INVARIANTS OF GRAPH DRAWINGS IN THE PLANE

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Abstract

We present a simplified exposition accessible to non-specialists of some classical results on graph drawings in the plane. These results are chosen so that they illustrate some spectacular recent higher-dimensional results on the border of topology and combinatorics. We define a mod2-valued self-intersection invariant, and construct a polynomial algorithm for recognizing graph planarity. Motivated by algorithmic, combinatorial and geometric problems, we introduce starting ideas of algebraic topology.

This talk is based on part of the paper under the same title, Arnold Math. J., 6 (2020) 21–55; full version: arXiv:1805.10237.

Introduction

The main contents of this survey is an introduction to starting ideas of algebraic topology (more precisely, to configuration spaces and cohomological obstructions) motivated by algorithmic, combinatorial and geometric problems. I present elementary formulations and arguments that do not involve configuration spaces and cohomological obstructions. Thus no knowledge of algebraic topology is required here. Important ideas are introduced in non-technical particular cases and then generalized. So this survey is accessible to mathematicians not specialized in the area.

I expose a classical polynomial algorithm for recognizing graph planarity, together with all the necessary definitions, some motivations and preliminary results. This algorithm is interesting because it can be generalized to graphs in surfaces, to higher dimensions, and to higher multiplicity intersections.

Planarity of graphs

Let me recall some standard results on graphs in the form convenient to higher-dimensional generalizations.

Proposition

From any 5 points in the plane one can choose two disjoint pairs such that the segment with the ends at the first pair intersects the segment with the ends at the second pair.

Theorem (General Position)

For any n there exist n points in 3-space such that no segment joining the points intersects the interior of any other such segment.

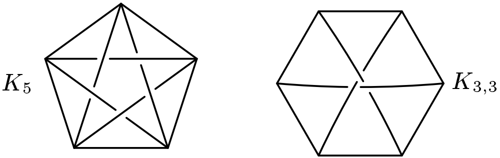


Figure: Nonplanar graphs K_5 and $K_{3,3}$

A **graph** (V, E) is a finite set V together with a collection $E \subset \binom{V}{2}$ of two-element subsets of V (i.e. of non-ordered pairs of elements).

A graph (V, E) is called **linearly realizable** in the plane if there exists $|V|$ points in the plane corresponding to the vertices so that no segment joining a pair (of points) from E intersects the interior of any other such segment.

A graph (V, E) is called **planar** (or piecewise-linearly realizable in the plane) if in the plane there exist

- $|V|$ points corresponding to the vertices, and
- simple polygonal lines joining pairs (of points) from E

such that no of the polygonal lines intersects the interior of any other polygonal line.

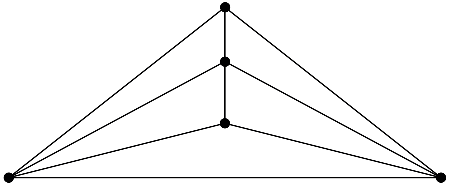


Figure: Linear realization of K_5 without one of the edges

Theorem

There is an algorithm for recognizing graph planarity, which is linear in the number of vertices n in the graph.

Observe that K. Kuratowski Theorem gives 'galactic complexity' algorithm, and the algorithm based on considering *thickenings* or *graphs with rotations* has 'exponential complexity'. The linear algorithm of J. Hopcroft, R. Tarjan (1974) is based on reduction to planarity of blocks. In this talk I present a (classical) polynomial algorithm which (as opposed to the above algorithms) generalizes

• to higher dimensions (see below),

• to *algebraic embeddability* of graphs to 2-dimensional surfaces, see papers by M. Schaefer (2013), R. Fulek, J. Kynčl (arXiv:1903.08637), and A. Bikeev (arXiv:2012.12070), and

• to higher multiplicity intersections, see I. Mabillard, U. Wagner (arXiv:1508.02349), S. Avvakumov, I. Mabillard, AS, U. Wagner (arxiv:1511.03501), I. Mabillard, U. Wagner (arXiv:1601.00876), AS (arxiv:1704.00143), and a survey AS (arXiv:1605.05141).

