jk \times pq := \{A_{jp}A_{kq}A_{jq}, A_{jp}A_{kq}A_{kp}\}.$$

This is a pair of triangles having a common side (Figure 9, left). Their union could be, but need not be, a plane quadrilateral. An (m, n) -**product** is a collection of triangles from

$$jk \times pq, \quad \text{where } 1 \leq j < k \leq m, \quad 1 \leq p < q \leq n.$$

(There are $mn(m-1)(n-1)/2$ such triangles.) The **body** of (m, n) -product is the union of its triangles. The body is a polyhedral and possibly self-intersecting

- square, if $m = n = 2$ (Figure 9, left);
- lateral surface of a cylinder, if $m = 3$ and $n = 2$ (Figure 9, middle);
- torus, if $m = n = 3$ (Figure 9, right).

Proposition 4.1. (a) Any $(4, 4)$ -product; (b) Any $(3, 5)$ -product in 3-space has a triangle and a side of a triangle which have disjoint vertices but intersect.

These folklore results are reduced to PL versions of Proposition 2.3.ab in §4.4.

Theorem 4.2 (Square; [Um78, Sk03]). Any $(5, 5)$ -product in 4-space has two triangles which have disjoint vertices but intersect.

The Square Theorem 4.2 is reduced to Theorem 2.7 in §4.5.

Example 4.3. The analogues of Proposition 4.1 and Theorem 4.2 are false for

- (a) $(2, n)$ -products in 3-space for every n (for $n \leq 4$ this is obvious; for $n = 5$ see Figure 10: the vertices of the parallelograms are the required 10 points; for $n \geq 6$ the construction is analogous, see §4.3; cf. [RSS', Theorem 1.5]);
- (b) $(3, n)$ -products in 3-space for every $n \leq 4$ (for $n \leq 3$ this is obvious, see Figure 9, right; for $n = 4$ the construction is analogous, see §4.3);
- (c) $(4, n)$ -products in 4-space for every n (see §4.3).

Remark 4.4. Denote

$$\widetilde{K}_5^2 := \left\{ \{(X, Y), (X', Y')\} : X, Y, X', Y' \in \binom{[5]}{2}, \text{ either } X \cap X' = \emptyset \text{ or } Y \cap Y' = \emptyset \right\}.$$

It would be interesting to find a subset $M \subset \widetilde{K}_5^2$ such that for any PL map $f : K_5 \times K_5 \rightarrow \mathbb{R}^4$ there is an odd number of pairs $\{(X, Y), (X', Y')\} \in M$ for which $|f(X \times Y) \cap f(X' \times Y')|$ is odd, where by X, Y, X', Y' we understand edges of K_5 .⁶

This is related to the following algebraic Menger problem [Pa20, Conjecture 2]: *Complexes K, L have non-trivial van Kampen obstructions to embeddability in \mathbb{R}^m and in \mathbb{R}^n , respectively (see definition e.g. in [Sk18, §1.5]). Does the cartesian product $K \times L$ of K and L has non-trivial van Kampen obstruction to embeddability in \mathbb{R}^{m+n} ?*

⁶For $M = \widetilde{K}_5^2$ the number of such pairs has apparently the same parity as the number of linked unordered pairs of cycles in $K_{4,4}$ embedded in \mathbb{R}^3 , which number is even. So either M should be different, or one should use integers (or residues modulo 4) instead of residues modulo 2.

4.3. Realization of products in 3- and 4-space.

Sketch of a proof of Example 4.3.a. Let A_{11}, \dots, A_{1n} be points in \mathbb{R}^3 of which no 4 lie in one plane. Take a vector v not parallel to any plane passing through some three of these points. For every $p \in [n]$ denote $A_{2p} := v + A_{1p}$. If v is small enough, then the points A_{jp} , $j \in \{1, 2\}$, $p \in [n]$, are as required: there are no triangle and a side of a triangle with vertices at these points, which have disjoint vertices but intersect.

