An alternative proof of the Conway-Gordon-Sachs Theorem[1]

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Abstract

In this paper we present a short and apparently new proof of the Conway-Gordon-Sachs Theorem on the complete graph on 6 vertices. We reduce this theorem to certain property of the complete graph on 5 vertices mapped to the plane.

Points in 3-dimensional space are in *general position*, if no four of them are in one plane. By a *triangle* we always mean a 2-dimensional triangle, i.e. the convex hull of three general position points. Two triangles in 3-dimensional space whose six vertices are in general position are *linked*, if the outline of the first triangle intersects the interior of the second triangle exactly at one point.

Linear Conway-Gordon-Sachs Theorem. *Assume that six points in 3-dimensional space are in general position. Then there exist two linked triangles with the vertices at these points.*

Consider a closed broken line a in 3-dimensional space. A finite set of triangles in 3-dimensional space is called a *membrane spanned on* a, if the following conditions hold:

• Any side of a is a side of exactly one triangle from this set;

• Any side of a triangle from this set that is not a side of a belongs to exactly two triangles from this set.

Two disjoint non-self-intersecting closed broken lines a and b in 3-dimensional space are *linked*, if there exists a membrane, denote it by A, spanned on a such that

• b do not intersect the outline of any triangle from A and no vertex of b belongs to some triangle from A;

• the number of triangles from A that intersect b is odd.

Denote by K_n the complete graph on n vertices.

Conway-Gordon-Sachs Theorem. *Assume that the graph K_6 is piecewise-linearly embedded in 3-dimensional space. Then there exist two linked cycles of length 3 in this graph.*

Remark. The statement of the theorem is meaningful because any cycle of length 3 in this graph is a closed broken line.

The original proof of the Conway-Gordon-Sachs Theorem [CG83] has two steps. In the first step it is proved that change of our six points does not change the parity of number of pairs of linked triangles.[2] In the second step one constructs an example when this number is odd.

Our proof is by reduction to a result for the plane. That result is proved in analogous two steps. They are easier, in particular, construction of planar example rather than spatial example is much easier. The main idea of our proof is shown on page 4 for the linear case and on page 6 for the piecewise-linear.

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[2]The proof of the first step uses either one of the following two facts:
(1) any broken line and 2-dimensional sphere that are in general position intersect in an even number of points;
(2) any two piecewise-linear embeddings of the graph K_6 in 3-dimensional space are related by isotopy and crossing changes where edges are allowed to pass through each other.

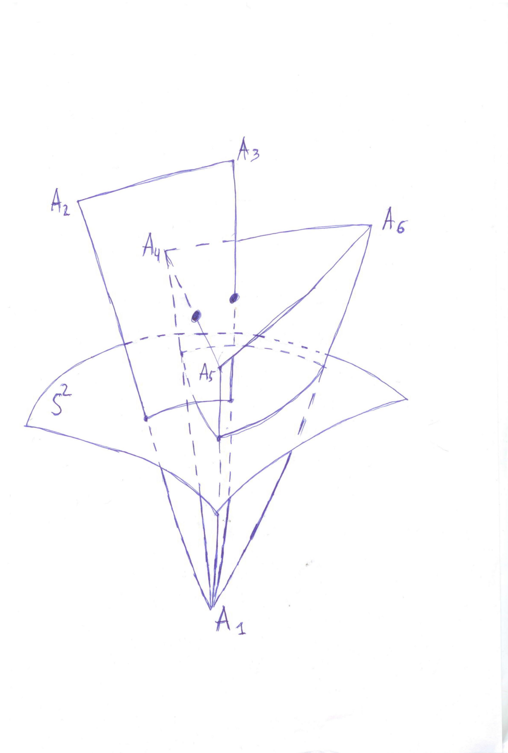

Figure 1: To Lemma 1

There is a shorter unpublished proof of the *linear case* invented by Alexander Shapovalov. That proof does not generalize to the proof of the *piecewise-linear* case.

Proof of the linear Conway-Gordon-Sachs Theorem.

In the proof we use the following definition and well-known lemmas whose proofs are presented by completeness.

Definition of the notions of being higher and lower. Let a, b be segments in 3-dimensional space, π be a plane such that segments a, b are in one half-space with respect to π and O be a point in other half-space with respect to π. A segment a is *lower* than a segment b with respect to plane π and point O, if there exists a half-line OX with the endpoint O that intersects segment a at a point $A := a \cap OX$, and segment b at a point $B := b \cap OX$, so that $A \neq B$ and A is in the segment OB. Analogously one can define the notion of a segment a being *higher* than a segment b.

