WHAT IS KNOT THEORY ABOUT [1]

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Abstract. We try to give a motivated definition of the Arf-Casson and Alexander-Conway invariants of links. We state the Vassiliev-Kontsevich Theorem in a way convenient for calculation of the invariants themselves, not only of the dimension of the space of the invariants.

Contents

1 Some basic problems and constructions

We use definitions from first sections of [P95], [PS96], [CDM12].

Theorem 1. (a) *The trefoil knot [P95, Fig. 2.1] is not isotopic [PS96, §1] to the standard knot.*

(b) *The 8-figure knot [P95, Fig. 2.3] is not isotopic to the standard knot.*

This and Theorem 15 is proved using *Arf invariant*, cf. §2. In the following problem we sketch an alternative proof for (a) (which does not work for (b)).

Problem 2. (a) Find the number of regular colorings [P95, §4] of the standard diagram of the standard knot.

(b) The same for the standard diagram of the trefoil knot [P95, Fig. 2.1].

(c) The same for the standard diagram of the 8-figure knot [P95, Fig. 2.3].

(d) The number of regular colorings of a planar knot diagram is preserved under the *Reidemeister moves* [P95, Fig. 4.1].

(e) The trefoil knot is not isotopic to the 8-figure knot.

(f) Neither of links in [P95, Fig. 4.5] is isotopic to the standard link.

Problem 3. (a) An oriented circle on the sphere is isotopic (on the sphere) to the same circle with the opposite orientation.

(b) Analogous assertion for the plane is false.

Any oriented circle on the torus is not isotopic (on the torus) to the same circle with the opposite orientation. (This is proved using homology.)

Problem 4. *Invertibility.* (a) The standard oriented circle in space is isotopic (in space) to the same circle with the opposite orientation.

(b) Two trefoil knots with the opposite orientations are isotopic.

(c) Two 8-figure knots with the opposite orientations are isotopic.

Theorem 5. (H. Trotter, 1964) *There exists an oriented knot which is not isotopic to the same knot with the opposite orientation.*

The proof is outside the purpose of these lecture notes.

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Problem 6. *Amphisphericity, or achirality.* The 8-figure knot is isotopic to its mirror image.

Theorem 7. *The trefoil knot is not isotopic to its mirror image.*

This is proved using *the Jones poynomial* [PS96], [CDM12].

Problem 8. *Crossing number.* (a) Every planar diagram of a knot with at most 2 crossing is a diagram of the trivial knot.

(b) The 8-figure knot has no planar diagrams with 3 crossings.

(c) The number of crossings in a reducible diagram can be decreased.

Theorem 9. (Tait conjecture of 1898) *An irreducible alternating diagram has the minimal number of crossings among all diagrams of a given knot.*

This was proved using *the Jones poynomial*; the proof is outside the purpose of these lecture notes.

Problem 10. The *stick number* of a knot K is the least number of rectilinear segments in a knot isotopic to K.

(a,b,c) Each of the stick numbers of the trefoil knot, the figure eight knot and the 5_1 knot [PS96, Fig 2.4, 3.13] does not exceed 6.

(a',b',c') These numbers equal 6.

Problem 11. (a) Each diagram of a knot can be transformed to a diagram of the trivial knot by *crossing changes*. [PS96, Theorem 3.8]

The *unknotting number* of a knot K is the least number of crossing changes in some diagram of K which transform the diagram to a diagram of the trivial knot.

(b,c) Find the unknotting number of the trefoil knot and the figure eight knot.

(d) Give an example of a knot whose unknotting number is 2.

The *connected sum* # of knots is defined in [PS96, §1, p.9], cf. [S, Remark 2.3.a].

Problem 12. (a) $K\#O = K$. (b) $K\#L = L\#K$. (c) $(K\#L)\#M = K\#(L\#M)$.

(d) The knots in [PS96, §1, Figure 1.17] are composite [PS96, §1, p.14].

(e)* $K\#L$ is non-trivial if K is not. (f)* If $K\#L = K\#M$, then $L = M$.

(g) $\operatorname{lk}(K\#L) = \operatorname{lk} K + \operatorname{lk} L$ for links K, L with 2 components.

