Whitney trick for eliminating multiple intersections

A. Skopenkov, https://users.mccme.ru/skopenko/

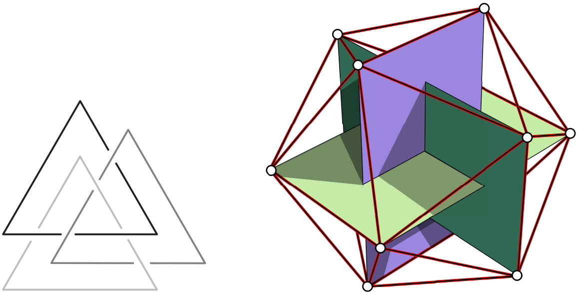

Figure: Borromean triangles (Valknut) and quadrilaterals (icosahedron)

Abstract

The Whitney trick for cancelling *double* intersections is one of the main tools in the topology of manifolds. Analogues of the Whitney trick for *multiple* intersections were 'in the air' since 1960s. However, only in this century they were stated, proved and applied to obtain interesting results.

I shall describe multiple Whitney trick in *codimension* ≥ 3 (Mabillard-Wagner 2015), in *codimension 2* (Avvakumov-Mabillard-Skopenkov-Wagner 2015 [AMSW]) and when *general position multiple intersections have positive dimension* (Mabillard-Wagner 2016 and Skopenkov 2017). These were most difficult steps of recent counterexamples to the topological Tverberg conjecture (found in a series of papers by M. Özaydin, M. Gromov, P. Blagojević, F. Frick, G. Ziegler, I. Mabillard and U. Wagner) and of stronger counterexamples ([AMSW], Avvakumov-Karasev-Skopenkov 2019).

[Sk16] *A. Skopenkov, A user's guide to the topological Tverberg Conjecture, Russian Math. Surveys, 73:2 (2018), 323–353; full version: arXiv:1605.05141.*

1. The Whitney trick for multiple intersections

Theorem (Strong Whitney Embedding theorem 1944)

Any compact k-dimensional manifold embeds into $\mathbb{R}^{2k}$.

For references to classical results and their discussion see surveys:

[Sk06] *A. Skopenkov, Embedding and knotting of manifolds in Euclidean spaces, London Math. Soc. Lect. Notes, 347 (2008) 248–342; arXiv:math/0604045.*

[Sk16c] *A. Skopenkov, Embeddings in Euclidean space: an introduction to their classification, to appear in Boll. Man. Atl.* `http://www.map.mpim-bonn.mpg.de/Embeddings_in_Euclidean_space:_an_introduction_to_their_classification`

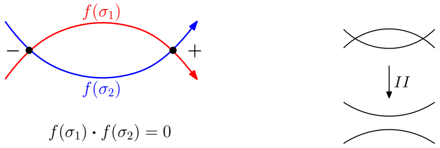

Figure: Algebraic intersection number and ‘Whitney trick’ in the plane

Strong Whitney Embedding theorem is proved using the following lemma.
We work in the PL category which we omit; in particular, all disks, balls and maps are PL. Analogous results hold in the smooth category.
A map $f\colon D^k \to B^d$ is called *proper*, if $f^{-1}S^{d-1} = S^{k-1}$.

Lemma (Whitney 1944)

If $f : D_1 \sqcup D_2 \to B^{2k}$ is a proper general position map of disjoint union of k-dimensional disks such that the algebraic intersection number $fD_1 \cdot fD_2$ is zero, then there is a proper general position map

$$f' : D \to B^{2k} \quad \text{such that} \quad f' = f \text{ on } \partial(D_1 \sqcup D_2) \quad \text{and} \quad f'D_1 \cap f'D_2 = \emptyset.$$

The case $k \geq 3$ is a version of the Whitney trick; the case $k = 2$ is an exercise on elementary link theory; the case $k = 1$ is trivial.
Whitney trick fails for 2-surfaces in 4-manifolds (Kervaire-Milnor 1961).

Denote by $D = D_1 \sqcup \ldots \sqcup D_r$ the disjoint union of r disks of dimension $k(r-1)$.

Lemma (r-fold Local Disjunction 2015)

If $k \geq 2$ and $f : D \to B^{kr}$ is a proper general position map such that the r-fold algebraic intersection number $fD_1 \cdot \ldots \cdot fD_r$ is zero, then there is a proper general position map

$$f' : D \to B^{kr} \quad \textit{such that} \quad f' = f \ \textit{on} \ \partial D \quad \textit{and} \quad f'D_1 \cap \ldots \cap f'D_r = \emptyset.$$

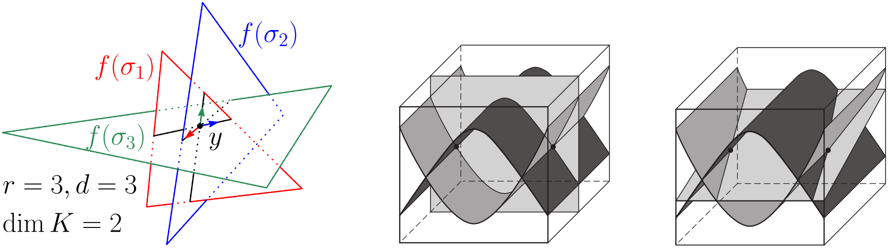

Figure: Triple intersection sign and Whitney trick

This is a simplified r-fold Whitney trick.
The case $r = 2$ is the above Local Disjunction lemma.
The case $r, k \geq 3$ is essentially due to Mabillard-Wagner [MW15].
The case $r \geq 3$, $k = 2$ is due to Avvakumov-Mabillard-Skopenkov-Wagner [AMSW].