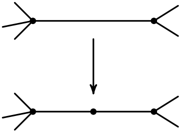


Figure: Subdivision of edge

The *subdivision of edge* operation for a graph is shown in the figure. Two graphs are called *homeomorphic* if one can be obtained from the other by subdivisions of edges and inverse operations. This is equivalent to the existence of a graph that can be obtained from each of these graphs by subdivisions of edges. *A graph is planar if and only if some graph homeomorphic to it is linearly realizable in the plane.*

Hypergraphs and complexes

A k-**hypergraph** (more precisely, k-dimensional, or $(k+1)$-uniform, hypergraph) (V, F) is a finite set V together with a collection $F \subset \binom{V}{k+1}$ of $(k+1)$-element subsets of V.

In topology it is more traditional (because sometimes more convenient) to work not with hypergraphs but with *complexes*. The following results are stated for complexes, although some of them are correct for hypergraphs.

A **complex** $K = (V, F)$ is a finite set $V = V(K)$ together with a collection $F = F(K) \subset 2^V$ of subsets of V such that if a subset σ is in the collection, then each subset of σ is in the collection. In an equivalent geometric language, a complex is a collection of closed faces of some simplex. A k-**complex** is a complex containing at most $(k+1)$-element subsets, i.e. at most k-dimensional simplices. Elements of V and of F are called **vertices** and **faces**. An *edge* is a 2-element (t.e., 1-dimensional) face.

For instance, the **complete k-complex on n vertices** (or the k-skeleton of the $(n-1)$-simplex) $\Delta_{n-1}^k := ([n], \binom{[n]}{\le k+1})$ is the collection of all at most $(k+1)$-element subsets of an n-element set. For $k = 0$ we denote this complex by $[n]$, for $n = k+1$ by D^k (k-simplex or k-disk), and for $n = k+2$ by S^k (k-sphere).

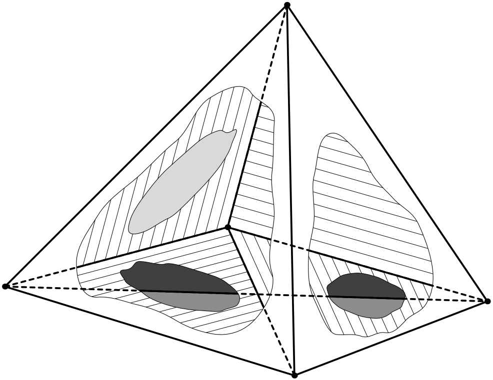


Figure: Realization in $\mathbb{R}^3$ of the complete two-homogeneous hypergraph on 5 vertices, i.e. of the union of 2-faces of 4-dimensional simplex.

Simplicial embeddability of complexes

Realizability of hypergraphs or complexes in $\mathbb{R}^d$ is defined similarly to the realizability of graphs in the plane. E.g. for 2-complex one 'draws' a triangle for every three-element subset.

A complex (V, F) is **simplicially** (or linearly) **embeddable** in $\mathbb{R}^d$ if there is a set V' of distinct points in $\mathbb{R}^d$ corresponding to V such that for any subsets $\sigma, \tau \subset V'$ corresponding to elements of F the convex hull $\langle\sigma\rangle$ is a simplex of dimension $|\sigma| - 1$ and $\langle\sigma\rangle \cap \langle\tau\rangle = \langle\sigma \cap \tau\rangle$.

Theorem (General Position)

Any k-complex is simplicially embeddable in $\mathbb{R}^{2k+1}$.

Here the number $2k + 1$ is the least possible:

Theorem (E. van Kampen 1932)

For any continuous (PL) map of the $(2k + 2)$-dimensional simplex to $\mathbb{R}^{2k}$ there are two disjoint k-dimensional faces whose images intersect.