Indeed, $12 \times pq$ is a parallelogram for every $p \neq q$. Since no 4 of the points A_{11}, \dots, A_{1n} lie in one plane, and by the choice of v , for any distinct p, q, r, s the segments $A_{1p}A_{1q}$ and $A_{1r}A_{1s}$ are disjoint. Since v is small enough, the same holds for 1 replaced by 2. Then any two (convex hulls of) parallelograms $12 \times pq$ and $12 \times rs$ that have no common side are disjoint. Now one can check that the points A_{jp} are as required. \square

Sketch of a proof of Example 4.3.b. Let

$$A_{11} = (1, 0, 1), \quad A_{12} = (-1, 0, 1), \quad A_{13} = (0, 0, 2), \quad A_{14} = (0, 0, 3).$$

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation through $\frac{2\pi}{3}$ w.r.t. x -axis. Let $A_{2p} = f(A_{1p})$ and $A_{3p} = f(f(A_{1p}))$ for every $p \in [4]$ (Figure 9, right). Then the points A_{jp} , $j \in [3]$, $p \in [4]$, are as required.

Indeed, $jk \times pq$ is a parallelogram for every $j \neq k$, $p \neq q$. Since every two segments joining points A_{1p} either are disjoint or intersect at a common vertex, any two of such parallelograms that have no common side are disjoint. Now one can check that the points A_{jp} are as required. \square

Sketch of a proof of the weaker version of Example 4.3.c: (3, 5)-product in 4-space. Take a 3-dimensional hyperplane in \mathbb{R}^4 (shown in Figure 11, left, as a plane in 3-space). In this hyperplane take 10 vertices A_{jp} , where $j \in [5]$, $p \in \{1, 2\}$, shown in Figure 10. Take a vector v not parallel to the hyperplane. Set $A_{j3} := A_{j1} + v$. (In Figure 11, left, we see the lateral surface of the prismoid $A_{41}A_{42}A_{43}A_{51}A_{52}A_{53}$.) Then the points A_{jp} , $j \in [5]$, $p \in [3]$, are as required: there are no two triangles with vertices at these points, which have disjoint vertices but intersect. \square

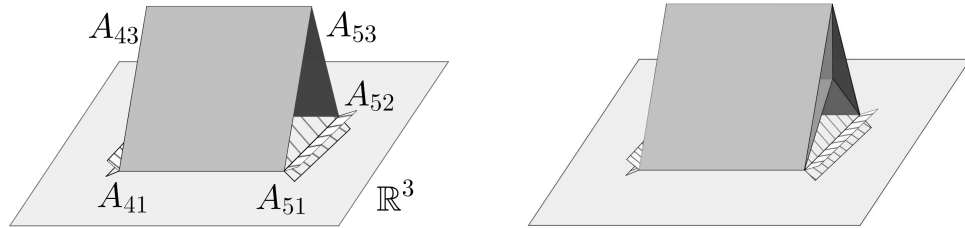


FIGURE 11. Left: to realization in \mathbb{R}^4 of the product $K_3 \times K_5$.
Right: to realization in \mathbb{R}^4 of the product $K_4 \times K_5$.

Sketch of a proof of Example 4.3.c. Take points $A_{jp} \in \mathbb{R}^3 \subset \mathbb{R}^4$, $j \in \{1, 2\}$, $p \in [n]$ from the proof of Example 4.3.a. Then $\overrightarrow{A_{1p}A_{1q}} = \overrightarrow{A_{2p}A_{2q}}$ for every $p \neq q$. Take non-collinear vectors $v_3, v_4 \in \mathbb{R}^4$ not parallel to the hyperplane $\mathbb{R}^3 \subset \mathbb{R}^4$. Denote $A_{jp} := A_{1p} + v_j$, $j \in \{3, 4\}$. We could have taken v_3, v_4 so that A_{14} is an interior point of the triangle $A_{11}A_{12}A_{13}$. See Figure 11, right. Then the points A_{jp} , $j \in [4]$, $p \in [n]$, are as required. \square

4.4. Non-realizability of products in 3-space.

Proof of Proposition 4.1.a. (The proof is analogous to Proposition 2.4.) There is a plane α such that one of the given points lies on one side of α , and the other 15 points lie on the other side of α (Figure 12, left). The intersection point of α

- with the segment $A_{jp}A_{kp}$ is colored in blue for every $k \in [4] - \{j\}$;
- with the segment $A_{jp}A_{jq}$ is colored in red for every $q \in [4] - \{p\}$.