Before reading the idea of proof (§3) a reader might want to look at striking applications (§2).
The analogue for $k = 1$ clearly holds when $r = 2$ and fails for each $r \geq 3$ [AMSW].

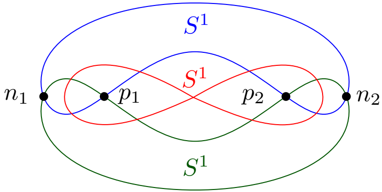

Figure: The boundary of a counterexample to the analogue for $k = 1$

1A. Some applications of double Whitney trick

Theorem (van Kampen 1932; Wu 1955; Shapiro 1957; Weber 1967; Čadek-Krčál-Vokřínek 2013)

If $2d \geq 3k + 3$ (in particular, $d = 2k \geq 6$), then there is an algorithm deciding if given finite k-dimensional simplicial complex (hypergraph) embeds into $\mathbb{R}^d$.

This result is proved using h-**principle for embeddings**, i.e. embeddability criterion involving $\mathbb{Z}_2$-equivariant maps to S^{d-1} from the configuration space $\widetilde{K}$ of ordered pairs of distinct points of a complex K (Haefliger-Weber 1963-67).
The *deleted product* $\widetilde{K} = K^{\underline{2}}$ is

$$\widetilde{K} := \{(x, y) \in K \times K \ : \ x \neq y\}.$$

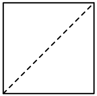

Figure: The deleted product

Suppose that $f : K \to \mathbb{R}^d$ is an embedding of a subset $K \subset \mathbb{R}^m$. Then the map $\widetilde{f} : \widetilde{K} \to S^{d-1}$ is well-defined by the Gauss formula

$$\widetilde{f}(x,y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

We have $\widetilde{f}(y,x) = -\widetilde{f}(x,y)$, i.e. this map is equivariant with respect to the 'exchanging factors' involution $(x,y) \mapsto (y,x)$ on $\widetilde{K}$ and the antipodal involution on S^{d-1}. Thus the existence of an equivariant map $\widetilde{K} \to S^{d-1}$ is a necessary condition for the embeddability of K in $\mathbb{R}^d$.

Theorem (Weber 1967; *h*-principle for embeddings)

If $2d \geq 3k+3$ and K is a finite k-dimensional simplicial complex, then this condition is also sufficient.

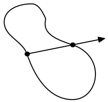

Figure: The Gauss map

Proof of h-principle for embeddings: generalization of Whitney trick to the case when general position multiple intersections have positive dimension (but still codimension $d - k \geq 3$).

Dream since the 1960's: prove analogue of h-principle for embeddings replacing pairs by triples and $d > 3k/2$ by $d > 4k/3$, etc (but still codimension $d - k \geq 3$).

Counterexamples by Segal-Spież 1992, Freedman-Kruskal-Teichner 1994, Segal-S-Spież 1998, Gonçalves-S 2006, see survey [Sk06, §5].

The general h-principle was introduced by Gromov in 1969as a generalizationof the 1959 Smale-Hirsch classification of immersions.

Another problem: the homotopy classes of equivariant maps $\widetilde{K} \to S^{d-1}$ are in general hard to calculate. (The same is true for isovariant maps.) This is formalized by the following result.

Theorem (Matoušek-Tancer-Wagner 2011, de Mesmay-Rieck-Sedgwick-Tancer, 2017)

For every fixed d, k such that $3 \le d \le \frac{3k}{2} + 1$ the algorithmic problem of recognizing PL embeddability of k-dimensional simplicial complexes into $\mathbb{R}^d$ is NP-hard.

The paper [FWZ] announces the analogous result with '*NP*-hard' replaced by 'undecidable' (for $k + 3 \le d < \frac{3k}{2} + 1$). However, there is a mistake exhibited in [Sk20e] and recognized by the authors of [FWZ].

[FWZ] *M. Filakovský, U. Wagner, S. Zhechev, Embeddability of simplicial complexes is undecidable. Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms,* `https://epubs.siam.org/doi/pdf/10.1137/1.9781611975994.47`

[Sk20e] *A. Skopenkov, Extendability of simplicial maps is undecidable, arXiv:2008.00492.*

The h-principle for embeddings for combinatorial (=PL) manifolds *does work* under dimension restrictions weaker than $2d \geq 3k+3$ (S 1997, 2002).

The Haefliger classification of higher-dimensional links (in codimension ≥ 3) includes higher-dimensional Borromean and Brunnian links, i.e. 3-linkings and r-linkings.

There are readily calculable isotopy classifications results for k-manifolds in $\mathbb{R}^d$ when $2d < 3k+3$ (S 2008, 2011, 2019, Crowley-S 2008, 2016, Cencelj-Repovš-M.Skopenkov 2012, M.Skopenkov 2015). Their proofs *do not use* use multiple Whitney trick (although for links perhaps it may be recovered using multiple Whitney trick).

The Goodwillie-Weiss calculus of embeddings [We96] might have a version of Whitney trick behind it [Go90]. So far that approach has not led to any new readily calculable classifications [Sk16c, Remark 2.1], but it gives results on higher homotopy groups of the space of embeddings.

[Go90] *T. Goodwillie, A multiple disjunction lemma for smooth concordance embeddings Memoirs of the AMS, 1990.*

[We96] *M. Weiss, Calculus of embeddings, Bull. of the AMS, 33:2 (1996) 177-187.*

Classification of higher-dimensional ornaments [Me17] (circulated earlier) used a simple version of triple Whitney trick.
[Me17] *S. Melikhov, Gauss type formulas for link map invariants, arXiv:1711.03530.*

The *h*-principle for *almost embeddings* [MW15] (see §2) uses a multiple Whitney trick and was motivated by the topological Tverberg conjecture.
[MW15] *I. Mabillard and U. Wagner, Eliminating Higher-Multiplicity Intersections, I. A Whitney Trick for Tverberg-Type Problems. arXiv:1508.02349.*

We need to speak about PL balls of different dimensions and we will use the word 'disk' for lower-dimensional objects and 'ball' for higher-dimensional ones in order to clarify the distinction (even though, formally, the disk D^d is the same as the ball B^d).