For every fixed d, k there is an algorithm for recognizing the simplicial embeddability of k-complexes in $\mathbb{R}^d$. For estimations of complexity see M. Abrahamsen, L. Kleist, T. Miltzow, arXiv:2108.02585.

Piecewise linear (PL) embeddability of complexes

The *subdivision of an edge* operation is shown in the figure. Exercise: represent the *subdivision of a face* operation as composition of several subdivisions of an edge and inverse operations. A **subdivision** of a complex K is any complex which can be obtained from K by several subdivisions of its edges. A complex is **PL embeddable** in $\mathbb{R}^d$ if some its subdivision is simplicially embeddable in $\mathbb{R}^d$.

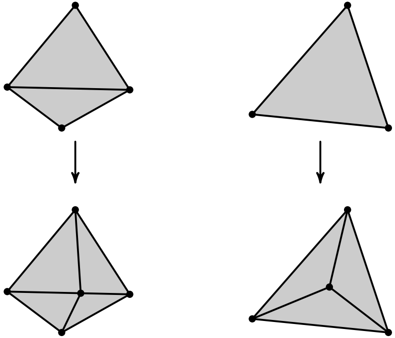


Figure: Subdivision of an edge (left) and of a face (right)

Algorithmic results on PL embeddability

Theorem

For every fixed d, k such that either $k = 2 \neq d - 2$ or $d \geq \frac{3k+3}{2}$ there is an algorithm for recognizing the PL embeddability of k-complexes in $\mathbb{R}^d$.

This theorem

• for $k = d = 2$ (even with linear algorithm) follows from the Kuratowski-type Halin-Jung planarity criterion for 2-complexes.

• for $k = d - 1 = 2$ is proved by J. Matoušek, E. Sedgwick, M. Tancer, U. Wagner, arXiv:1402.0815.

• for $d \geq \frac{3k+3}{2}$ (even with polynomial algorithm) follows from and the Haefliger-Weber 'configuration spaces' criterion for embeddability of complexes (stated e.g. in the survey arXiv:math/0604045, §5) and a result of M. Čadek, M. Krčál, L. Vokřínek, arXiv:1307.6444.

The algorithm for $d = 2k \geq 6$ generalizes the algorithm for $d = 2k = 2$ presented below.

Theorem

For every fixed d, k such that $5 \leq d \in \{k, k+1\}$ there is no algorithm recognizing PL embeddability of k-complexes in $\mathbb{R}^d$.

This is deduced in arXiv:0807.0336 from S. Novikov's theorem on unrecognizability of the sphere. The analogue of this for $8 \leq d \leq \frac{3k+1}{2}$ is published by M. Filakovský, U. Wagner, S. Zhechev but contains a mistake (see survey arXiv:2008.00492, §3; see also R. Karasev, AS, arXiv:2008.02523).

Theorem

For every fixed d, k such that $3 \leq d \leq \frac{3k}{2} + 1$ the algorithmic problem of recognizing PL embeddability of k-complexes in $\mathbb{R}^d$ is NP-hard.

This is proved for $d \geq 4$ and $d = 3$ by J. Matoušek, M. Tancer, U. Wagner (arXiv:0807.0336) and by A. de Mesmay, Y. Rieck, E. Sedgwick, M. Tancer (arXiv:1708.07734), respectively. A simpler exposition (and a generalization) for $d \geq 4$ is given by AS, M. Tancer (arXiv:1703.06305). The proof for $d \geq 4$ uses the construction of counterexamples to the 'configuration space' criterion for embeddability of complexes (J. Segal, S. Spież, 1992; M. Freedman, V. Krushkal and P. Teichner, 1994; J. Segal, AS, S. Spież, 1998).