The intersection of a product $jk \times pq$ with α is called an *arc*. Then arcs have ends of different color. (The intersection of α with the body of the $(4, 4)$ -product is a PL drawing, possibly with self-intersections, of $K_{3,3}$ in α , i.e., the image of a *PL map* $K_{3,3} \rightarrow \alpha$.)

Then by the PL analogue of Proposition 2.3.a⁷ there are intersecting arcs without common edges. Then there are $k, k' \in [4] - \{j\}$ and $q, q' \in [4] - \{p\}$ such that $k \neq k'$, $q \neq q'$, and some two triangles, one from $jk \times pq$ and the other from $jk' \times pq'$, have a common point distinct from the common vertex A_{jp} of the triangles. Hence one of these triangles intersects a side of the other not passing through A_{jp} , hence not having any common vertices with the first triangle. \square

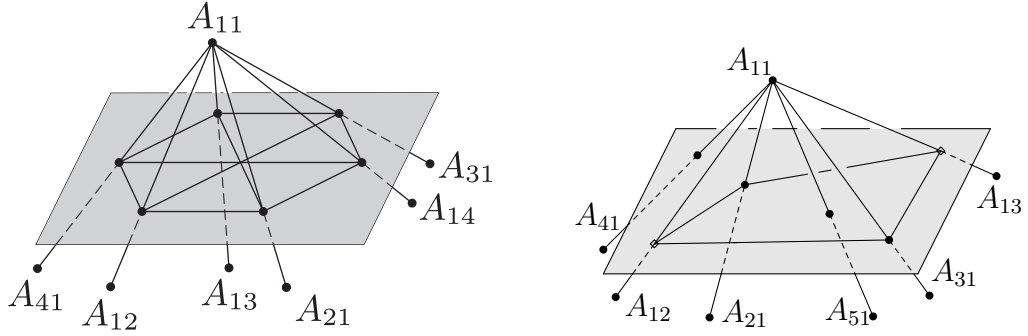


FIGURE 12. To the proofs of Proposition 4.1.a (left) and 4.1.b (right)

Given 9 points A_{jp} , $j \in \{u, v, w\}$, $p \in \{u', v', w'\}$, in 3- or in 4-space denote by $uvw \times u'v'w'$ the corresponding $(3, 3)$ -product (Figure 9, right; as opposed to the figure, the $(3, 3)$ -product can have self-intersections).

Proof of Proposition 4.1.b. We may assume that no 4 of given 15 points A_{jp} , $j \in [3]$, $p \in [5]$, lie in one hyperplane (analogously to Remark 2.5.b). There is a plane α such that one of the given points, say $A_{j,p}$, lies on one side of α , and the other 15 points lie on the other side of α (Figure 12, right). The intersection point of α

- with the segment $A_{jp}A_{kp}$ is colored in blue for every $k \in [3] - \{j\}$;
- with the segment $A_{jp}A_{jq}$ is colored in red for every $q \in [5] - \{p\}$.

The intersection of a product $jk \times pq$ with α is called an *arc*. Then arcs have ends of different color. (The intersection of α with the union of the triangles of the $(5, 3)$ -product is a PL drawing, possibly with self-intersections, of $K_{2,4}$ in α .)

Analogously to the last paragraph of the proof of Proposition 4.1.a either

(1) the $(5, 3)$ -product has a triangle and a side of a triangle which have disjoint vertices but intersect, or

(2) any two arcs intersect only at their common vertex (if there is one).

In the second case denote the blue points by K, L . By the PL analogue of Proposition 2.3.b there are two red points Q, R such that the remaining two red points S, T lie on different sides w.r.t. the closed polygonal γ formed by arcs QK, KR, RL, LQ . Take $k, l \in [3] - \{j\}$

⁷This analog is proved analogously, see [Sk18, Remark 1.4.4b].

and $q, r, s, t \in [5] - \{p\}$ such that the points K, L , and Q, R, S, T belong to the segments joining A_{jp} to A_{kp}, A_{lp} , and to $A_{jq}, A_{jr}, A_{js}, A_{jt}$, respectively. Then α intersects

- the outline of the triangle $j \times pst := A_{jp}A_{js}A_{jt}$ by S and T (note that the triangle is not contained in the $(3, 5)$ -product);
- the body of the $(3, 3)$ -subproduct $jkl \times pqr$ (contained in the $(3, 5)$ -product) by γ .