2. Counterexamples to the topological Tverberg conjecture

Any $d+2$ points in d-space can be split into 2 disjoint subsets whose convex hulls intersect (Radon theorem).
Denote by Δ_N the N-dimensional simplex.

Theorem (topological Radon theorem; Bajmóczy-Bárány 1979)

For any d and continuous map $\Delta_{d+1} \to \mathbb{R}^d$ there are two disjoint faces whose images intersect.

This is interesting as a simplicial version of the Borsuk-Ulam theorem.

Usual proof: apply Borsuk-Ulam theorem for certain configuration space, see e.g. excellent book of Matousek 2003.
Simple proof: for a generic map $f : \Delta_{d+1} \to \mathbb{R}^d$ the parity of the number of intersection points in $f\sigma \cap f\tau$ for all non-ordered pairs $\{\sigma, \tau\}$ of disjoint faces does not depend on f, see e.g. the survey [Sk18].
[Sk18] *A. Skopenkov. Invariants of graph drawings in the plane, Arnold Math. J., 6 (2020) 21–55; full version: arXiv:1805.10237.*

Any $3d-2$ points in $\mathbb{R}^d$ can be decomposed into 3 groups such that all the 3 convex hulls of the groups have a common point (Tverberg theorem for $r=3$, 1966).

Conjecture (topological Tverberg conjecture)

For any r,d and any continuous map $f:\Delta_{(d+1)(r-1)}\to\mathbb{R}^d$ there are pairwise disjoint faces $\sigma_1,\dots,\sigma_r\subset\Delta_{(d+1)(r-1)}$ such that $f(\sigma_1)\cap\ldots\cap f(\sigma_r)\neq\emptyset$.

This conjecture was considered a central unsolved problem of topological combinatorics.
The conjecture was proved for r a prime by Bárány-Shlosman-Szűcs (1981), and then for r a prime power by Özaydin (unpublished, 1987) and A. Volovikov (1996). See a short exposition accessible to non-specialists in the survey [Sk16].

Recently and somewhat unexpectedly, it turned out that there are counterexamples for r not a prime power. For the counterexample papers by Özaydin (1987), Gromov (2010), Blagojević-Frick-Ziegler (2014), Frick (2015) and Mabillard-Wagner (2015) are important.

I do not distribute the credits for the counterexample, because the exact description of contribution of particular authors is complex. I describe here the structure of the proof so that a reader could make his/her own opinion. See the details in §2A and in survey [Sk16].

Counterexamples were constructed first for $d \geq 3r+1$ (see above) and then for $d \geq 2r+1$ [AMSW].

For $d < 2r$ (e.g. $d = 2$) and r not a prime power this conjecture is still open. (I did not check the counterexample for $d = 3r$ presented in [MW15]. If correct, it presumably allows to improve the counterexample for $d \geq 2r+1$ to $d = 2r$.)

Counterexamples use a relation to the following r-fold van Kampen–Flores conjecture.

From any $2k+3$ points in $2k$-space one can choose 2 pairwise disjoint collections of $k+1$ points whose convex hulls intersect (linear van Kampen-Flores theorem, 1932).

Theorem (van Kampen-Flores 1932)

For any continuous map $\Delta_{2k+2} \to \mathbb{R}^{2k}$ there are two disjoint k-dimensional faces whose images intersect.

Denote by Δ_N^k the union of k-faced of Δ_N (i.e. the complete $(k+1)$-regular hypergraph on $N+1$ vertices). The theorem implies that Δ_{2k+2}^k is not embeddable into $\mathbb{R}^{2k}$.

Usual proof: apply Borsuk-Ulam theorem for certain configuration space, see e.g. excellent book of Matousek 2003.

Simple proof: for a generic map $f : \Delta_{2k+2}^k \to \mathbb{R}^{2k}$ the parity of the number of intersection points in $f\sigma \cap f\tau$ for all non-ordered pairs $\{\sigma, \tau\}$ of disjoint faces does not depend on f, see e.g. the survey [Sk18].

From any 11 points in 3-space one can choose 3 pairwise disjoint triples whose 3 convex hulls have a common point (linear Sarkaria theorem, 1991).

Conjecture (r-fold van Kampen–Flores)

For any r, k and any continuous map $f\colon \Delta_{(kr+2)(r-1)} \to \mathbb{R}^{kr}$ there are pairwise disjoint $k(r-1)$-dimensional faces $\sigma_1, \ldots, \sigma_r \subset \Delta_{(kr+2)(r-1)}$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

This is true for a prime power r (Sarkaria 1991, Volovikov 1996) and is false for other r (Mabillard-Wagner [MW15] + Özaydin 1987).

Counterexamples were constructed using the multiple Whitney trick (see §1), first for $k \geq 3$ [MW15] and then for $k = 2$ [AMSW].
For $k = 1$ and r not a prime power this conjecture is still open.

A continuous map $f\colon K \to \mathbb{R}^d$ of a simplicial complex K is an **almost r-embedding** if $f\sigma_1 \cap \ldots \cap f\sigma_r = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of K.