The van Kampen (self-intersection) number

We shall consider plane drawings of a graph such that the edges are drawn as polygonal lines and intersections are allowed. Let us formalize this for graph K_n. **A PL map** $f : K_n \to \mathbb{R}^2$ of the graph K_n to the plane is a collection of $\binom{n}{2}$ (non-closed) polygonal lines pairwise joining some n points in the plane. **The image** $f(\sigma)$ **of edge** σ is the corresponding polygonal line. **The image of a collection of edges** is the union of images of all the edges from the collection.

Theorem

For any PL (or continuous) map $K_5 \to \mathbb{R}^2$ there are two non-adjacent edges whose images intersect.

This theorem is deduced from its 'quantitative version': for 'almost every' drawing of K_5 in the plane the number of intersection points of non-adjacent edges is odd.

A PL map $f: K_n \to \mathbb{R}^2$ is called **a general position** PL map if all the vertices of the polygonal lines are in general position. Then the images of any two non-adjacent edges intersect by a finite number of points.
Let **the van Kampen number** (or the self-intersection invariant) $v(f) \in \mathbb{Z}_2$ be the parity of the number of all such intersection points, for all pairs of non-adjacent edges.
E.g. a convex pentagon with the diagonals forms a general position PL map $f: K_5 \to \mathbb{R}^2$ such that $v(f) = 1$.

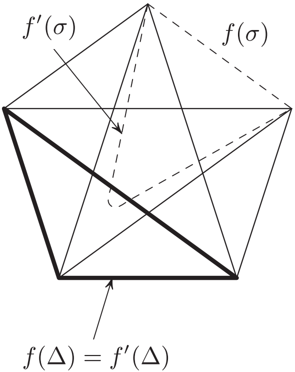
Lemma

For any general position PL map $f: K_5 \to \mathbb{R}^2$ the van Kampen number $v(f)$ is odd.

Proof of the lemma. It suffices to prove that $v(f) = v(f')$ for each two general position PL maps $f, f' : K_5 \to \mathbb{R}^2$ coinciding on every edge except an edge σ, and such that $f\sigma$ is a segment (see figure). The edges of K_5 non-adjacent to σ form a cycle Δ. Then

$$v(f) - v(f') = \sum_{\{\sigma,\tau\},\ \sigma\cap\tau=\emptyset} \left(|f\sigma \cap f\tau|_2 - |f'\sigma \cap f'\tau|_2\right) =$$

$$= |(f\sigma \cup f'\sigma) \cap f\Delta|_2 = 0.$$



Here the last equality is implied by the following Parity Lemma.

Lemma (Parity)

Any two closed polygonal lines in the plane whose vertices are in general position intersect at an even number of points.

This is not trivial because the polygonal lines may have self-intersections and because the Jordan Theorem is not obvious. It is not reasonable to deduce the Parity Lemma from the Jordan Theorem or the Euler Formula because this could form a vicious circle.

Van Kampen-Hanani-Tutte planarity criterion

A polynomial algorithm for recognizing graph planarity is obtained using the following equivalence of planarity and solvability of certain system of linear equations with coefficients in $\mathbb{Z}_2$.

Proposition (van Kampen-Hanani-Tutte)

Take any ordering of the vertices of a graph. The graph is planar if and only if the following system of linear equations over $\mathbb{Z}_2$ is solvable. To every pair A, e of a vertex and an edge such that $A \notin e$ assign a variable $x_{A,e}$. For every non-ordered pair of non-adjacent edges σ, τ denote by $b_{\sigma,\tau} \in \mathbb{Z}_2$ the number of those endpoints of σ whose numbers lie between the numbers of the endpoints of τ. For every such pairs (A, e) and $\{\sigma, \tau\}$ let

$$a_{A,e,\sigma,\tau} = \begin{cases} 1 & \text{either} \quad (A \in \sigma \text{ and } e = \tau) \quad \text{or} \quad (A \in \tau \text{ and } e = \sigma) \\ 0 & \text{otherwise} \end{cases}.$$

For every such pair $\{\sigma, \tau\}$ take the equation $\sum\limits_{A \notin e} a_{A,e,\sigma,\tau} x_{A,e} = b_{\sigma,\tau}$.