If γ has self-intersections, then we obtain the property (1). If not, then we obtain the property (1) by Lemma 4.5 below because S, T lie on different sides w.r.t. γ . \square

Lemma 4.5 (lowering of dimension). *In 3-space the outline $\partial\Delta$ of a triangle Δ and a $(3, 3)$ -product τ have a unique common vertex O . No 4 vertices of Δ and τ lie in one plane. A plane α splits O from the base of Δ , and from the remaining 8 points of τ . The plane α intersects*

- the body $|\tau|$ by a closed polygonal line without self-intersections;
- $\partial\Delta$ by two points S, T lying on different sides w.r.t. the polygonal line.

Then τ and $\partial\Delta$ have a triangle and a side of a triangle which have disjoint vertices but intersect.

Proof. Denote by $\langle\Delta\rangle$ the plane of Δ . Then $\alpha \cap \langle\Delta\rangle$ is a line. The intersection $\alpha \cap \Delta$ is the segment ST . The points S, T lie in α on different sides w.r.t. the closed polygonal line $\alpha \cap |\tau|$ which does not have self-intersections, and no 3 points among S, T and the vertices of $\alpha \cap |\tau|$ lie in one line. Hence the segment $\alpha \cap \Delta$ intersects $|\tau| \cap \langle\Delta\rangle$ at an odd number of points. Analogously, the line $\alpha \cap \langle\Delta\rangle$ intersects $|\tau| \cap \langle\Delta\rangle$ at an even number of points. Since also no 4 vertices of Δ and τ lie in one plane, it follows that $\tau \cap \langle\Delta\rangle := \{\Gamma \cap \langle\Delta\rangle : \Gamma \in \tau\}$ is a 1-cycle, i.e., is a set of segments in the plane $\langle\Delta\rangle$ such that every point of $\langle\Delta\rangle$ is the endpoint of an even number (possibly, zero) of the segments. By the assumption on α we may choose the segments so that the line $\alpha \cap \langle\Delta\rangle$ splits O from S, T , and from all the vertices of the segments distinct from O (cf. Figure 6, left). Hence by an analogue of Lemma 2.2 for a triangle and a 1-cycle, some two segments from $\partial\Delta$ and from $\tau \cap \langle\Delta\rangle$ intersect but do not have common vertices. At most one of these segments contains O . Hence the obtained segment of the 1-cycle $\tau \cap \langle\Delta\rangle$ is the intersection with $\langle\Delta\rangle$ of a triangle from τ , which intersects a side of Δ , but does not have common vertices with the side. \square

4.5. Non-realizability of products in 4-space. A *Seifert chain* (or a coboundary) of a closed polygonal line a in 3-space is a finite collection S of triangles (non-degenerate to a segment or a point) in 3-space such that

- every edge of a is the side of exactly one triangle from S ;
- every segment that is not an edge of a is the side of an even number (possibly, zero) of triangles from S .

Two disjoint closed polygonal lines a and a' in 3-space *linked modulo 2* if for any Seifert chains S of a and S' of a' such that the outline of any triangle of S is disjoint from the outline of any triangle of S' , the number of linked modulo 2 pairs (Δ, Δ') of triangles Δ of S and Δ' of S' is odd. The equivalence to other definitions of being linked modulo 2 (in particular, to the definition before Theorem 2.7) is proved in [Sk, Lemma 4.8.3].

Proof of the Square Theorem 4.2. We may assume that no 5 of the given 25 points A_{jp} , $j, p \in [5]$, lie in one 3-dimensional hyperplane (analogously to Remark 2.5.b). There is a 3-dimensional hyperplane α such that one of the given points, say $A_{j,p}$, lies on one side of α , and the other 24 points lie on the other side of α (Figure 13). The intersection point of α

- with the segment $A_{jp}A_{kp}$ is colored in blue for every $k \in [5] - \{j\}$;
- with the segment $A_{jp}A_{jq}$ is colored in red for every $q \in [5] - \{p\}$.