Theorem (Counterexample to the r-fold van Kampen–Flores conjecture; Ozaydin 1987 + Mabillard-Wagner 2015)

If r is not a prime power and $k \geq 3$, then there is an almost r-embedding $\Delta^{k(r-1)}_{(kr+2)(r-1)} \to \mathbb{R}^{kr}$.

Lemma (Constraint; Gromov 2010, Blagojević-Frick-Ziegler 2014, Frick 2015)

If there is an almost r-embedding $\Delta^{k(r-1)}_{(kr+2)(r-1)} \to \mathbb{R}^{kr}$, then there is an almost r-embedding $\Delta_{(kr+2)(r-1)} \to \mathbb{R}^{kr+1}$.

Taking $k = 3$ we obtain

Theorem (Counterexample to the topological Tverberg conjecture)

If r is not a prime power, then there is an almost r-embedding $\Delta_{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$.

Let us prove the Constraint Lemma for $r = 6$ and $k = 3$ (this makes the argument more accessible; the general case is analogous).
Take an almost 6-embedding $\Delta_{100}^{15} \to \mathbb{R}^{18}$ of the union of 15-dimensional faces of Δ_{100}. Extend it arbitrarily to a map $f : \Delta_{100} \to \mathbb{R}^{18}$. Denote by $\rho(x)$ the distance from $x \in \Delta_{100}$ to Δ_{100}^{15}.
It suffices to prove that $f \times \rho : \Delta_{100} \to \mathbb{R}^{19}$ is an almost 6-embedding.
Suppose to the contrary that 6 points $x_1, \dots, x_6 \in \Delta_{100}$ lie in pairwise disjoint faces and are mapped to the same point under $f \times \rho$.
Dimension of one of those faces does not exceed $\frac{101}{6} - 1$, so it is at most 15. W.l.o.g. this is the first face, hence $\rho(x_1) = 0$. Then $\rho(x_2) = \ldots = \rho(x_6) = \rho(x_1) = 0$, i.e. $x_1, \dots, x_6 \in \Delta_{100}^{15}$. Now the condition $f(x_1) = \ldots = f(x_6)$ contradicts the fact that $f|_{\Delta_{100}^{15}}$ is an almost 6-embedding.

The following straightforward generalization is required for stronger counterexamples of [AKS].
Let $N = (s+2)r - 2$. If there is an almost r-embedding of the union of s-faces of Δ_N in $\mathbb{R}^{d-1}$, then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.

Theorem ([AKS])

If r is not a prime power and $N := (d+1)r - r\left\lceil \frac{d+2}{r+1} \right\rceil - 2$, then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.

[AKS] *S. Avvakumov, R. Karasev and A. Skopenkov, Stronger counterexamples to the topological Tverberg conjecture, arxiv:1908.08731.*

According to a private communication by F. Frick the bound of [BFZ15, Theorem 5.4] together with the counterexample in [AMSW, Theorem 1.1] gives an almost r-embedding $\Delta_F \to \mathbb{R}^d$ for r not a prime power, d sufficiently large, and F some integer close to $(d+1)r - \frac{r+\frac{1}{2}}{r+1}(d+1)$.

The above theorem provides even stronger counterexamples to the topological Tverberg conjecture: *for d large compared to r we have $N > (d+1)(r-1)$ and even $N > F$.*

Conjecture 5.5 of [BFZ15] states that *for $r < d$ not a prime power there is an almost r-embedding $\Delta_{(d+1)r-2} \to \mathbb{R}^d$ and there are no almost r-embeddings $\Delta_{(d+1)r-1} \to \mathbb{R}^d$.*

(The case $d \le r$ of the conjecture is trivially covered by known results.)

Theorem 5.4 from [BFZ15] is based on the following construction of higher-dimensional counterexamples by taking k-fold join power of lower-dimensional ones.

Lemma (Blagojević-Frick-Ziegler [BFZ15, Lemma 5.2])

If there is an almost r-embedding $\Delta_a \to \mathbb{R}^d$, then for each k there is an almost r-embedding $\Delta_{k(a+1)-1} \to \mathbb{R}^{k(d+1)-1}$.

Proof. For two maps $f : \Delta_a \to B^p$ and $g : \Delta_b \to B^q$ define the *join*

$$f * g : \Delta_{a+b+1} = \Delta_a * \Delta_b \to B^p * B^p = B^{p+q+1}$$

by the formula $(f * g)(\lambda x \oplus \mu y) := \lambda f(x) \oplus \mu f(y)$. A join of almost r-embeddings is an almost r-embedding. Hence the k-fold join power of an almost r-embedding $\Delta_a \to B^d$ is an almost r-embedding $\Delta_{k(a+1)-1} \to B^{k(d+1)-1}$. □

This is analogous to the well-known [Sk16, Lemma 1.5].
We think our counterexamples are mostly interesting because their proof requires non-trivial ideas, see below. Thus we do not spell out even stronger counterexamples which presumably could be obtained by combining our counterexamples with the above procedure of [BFZ15].

Our theorem follows from the above generalization of the Constraint Lemma and the following stronger counterexample to the r-fold van Kampen-Flores conjecture.

By general position, any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\lceil\frac{k+1}{r-1}\rceil}$. A counterexample to the r-fold van Kampen–Flores conjecture asserts that *if r is not a prime power and k is divisible by $r-1$, then any k-complex admits an almost r-embedding in* $\mathbb{R}^{k+\frac{k}{r-1}}$.

Theorem ([AKS])

If r is not a prime power, then any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\lceil\frac{k+3}{r}\rceil}$.

This follows from the Equivariant map Theorem and Metastable global disjunction Theorem below.