Problem 13. Is the connected sum well-defined for two-component links

(a) non-oriented non-ordered? (b) oriented non-ordered?

(c)* non-oriented ordered? (d)* oriented ordered?

Problem 14. (a) A knot spans an embedded disk if and only if the knot is trivial.

The *genus* of a knot is the minimal genus of its *Seifert surfaces.*

(b,c*) Find the genus of the trefoil knot and the figure eight knot.

Theorem 15. *If the graph K_7 is piecewise linearly embedded in space, then there exists a knotted cycle in this graph.*

2 Gauss, Arf-Casson and Alexander-Conway invariants

Denote by K_+, K_-, K knots and link from [CDM12, 2.3.1].

The reader is recommended to prove first mod2-analogues of the following results and problems.

Theorem 16. (Gauss) *There is a unique integer-valued invariant* lk *of oriented links that assumes value 0 on the trivial link and for which* $\operatorname{lk}(K_+) - \operatorname{lk}(K_-) = 1$.

This invariant is called *the linking coefficient.*

Hint for the existence: lk is defined in [Sk, §4.2, §.3 'Linking modulo 2' and 'Linking number'].

Hint for the uniqueness: use analogue of Problem 11.a.

Theorem 17. (Arf-Casson) *There is a unique integer-valued invariant c_2 of non-oriented knots that assumes value 0 on the trivial knot and for which* $c_2(K_+) - c_2(K_-) = \mathrm{lk}(K)$

This invariant is called *Casson invariant* and its reduction modulo 2 is called *Arf invariant*.

Arf invariant of a knot is the parity of the number of all those non-ordered pairs of crossing arrows (in a based *Gauss diagrams* of a knot [CDM12, 1.8.4]) for which the base point is situated between the ends of the arrows. Casson invariant is defined analogously.

Hint for the uniqueness: use Problem 11.a.

In this section Theorem 17 can be used without proof.

Problem 18. Find the Casson invariant of the following knots.

(a) the trefoil knot; (b) the 8-figure knot.

Problem 19. Draw a knot whose Casson invariant is 10.

Problem 20. (a) The Casson invariants of mirror-symmetric knots are the same.

(b) $c_2(K\#L) = c_2(K) + c_2(L)$.

Problem 21. If C is a $\mathbb{Z}[t]$-valued invariant of oriented links that assumes value 1 on the trivial knot and $a, b, c \in \mathbb{Z}[t]$ are such that $aC(K_+) + bC(K_-) = cC(K)$, then $a = -b = 1$ and $c = t$.

Hint: analogously to [PS96, §3.1].

Theorem 22. (Alexander-Conway) *There is a unique $\mathbb{Z}[t]$-valued invariant C of oriented links that assumes value 1 on the trivial knot and for which $C(K_+) - C(K_-) = tC(K)$.*

This invariant is called *Conway polynomial*.

Hint for the uniqueness: [PS96, Theorem 3.8].

In this section Theorem 22 can be used without proof. See also `http://en.wikipedia.org/wiki/Alexander_polynomial#Geometric_significance_of_the_polynomial`

Problem 23. Find the Conway polynomials of the following links, with some orientation on their components.

(a) the unlink with 2 components; (b) the unlink with n components;

(c) the Hopf link; (d) the trefoil knot; (e) the 8-figure knot;

(f) the Whitehead link; (g) the Borromean rings.

Problem 24. (a) The Conway polynomials of inverse knots are the same.

(b) The Conway polynomials of mirror-symmetric knots are the same.

Problem 25. (a) The link of Fall 2013-2-11 is *split* [CDM12, Ex. 4 in p. 64].

(b) The Conway polynomials of the standard link is trivial.

(c) The Whitehead link is not isotopic to the standard link.

3 Vassiliev invariants

Denote by

• Σ the set of isotopy classes of singular knots [PS96, 4.1],

• δ_n the set of all chord diagrams that have n chords [PS96, 4.8];

• $\sigma(K)$ the *chord diagram* of a singular knot K [PS96, 4.8], [CDM12, 3.4.1] (not to be confused with *Gauss diagrams* for a non-singular knot K [CDM12, 1.8.4]).