Linear and PL maps of graph to the plane

A linear map $f : K \to \mathbb{R}^2$ of a graph $K = (V, E)$ to the plane is a map $f : V \to \mathbb{R}^2$. *The image $f(AB)$ of edge AB* is the segment $f(A)f(B)$.

A PL map $f : K \to \mathbb{R}^2$ of a graph $K = (V, E)$ to the plane is a collection of polygonal lines corresponding to the edges, whose endpoints correspond to the vertices. A PL map of a graph K to the plane is 'the same' as a linear map of some graph homeomorphic to K. *The image* of an edge, or of a collection of edges, is defined analogously to the case of K_n.

A linear map of a graph to the plane is called **a general position** linear map if the images of all the vertices are in general position. A PL map $f : K \to \mathbb{R}^2$ of a graph K is called **a general position** PL map if there exist a graph H homeomorphic to K and a general position linear map of H to the plane such that this map 'corresponds' to the map f.

Hanani-Tutte Theorem

A graph is called $\mathbb{Z}_2$-**planar** if there exists a general position PL map of this graph to the plane such that images of any two non-adjacent edges intersect at an even number of points.

We essentially proved above that K_5 is not $\mathbb{Z}_2$-planar. Analogously, $K_{3,3}$ is not $\mathbb{Z}_2$-planar. Hence, if a graph K is homeomorphic to K_5 or to $K_{3,3}$, then K is not $\mathbb{Z}_2$-planar. Then using Kuratowski Theorem one obtains the following result.

Theorem (Hanani-Tutte)

A graph is planar if and only if it is $\mathbb{Z}_2$*-planar.*

Example

Suppose a graph and an arbitrary ordering of its vertices are given. Put the vertices on a circle, preserving the ordering. Take the chord for each edge. We obtain a general position linear map of the graph to the plane. For any non-adjacent edges σ, τ the number of intersection points of their images has the same parity as the number of endpoints of σ that lie between the endpoints of τ.

The intersection cocycle

Let $f : K \to \mathbb{R}^2$ be a general position PL map of a graph K. Take any pair of non-adjacent edges σ, τ. The intersection $f\sigma \cap f\tau$ consists of a finite number of points. Assign to the pair $\{\sigma, \tau\}$ the residue

$$|f\sigma \cap f\tau| \mod 2.$$

Denote by K^* the set of all unordered pairs of non-adjacent edges of the graph K. The obtained map $K^* \to \mathbb{Z}_2$ is called **the intersection cocycle** (modulo 2) of f. Maps $K^* \to \mathbb{Z}_2$ are identified with subsets of K^* consisting of pairs going to $1 \in \mathbb{Z}_2$.

Maps $K^* \to \mathbb{Z}_2$ can also be regarded as 'partial matrices', i.e., symmetric arrangements of zeroes and ones in those cells of an $e \times e$-matrix that correspond to the pairs of non-adjacent edges, where e is the number of edges.

A graph is $\mathbb{Z}_2$-planar if and only if the intersection cocycle is zero for some general position PL map of this graph to the plane.

Take a linear map $f : K_n \to \mathbb{R}^2$ such that $f(1)f(2)\dots f(n)$ is a convex n-gon. For $n = 4$ and $n = 5$ the intersection cocycles correspond to the subsets $\big\{\{13, 24\}\big\}$ and $\big\{\{13, 24\}, \{24, 35\}, \{35, 41\}, \{41, 52\}, \{52, 13\}\big\}$.

Intersection cocycles of different maps

Proposition

(a) The intersection cocycle does not change under the first four Reidemeister moves in figure I-IV. (The graph drawing changes in the disk as in figures, while out of this disk the graph drawing remains unchanged. No other images of edges besides the pictured ones intersect the disk.)