Denote by Σ_r the permutation group of r elements. Let $\mathbb{R}^{d\times r} := (\mathbb{R}^d)^r$ be the set of real $d\times r$-matrices. The group Σ_r acts on $\mathbb{R}^{d\times r}$ by permuting the columns. Denote

$$\operatorname{diag}_r = \operatorname{diag}_{r,d} := \{(x, x, \dots, x) \in \mathbb{R}^{d\times r} \mid x \in \mathbb{R}^d\}.$$

Theorem (Equivariant map; [AKS])

If r is not a prime power and X is a complex with a free PL action of Σ_r, then there is a Σ_r-equivariant map $X \to \mathbb{R}^{2\times r} - \mathrm{diag}_r$.

This improves the Özaydin Theorem (see the survey [Sk16, Theorem 3.5]). The proof is analogous to Theorem 5.1 of [AK19]: this Theorem follows by Lemmas below in §2B.

[AK19] *S. Avvakumov, R. Karasev. Envy-free division using mapping degree, arXiv:1907.11183.*

Theorem (Metastable global disjunction; [MW16, Sk17])

Assume that K is a k-complex and $rd \geq (r+1)k + 3$. There exists an almost r-embedding $f : K \to \mathbb{R}^d$ if and only if there exists a Σ_r-equivariant map to $\mathbb{R}^{d\times r} - \mathrm{diag}_r$ from

$$K_\Delta^{\times r} := \bigcup \{\sigma_1 \times \dots \times \sigma_r \ : \sigma_i \text{ a simplex of } K,\ \sigma_i \cap \sigma_j = \emptyset \text{ for every } i \neq j\}.$$

[MW16] *I. Mabillard and U. Wagner. Eliminating Higher-Multiplicity Intersections, II. The Deleted Product Criterion in the r-Metastable Range. arxiv:1601.00876.*
[Sk17] *A. Skopenkov, Eliminating higher-multiplicity intersections in the metastable dimension range, arxiv:1704.00143.*

Proof of the stronger counterexample to the r-fold van Kampen-Flores conjecture. Let K be any k-complex and $d := k + \left\lceil \dfrac{k+3}{r} \right\rceil$. Since r is not a prime power, by the Equivariant map Theorem there is a Σ_r-equivariant map $K_\Delta^{\times r} \to \mathbb{R}^{2\times r} - \mathrm{diag}_r$. The composition of this map with the r-th power of the inclusion $\mathbb{R}^2 \to \mathbb{R}^d$ gives a Σ_r-equivariant map $K_\Delta^{\times r} \to \mathbb{R}^{d\times r} - \mathrm{diag}_r$. We have $rd \geq (r+1)k + 3$. Hence by the Metastable global disjunction Theorem there is an almost r-embedding $K \to \mathbb{R}^d$.

2A. Can anonymous peer review system be used to promote unreasonable opinions which do not stand open discussion?

[BZ16] P. Blagojević, G. Ziegler, in: A Journey Through Discrete Mathematics, 2016. arXiv:1605.07321.

§**1:** In a spectacular recent development, Isaac Mabillard and Uli Wagner [33] [34] have developed an r-fold version of the classical 'Whitney trick' (cf. [50]), which yields the failure of the generalized Van Kampen-Flores theorem when $r \geq 6$ is not a prime power. Then Florian Frick observed that this indeed implies the existence of counterexamples to the topological Tverberg conjecture [26] [13] by a lemma of Gromov [27, p. 445] that is an instance of the constraint method of Blagojević, Frick and Ziegler [12, Lemma 4.1(iii) and Lemma 4.2].

§**5.2:** Based on the work of Mabillard and Wagner [33] [34] we will prove that the generalized Van Kampen-Flores theorem for any r that is not a prime power fails, as demonstrated by Frick [26] [13].

[26] = Frick, Oberwolfach reports, arXiv:1502.00947, and
[13] = Blagojević-Frick-Ziegler, J. Eur. Math. Soc. arXiv:1510.07984.

Explanation why this is misleading: [Sk16] = arXiv:1605.05141, version 2 or higher, §4 (not cited in [BZ16]).

Remark. The survey [Sk16] was sent to Blagojević-Frick-Ziegler / Shlosman before arxiv / journal publication for their potential comments. The criticism of [BZ16] was sent to the authors two months before making it public (and was repeated in a referee report to [BZ16]). Since the papers below were not sent to me before arxiv / journal publication, the authors have chosen my criticism of their papers to be public not private.

[BFZ15] Blagojević-Frick-Ziegler, in: J. Eur. Math. Soc. arXiv:1510.07984, last revised 20 Feb 2019: Using the authors' 2014 "constraints method," we ... show that the "r-fold Whitney trick" of Mabillard and Wagner (2014/2015) implies that the Topological Tverberg Conjecture for r-fold intersections fails dramatically for all r that are not prime powers.

This is misleading because it is not mentioned that Gromov proved the same implication in

[Gro10] *M. Gromov, Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry, Geometric and Functional Analysis 20 (2010), no. 2, 416–526.*

See more criticism of arXiv:1510.07984 in Math Review.

I. Bárány, P. Soberón, arXiv:1712.06119, pp. 10-11:
P. 8: Recently it has been shown by Blagojević, Frick and Ziegler [BFZ14] that the topological Radon theorem implies Van Kampen-Flores. The same implication is mentioned (somewhat implicitly) in Gromov [Gro10] as well.
The 'implicitly' is misleading because Gromov [Gro10] *explicitly* stated a more general result 'The topological Tverberg theorem, whenever available, implies the (generalized) Van Kampen-Flores theorem' (cited in arXiv:1712.06119, p. 10).
P. 10: In 2010, in a groundbreaking paper, Gromov [Gro10] states that 'The topological Tverberg theorem, whenever available, implies the (generalized) Van Kampen-Flores theorem'. This implication holds for any r, prime or not. Gromov also gives a proof (or rather a sketch of proof) in three lines. A detailed proof can be found in [BFZ14].
This is misleading because [BFZ14] (as opposed to [Gro10]) does not even contain a statement of the implication, although it does contain its implicit proof.