Theorem 26 (Vassiliev-Kontsevich, [PS96], [CDM12])**.** *For any map $\lambda : \delta_n \to \mathbb{R}$ there exists a map $v : \Sigma \to \mathbb{R}$ such that*

(1) $v(K_+) - v(K_-) = v(K_0)$ for each singular knots K_+, K_- and K_0 from [PS96, (4.1)],

(2_n) $v(K) = 0$ for each singular knot that has more than n double points, and

(3) $v(K) = \lambda(\sigma(K))$ for each singular knot that has exactly n double points,

if and only if λ satisfies to the 1-term and the 4-term relations [PS96, (4.5),(4.6)].

A map $v : \Sigma \to \mathbb{R}$ such that (1) holds is called a *Vassiliev invariant.*

A map $v : \Sigma \to \mathbb{R}$ such that (2_n) holds is called a *map of order at most* n.

Problem 27. (a) The map v of Theorem 26 is unique up to Vassiliev invariant of order at most $n-1$. More precisely, the difference between maps $v, v' : \Sigma \to \mathbb{R}$ satisfying to (1), (2_n) and (3), satisfies to (1) and (2_{n-1}).

(b) Prove the 'only if' part of Theorem 26.

(0),(1),(2),(3)* Prove the 'if' part of Theorem 26 for $n = 0, 1, 2, 3$.

Hint: for $n = 2$ use Theorem 17, for $n = 3$ use the coefficient of h^3 in $J(e^h)$, where J is the Jones polynomial in t-parametrization [CDM12, 2.4.2, 2.4.3].

In the remaining problems Theorem 26 can be used without proof. Assertion '$v(K) = x$ for any singular knot K whose chord diagram is a' is shortened to '$v(a) = x$'.

Problem 28. (a) There exists a unique Vassiliev invariant $v_2 : \Sigma \to \mathbb{R}$ of order at most 2 such that

• $v_2(O) = 0$ for the trivial knot O, and

• $v_2(1212) = 1$ ((1212) is the 'non-trivial diagram with 2 chords' [PS96, Figure 4.4], 3rd diagram of the first line).

Warning: in this problem it is allowed to use Theorem 26 but not Theorem 17.

(b,b',c,d) Calculate v_2 for the right trefoil, left trefoil, figure 8 knot and the 5_1 knot.

Problem 29. (a) There exists a unique Vassiliev invariant $v_3 : \Sigma \to \mathbb{R}$ of order at most 3 such that

• $v_3(O) = 0$ for the trivial knot O and for the left trefoil O, and

• $v_3(123123) = 1$ ((123123) is the 'non-trivial most symmetric diagram with 3 chords', [PS96, Figure 4.4], 5th diagram of the second line).

(b,c,d*) Calculate v_3 for the right trefoil, figure 8 knot and the 5_1 knot.

Problem 30. (a) There exists a unique Vassiliev invariant $v_4 : \Sigma \to \mathbb{R}$ of order at most 4 such that

• $v_4(O) = 0$ for the trivial knot O, for the left trefoil O, and for the right trefoil O,

• $v_4(12341234) = 2$, $v_4(12341432) = 3$ and $v_4(12341423) = 5$ [PS96, Problem 4.4.b].

(c*,d*) Calculate v_4 for the figure 8 knot and the 5_1 knot.

References

[CDM12] *S. Chmutov, S. Duzhin, J. Mostovoy.* Introduction to Vassiliev knot invariants, Cambridge Univ. Press, 2012. `http://www.pdmi.ras.ru/~duzhin/papers/cdbook`

[P95] *V. V. Prasolov.* Intuitive topology. Amer. Math. Soc., Providence, R.I., 1995. Russian version: `ftp://ftp.mccme.ru/users/prasolov/nagltop/pra2.pdf`

[PS96] *V. V. Prasolov, A. B. Sossinsky.* Knots, Links, Braids, and 3-manifolds. Amer. Math. Soc. Publ., Providence, R.I., 1996. Russian version: `http://www.mccme.ru/prasolov`

[S15] *A. Skopenkov.* Classification of knotted tori, `http://arxiv.org/abs/1502.04470`

[S] *A. Skopenkov.* Algebraic Topology From Algorithmic Viewpoint, draft of a book, `www.mccme.ru/circles/oim/algor.pdf`