The intersection of a product $jk \times pq$ with α is called an *arc*. Then arcs have ends of different color.

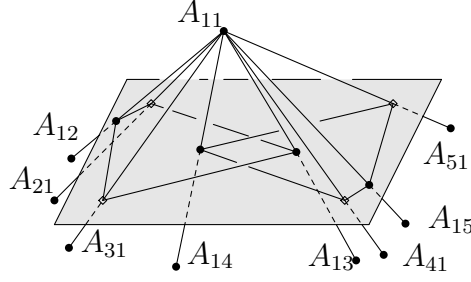


FIGURE 13. To the proof of the Square Theorem 4.2

Analogously to the last paragraph of the proof of Proposition 4.1.a either

- (1) the $(5, 5)$ -product has two triangles which have disjoint vertices but intersect, or
- (2) any two arcs intersect only at their common vertex (if there is one).

In the second case the intersection of α with the body of the $(5, 5)$ -product is a PL drawing without self-intersections of $K_{4,4}$ in α . Use the following PL analogue of Theorem 2.7: in any PL drawing without self-intersections of $K_{4,4}$ in \mathbb{R}^3 there are two cycles of length 4 which are linked modulo 2 (see proof in [Sa81, Zi13]). We obtain two linked modulo 2 closed polygonal lines in α , each consisting of four arcs. Take $\{a, b, a', b'\} = [5] - \{j\}$ and $\{c, d, c', d'\} = [5] - \{p\}$ such that the arcs

- of the first polygonal line belong to the products $ja \times pc$, $jb \times pc$, $ja \times pd$, $jb \times pd$,
- of the second polygonal line belong to the products $ja' \times pc'$, $jb' \times pc'$, $ja' \times pd'$, $jb' \times pd'$.

Then the polygonal lines are the intersections with the hyperplane of the bodies of the $(3, 3)$ -products $jab \times pcd$ and $ja'b' \times pc'd'$. So the required statement is implied by the following Lemma 4.6. \square

Lemma 4.6 (lowering of dimension). *Two $(3, 3)$ -products in 4-space have a unique common vertex O . No 5 of their 17 vertices lie in one 3-dimensional hyperplane. A 3-dimensional hyperplane α splits O from the remaining 16 points of the $(3, 3)$ -products. The hyperplane α intersects the bodies of the $(3, 3)$ -products by a pair of disjoint closed polygonal lines linked modulo 2 in α . Then the $(3, 3)$ -products have two triangles which have disjoint vertices but intersect.*

Proof. Denote by S (by S') the set of all triangles from the first (the second) $(3, 3)$ -product, which do not contain O . Since no 5 of the 17 vertices of the $(3, 3)$ -products lie in one 3-dimensional hyperplane, no 4 of their 16 projections to α with the center O lie in one plane. Then the outlines of the triangles $\alpha \cap O\Delta$ and $\alpha \cap O\Delta'$ are disjoint for any $\Delta \in S$ and $\Delta' \in S'$. Denote by γ and γ' the given disjoint closed polygonal lines. Since γ and γ' are linked modulo 2 in α , the number of linked modulo 2 pairs (Δ, Δ') of such triangles is odd. By Lemma 2.8'.a such triangles are linked modulo 2 if and only if the surfaces of the tetrahedra $O\Delta$ and $O\Delta'$ intersect at an even number of points (including O). Take any side MN of a triangle from S , and any side $M'N'$ of a triangle from S' . If MN is not contained in γ , and $M'N'$ is not contained in γ' , the intersection $OMN \cap OM'N'$ appears exactly in two intersections of lateral surfaces of tetrahedra $O\Delta$ and $O\Delta'$. Then the numbers of pairs of intersecting triangles having one of the following types is odd:

- pairs (Δ, Δ') for $\Delta \in S$ and $\Delta' \in S'$;
- pairs (OMN, Δ') for a side MN of γ , and a triangle $\Delta' \in S'$.
- pairs $(\Delta, OM'N')$ for a side $M'N'$ of γ' , and a triangle $\Delta \in S$.

Now the lemma follows because in any of these pairs triangles have disjoint vertices, and are contained in triangles of given $(3, 3)$ -products. \square

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