Pp. 10-11: Mabillard and Wagner almost succeeded in finding a counterexample to the topological Tverberg conjecture: what was missing was an example where the generalized Van Kampen-Flores theorem fails. It was Florian Frick [Fri15] who realized that the above theorem and Ozaydin's example combined with the constraint method (or Gromov's remark), gives a counterexample for every non-prime power r.

This is misleading because the Özaydin theorem was stated in [MW14, p, 174, Motivation for future work] as '$A(r, K)$ for non-prime power r and any complex K', whereas their own result was stated there as '$A(r, K) \Rightarrow B(r, K)$', where $B(r, \Delta^{k(r-1)}_{(kr+2)(r-1)})$ is a counterexample to the r-fold (=generalized) van Kampen–Flores conjecture.

[MW14] I. Mabillard and U. Wagner, Eliminating Tverberg Points, I. An Analogue of the Whitney Trick, Proceedings of the Thirtieth Annual Symposium on Computational Geometry (SoCG'14), pp. 171–180, ACM, New York, 2014.

So the failure of the r-fold Van Kampen-Flores conjecture is known since [MW14], although it was first explicitly stated by Frick [Fri15].

What was missing between [MW14] and [Fri15] was the Constraint Lemma, although it was explicitly stated in [Gro10] and implicitly proved in [BFZ14], and although the corresponding section [Gro10, 2.9.e] was discussed during the problem session at 2012 Oberwolfach Workshop on Triangulations.

S. Shlosman, in: Russian Math. Surveys, 2018.
arXiv:1804.03120, p. 8: Florian Frick came with counterexamples to TTT, in [F].
This is misleading because Mabillard-Wagner's, Gromov's and Özaydin's contributions are not mentioned.

Less important question: distribution of credits for the counterexample.
See more important questions in Remarks 1e, 2f, 6 of
[Sk21o] *A. Skopenkov. On different reliability standards in current mathematical research, arXiv:2101.03745.*

2B. Idea of proof

Lemma

Let G be a finite group acting on S^n. If there exists a degree zero G-equivariant self-map of S^n, then any complex X with a free action of G has a G-equivariant map $X \to S^n$.

See the historical remarks and a proof in [AK19, §5]. In particular, this lemma follows from [Ba93, Lemma 3.9], although to read the direct proof in [AK19, §5] is simpler than to find the notation required for [Ba93, Lemma 3.9] and make such a deduction.

[Ba93] *T. Bartsch. Topological methods for variational problems with symmetries, Lect. Notes in Math., 1560, Springer, Berlin, 1993.*

Denote by $S^{d(r-1)-1}_{\Sigma_r} \subset \mathbb{R}^{d\times r} - \delta_r$ the set formed by all $d \times r$-matrices in which the sum of the elements in each row is zero, and the sum of the squares of all the matrix elements is 1. This set is invariant under the action of Σ_r. This set is homeomorphic to the sphere of dimension $d(r-1)-1$.

Lemma (Main)

If r is not a prime power, then there is a degree zero Σ_r-equivariant self-map of $S^{2r-3}_{\Sigma_r} = S^{2(r-1)-1}_{\Sigma_r}$.

Proof of the Main Lemma. Since r is not a prime power, the greatest common divisor of the binomial coefficients $\binom{r}{k}$, $k = 1, \ldots, r-1$ is 1. Hence -1 is an integer linear combination of the binomial coefficients. Denote by $C \subset S^{2r-3}_{\Sigma_r}$ the set of $2 \times r$-matrices whose second row is zero, and the entries of the first row involve only two numbers. A *special* map is a Σ_r-equivariant self-map f of $S^{2r-3}_{\Sigma_r}$ which is a local homeomorphism in some neighborhood of C. The identity map of $S^{2r-3}_{\Sigma_r}$ is a special map of degree 1. Thus the lemma is implied by the following assertion.

For any r, any $k = 1, \ldots, r-1$ and any special map f there are special maps f_+, f_- such that $\deg f_\pm = \deg f \pm \binom{r}{k}$.

Idea of the proof of the assertion. Denote by $\mathbb{R}^{2r-2}_{\Sigma_r}$ the subspace of $\mathbb{R}^{2\times r}$ of $2 \times r$-matrices for which the sum in each row zero. We construct an equivariant homotopy

$$h : S^{2r-3}_{\Sigma_r} \times I \to \mathbb{R}^{2r-2}_{\Sigma_r}$$

of the composition h_0 of f with the standard inclusion.

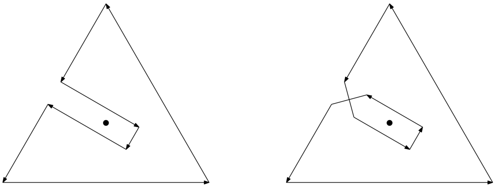

Figure: The homotopy 'pushes' certain point $c \in C$ towards the origin in $\mathbb{R}^{2\times r}$ so that the origin is a regular value of h

The images of h_0 and h_1 miss the origin. Apply the central projection from the origin to define for $t = 0, 1$ the equivariant map

$$f_t : S^{2r-3}_{\Sigma_r} \to S^{2r-3}_{\Sigma_r} \quad \text{by} \quad f_t(x) = \frac{h_t(x)}{|h_t(x)|}.$$

Then $f_0 = f$. We construct f_1 (i.e., c and h) in two ways $f_{1,+}, f_{1,-}$ so that the difference $\deg f_1 - \deg f_0$ is $\pm\binom{r}{k}$. More precisely, we obtain $\deg f_{1,\pm} - \deg f_0 = \pm\binom{r}{k} \deg_c f$, where $\deg_c f \in \{+1, -1\}$ is the local degree of f at c. We use the fact that this difference equals to the local degree of h at the origin, which is a regular value of h. The construction of $f_{1,-}$ is easier, while for $f_{1,+}$ we use the reflection w.r.t. a certain hyperplane.