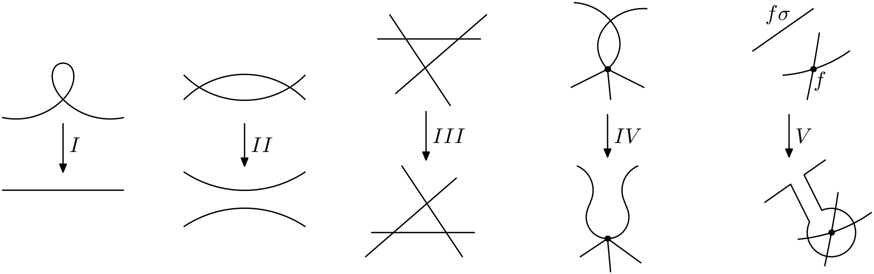


Figure: The Reidemeister moves for graphs in the plane

Addition of cocycles $K^* \to \mathbb{Z}_2$ is componentwise, i.e. is defined by adding modulo 2 numbers corresponding to the same pair. This corresponds to the sum modulo 2 of subsets of K^*.

Proposition

(b) Let K be a graph and A its vertex that is not the end of an edge σ. **An elementary coboundary** of the pair (A, σ) is the subset $\delta_K(A, \sigma) \subset K^*$ consisting of all pairs $\{\sigma, \tau\}$ with $\tau \ni A$. Under the Reidemeister move in figure V the intersection cocycle changes by adding $\delta_K(A, \sigma)$.

Example

(a) We have $\big\{\{13, 24\}\big\} = \delta_{K_4}(1, 24) = \delta_{K_4}(2, 13) = \delta_{K_4}(3, 24) = \delta_{K_4}(4, 13)$. So the intersection cocycle of the above map $K_4 \to \mathbb{R}^2$ is an elementary coboundary.
(b) We have $\delta_{K_5}(3, 12) = \big\{\{12, 34\}, \{12, 35\}\big\}$. So the intersection cocycle of the above map $K_5 \to \mathbb{R}^2$ is not a sum of elementary coboundaries.

Cocycles $\nu_1, \nu_2 : K^* \to \mathbb{Z}_2$ (or $\nu_1, \nu_2 \subset K^*$) are called **cohomologous** if

$$\nu_1 - \nu_2 = \delta_K(A_1, \sigma_1) + \ldots + \delta_K(A_k, \sigma_k)$$

for some vertices $A_1, \ldots, A_k$ and edges $\sigma_1, \ldots, \sigma_k$ (not necessarily distinct).

Lemma

The intersection cocycles of different general position PL maps of the same graph to the plane are cohomologous.

Proof of the lemma is a non-trivial generalization of the proof that $v(f) = 1$. This lemma is implied by, but is easier than, the following fact: any general position PL map of a graph to the plane can be obtained from any other such map using Reidemeister moves.

Proposition

A graph is $\mathbb{Z}_2$-planar if and only if the intersection cocycle of some (or, equivalently, of any) general position PL map of this graph to the plane is cohomologous to the zero cocycle.

Historical notes

This talk was presented at seminars 'Selected Topics in Mathematics' (NUI Galway, University of Liverpool, TU Wien, and Worcester PI) and 'Dynamical systems' (Department of Mathematics, HSE, Moscow), as well as in courses at Independent University of Moscow, at Moscow Institute of Physics and Technology, and at various summer mathematical schools. A course under the same title was eternally rejected from Summer School 'Modern Mathematics' (by president of Scientific Committee V. Kleptsyn), see `https://www.mccme.ru/circles/oim/home/ssmm_teaching.htm`. This talk is based on part of the paper under the same title, arXiv:1805.10237. During the refereeing process one of the referees misused anonymous peer review system to promote an unreasonable opinion which does not stand open discussion. In my reply to the Editors I justified this judgement by considering the referee's comments one by one. I am glad that the Editors made their (acceptance) decision carefully and critically studying the referee reports and the author's responses, not blindly believing the referees. See Example 5.5 of `https://www.mccme.ru/circles/oim/rese_inte.pdf`.