Let us realize the above idea. The objects we construct depend on r, k but we suppress r, k from their notation. Define the vector

$$M := (\underbrace{k-r, \ldots, k-r}_{k}, \underbrace{k, \ldots, k}_{r-k}).$$

Define the $2 \times r$-matrix $c := \begin{pmatrix} M/|M| \\ 0 \end{pmatrix} \in S^{2r-3}_{\Sigma_r}$. The orbit $\Sigma_r c$ of c contains $\binom{r}{k}$ points. The stabilizer of c is $G := \Sigma_k \times \Sigma_{r-k} \subset \Sigma_r$. The standard metric on the sphere is Σ_r-invariant. Hence there is a small ball U centered at c such that $U \cap \sigma U = \emptyset$ for any $\sigma \in \Sigma_r - G$ and $\sigma U = U$ for any $\sigma \in G$. Take a smooth function $\rho : S^{2r-3}_{\Sigma_r} \to [0,1]$ which is zero outside $\Sigma_r U$, invariant with respect to the Σ_r-action, and equal to 1 in a neighborhood of $\Sigma_r c$.

Construction of f_{1-}. Define

$$h(x,t) = h_t(x) := \begin{cases} f(x) & x \notin \Sigma_r U \\ f(x) - 2t\rho(x) f(\sigma c) & x \in \sigma U, \ \sigma \in \Sigma_r. \end{cases}$$

Clearly, h is well-defined and is continuous.
If $h_t(x) = 0$, then $x \in \Sigma_r U$. Since f is a local homeomorphism, we have $2t\rho(x) = 1$ and $x = \sigma c$ for some $\sigma \in \Sigma_r$. Then $t = 1/2$. Therefore

$$\deg f_1 - \deg f_0 = \binom{r}{k} \deg_{(c,1/2)} h = -\binom{r}{k} \deg_c f.$$

Here the second equality holds because $\rho = 1$ in a neighborhood of c. Since $f_0 = f$ is a local homeomorphism in some neighborhood of C, the map $f_{1-} := f_1$ is such in a neighborhood of $C - \Sigma_r c$. In a neighborhood of σc the map f_1 is a shift by $-2f(\sigma c)$ composed with the central projection back to the sphere. This is clearly a homeomorphism in a neighborhood of σc.

3. Idea of proof of r-fold Local Disjunction Lemma

Before I sketch the idea of proof, let me recall the statement.
Denote by $D = D_1 \sqcup D_2 \sqcup D_3$ the disjoint union of three $2k$-disks.

Lemma (3-fold Local Disjunction 2015)

If $k \geq 2$ and $f : D \to B^{3k}$ is a proper general position map such that $fD_1 \cdot fD_2 \cdot fD_3 = 0$, then there is a proper general position map

$$f' : D \to B^{3k} \quad \text{such that} \quad f' = f \text{ on } \partial D \quad \text{and} \quad f'D_1 \cap f'D_2 \cap f'D_3 = \emptyset.$$

If two triple points of opposite signs in $fD_1 \cap fD_2 \cap fD_3$ are contained in one connected component of $fD_1 \cap fD_2$, then we can 'cancel' them by double Whitney trick applied to $fD_1 \cap fD_2$ and fD_3 (see fig. left, where fD_1 is the square section and fD_2, fD_3 are curvilinear sections). If not (fig. right), then we need to first achieve this property by an analogue of double Whitney trick applied to fD_1 and fD_2.
This is analogous to 'surgery of the intersection' $fD_1 \cap fD_2$ as described by Haefliger 1963, Habegger-Kaiser 1998, Cencelj-Repovš-M.Skopenkov 2012, Melikhov 2017. Application of this construction is non-trivial and is an important achievement of Mabillard and Wagner.

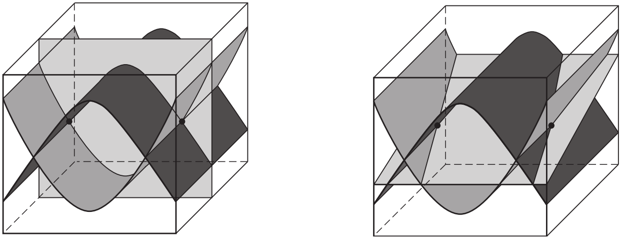

Figure: Why triple Whitney trick is non-trivial?

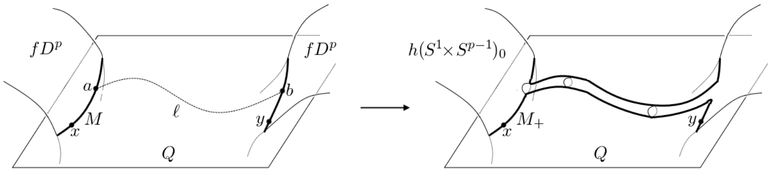

Figure: Surgery of the intersection $fD_1 \cap fD_2$: piping

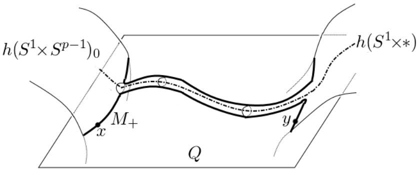

Figure: Surgery of the intersection $fD_1 \cap fD_2$: unpiping

Let me present the r-fold Whitney trick when general position r-tuple intersections have positive dimension. See applications in §2.
Denote by $D = D_1 \sqcup \ldots \sqcup D_r$ the disjoint union of r disks of dimension $k(r-1)$.
Let $B^{d\times r} := (B^d)^r$. Denote

$$\delta_r := \{(x, x, \ldots, x) \in \mathbb{R}^{d\times r} \mid x \in \mathbb{R}^d\}.$$

Theorem (Metastable r-fold local disjunction; Mabillard-Wagner 2016, S 2017)

Assume that $rd \geq (r+1)k + 3$ and $f : D \to B^d$ a proper map such that $f\partial D_1 \cap \ldots \cap f\partial D_r = \emptyset$. If the map

$$f^r : \partial(D_1 \times \ldots \times D_r) \to B^{d\times r} - \delta_r$$

extends to $D_1 \times \ldots \times D_r$, then there is a proper

$$\overline{f} : D \to B^d \quad \text{such that} \quad \overline{f} = f \quad \text{on} \quad \partial D \quad \text{and} \quad \overline{f} D_1 \cap \ldots \cap \overline{f} D_r = \emptyset.$$

Passage from the case $(r-1)d = rk$ of [MW15, AMSW] to the case $rd \geq (r+1)k+3$ considered here is non-trivial because here general position r-tuple intersections are no longer isolated points. This makes surgery of intersection more complicated. More importantly, this brings in 'extendability of f^r' obstruction, which is harder to work with than the 'sum of the signs of the global r-fold points' integer obstruction.

The proof uses an application to PL topology of differential topology (which is unusual). Generalization to enough highly-connected stably parallelizable manifolds instead of disks, and to a realization theorem. Smale-Hirsch immersion theory; studies of homotopy classes as framed bordism classes (Pontryagin construction).

Denote CAT = PL or Diff.
A **CAT** $(n_1, \dots, n_r)$-**Whitney map** is a proper CAT map $f : N \to B^d$ of disjoint union $N = N_1 \sqcup \dots \sqcup N_r$ of r smooth compact manifolds, possibly with boundary, of dimensions $n_1, \dots, n_r$ such that

$$(*) \qquad rd \geq (n_1 + n_2 + \dots + n_r) + n_i + 3 \quad \text{for each } i.$$

Denote

$$w := n_1 + n_2 + \dots + n_r - (r-1)d.$$

This is the dimension of general position r-tuple intersection $fN_1 \cap \dots \cap fN_r$. The inequality (*) is equivalent to $d \geq n_i + w + 3$ (which is more convenient to use).

For a CAT Whitney map $f : N \to B^d$ set $N^\times := N_1 \times \ldots \times N_r$ and define a map

$$f^r : N^\times \to B^{d\times r} \quad \text{by} \quad f^r(x_1, \ldots, x_r) := (fx_1, \ldots, fx_r).$$

We use the same notation for maps defined by the same formula on subsets $N^\times$ and assuming values in $B^{d\times r} - \delta_r$; since the domain and the range are specified, no confusion would appear.

Theorem (Metastable Local Disjunction)

Let $f : N \to B^d$ be a CAT $(n_1, \ldots, n_r)$-Whitney map of disjoint union of $(w+1)$-connected stably parallelizable manifolds such that $f\partial N_1 \cap \ldots \cap f\partial N_r = \emptyset$. There exists a CAT map $\overline{f} : N \to B^d$ such that

$$\overline{f} = f \quad \text{on} \quad \partial N \quad \text{and} \quad \overline{f}N_1 \cap \ldots \cap \overline{f}N_r = \emptyset$$

if and only if the map $f^r : \partial(N^\times) \to B^{d\times r} - \delta_r$ extends to $N^\times$.

An s-**frimmersion** is a proper smooth framed immersion $N_1 \sqcup \ldots \sqcup N_r \to B^d$ of disjoint union of smooth manifolds whose restrictions to the components are transverse to each other, $f|_{N_i}$ is self-transverse for each $i = 1, \ldots, r-1$ and

$$\Sigma(f|_{N_r}) := \{x \in N_r \ : \ |f|_{N_r}^{-1} fx| > 1\}$$

is the image of a self-transverse immersion of a finite disjoint union of manifolds of dimensions at most s.

Proposition

Let $f : N \to B^d$ be an $(n_1, \ldots, n_r)$-Whitney $(n_r - w - 3)$-frimmersion such that
(1) $f\partial N_1 \cap \ldots \cap f\partial N_r = \emptyset$,
(2) the map $f^r : \partial N^\times \to B^{d\times r} - \delta_r$ extends to $N^\times$, and
(3) if $r \geq 3$, then every N_i is $(w+1)$-connected.
Then there exists an immersion $\overline{f} : N \to B^d$ such that

$$\overline{f} = f \quad \textit{on} \quad N_r \cup \partial N \quad \textit{and} \quad \overline{f} N_1 \cap \ldots \cap \overline{f} N_r = \emptyset.$$

Lemma (Surgery of Intersection)

Let P and Q be k-connected smooth p- and q-manifolds, possibly with boundary. Assume that $f : P \sqcup Q \to B^d$ is a $(q-k-2)$-frimmersion and

$$k \leq \min\{d-p-2, (p+q-d-2)/2\}.$$

Then there is a $(q-k-2)$-frimmersion $f_1 : P \sqcup Q \to B^d$ such that $f_1 = f$ on $\partial P \sqcup Q$, and

$$M(f_1) := \{(x,y) \in P \times Q \ : \ f_1 x = f_1 y\}$$

is a smooth k-connected $(p+q-d)$-manifold.