Lemma 1 (see Figure 1)**.** *Assume that the vertices of two triangles are in general position. Denote by $A_1A_2A_3$ the first triangle. We may assume that A_1 is the unique point among them whose first coordinate a is maximal. Consider a plane $x = b$, denote it by π, where b is slightly smaller than a. If the number of the sides of the second triangle that are lower than A_2A_3 with respect to plane π and point A_1 is odd then these two triangles are linked.*

Proof of Lemma 1. Denote by A_4, A_5, A_6 the vertices of the second triangle. We will say that point A_1 is in first half-space with respect to π and points $A_2, ..., A_6$ are in second half-space with respect to π. Let $f : \mathbb{R}^3 - \{A_1\} \to \pi$ be the central projection with the center A_1. For any two segments a and b that are in second half-space with respect to π, if a is higher than b then $f(a)$ intersects $f(b)$. Moreover, if $f(a)$ intersects $f(b)$ then a is higher than b or b is higher than a. By the assumption of the lemma there exists a side, say, A_4A_5, of the triangle $A_4A_5A_6$ such that A_2A_3

is higher than A_4A_5. Then the point $f^{-1}(f(A_2A_3)) \cap A_4A_5$ is in the interior of the triangle $A_1A_2A_3$. Since $f(A_2A_3)$ is a segment in π and $f(A_4A_5A_6)$ is a triangle in π, $f(A_2A_3)$ intersects the outline of $f(A_4A_5A_6)$ in at most 2 points. So A_4A_5 is the unique side of the triangle $A_4A_5A_6$ that is lower than A_2A_3. This implies that the outline of the triangle $A_4A_5A_6$ intersects the interior of the triangle $A_1A_2A_3$ at a unique point $f^{-1}(f(A_2A_3)) \cap A_4A_5$. Since the vertices of these two triangles are in general position it follows that these two triangles are linked. *QED.*

Consider a set A of points in the plane. We will say that the set A is *in general position* in the plane, if the following conditions hold:

- no three points from A lie on one line;
- no three segments joining some three pairs of points from A have a common point.

Lemma 2. *[Sk] Let a collection f of five points in general position in the plane be given. Then the sum of numbers of intersection points of the segments AB and CD for all unordered pairs $\{\{A,B\},\{C,D\}\}$ of disjoint two-element subsets $\{A,B\},\{C,D\} \subset f$ is odd.*

Proof . For any four distinct points A, B, C, D of the collection f, the segments AB and CD either are disjoint or have a unique common point. Define $v(f)$ to be the parity of the sum of numbers of intersection points of the segments AB and CD for all unordered pairs $\{\{A,B\},\{C,D\}\}$ of disjoint two-element subsets $\{A,B\},\{C,D\} \subset f$.

$$v(f) := \sum\{|AB \cap CD| \ : \ \{\{A,B\},\{C,D\}\} \subset \binom{f}{2},\ \{A,B\} \cap \{C,D\} = \emptyset\} \mod 2.$$

This lemma is implied by the following two assertions.

(a) For the collection f_0 of five vertices of a regular pentagon we have $v(f_0) = 1$.

(b) $v(f)$ does not depend on f.

Assertion (a) is clear. Let us prove (b).

It suffices to prove that if we change the first point keeping the remaining four fixed then the number $v(f)$ is not changed. Suppose that we change point $K \in f$ to K'. Denote by f' the obtained collection.

There exists a point K'' such that points from each of two sets $\{K''\} \cup f$ and $\{K''\} \cup f'$ are in general position. Denote $f'' := (f - \{K\}) \cup \{K''\}$. Then it suffices to prove that $v(f) = v(f'')$ and $v(f') = v(f'')$. So it suffices to prove the item (b) for the case when points from the set $f \cup f'$ are in general position. Assume that the points from the set $f \cup f'$ are in general position. For each $A \in f - \{K\}$ denote by $\triangle_A$ the triangle with vertices from $f - \{A, K\}$. Then assertion (b) follows from

$$v(f') - v(f) = \sum_{A \in f - \{K\}} (|KA \cap \triangle_A| - |K'A \cap \triangle_A|) = \sum_{A \in f - \{K\}} |KK' \cap \triangle_A| = 0 \mod 2.$$

- Here the first equality is clear.
- The second equality holds because $|KK'A \cap \triangle_A|$ is even for each $A \in f - \{K\}$ because the outlines of two triangles on the plane whose vertices are in general position intersect each other in an even number of points, see, e.g., [BE82].
- The last equality holds because for each unordered pair $\{A,B\} \subset f - \{K\}$ there exist exactly two triangles with vertices from $f - \{K\}$ containing the segment AB. So for each unordered pair $\{A,B\} \subset f - \{K\}$ the number $|KK' \cap AB|$ appears in the sum twice for two triangles $\triangle_A$, $\triangle_B$. *QED.*

Deduction of the linear Conway-Gordon-Sachs Theorem from Lemma 1 and Lemma 2. Suppose that points $A_1, A_2, A_3, A_4, A_5, A_6$ are in general position in 3-dimensional space. We may assume that A_1 is the unique point among them whose first coordinate a is maximal. Consider a plane $x = b$, where b is slightly smaller than a. Denote by E the set of segments joining pairs of points A_2, A_3, A_4, A_5, A_6. Consider the central projection $f : \mathbb{R}^3 - \{A_1\} \to \pi$ with the center A_1. For any ordered pair $(e, e') \in E^2$ denote

$$lk(e,e') := \begin{cases} 1, & \text{if } e \text{ is higher than } e' \text{ with respect to plane } \pi \text{ and point } A_1; \\ 0, & \text{otherwise.} \end{cases}.$$

For any segment $e \in E$ define the number

$$S_e := \sum_{e' \in (E - \{e\})} lk(e, e').$$

For any segment e from E we have that $f(e)$ is a segment in π. Define $v(f)$ to be the sum of numbers of intersection points of segments $f(e)$ snd $f(e')$ for all unordered pairs $\{e, e'\} \subset E$ of disjoint segments e, e'. Since our six points are in general position it follows that for any two segments e, e' from E we have $|f(e) \cap f(e')| \le 1$. So the number $v(f)$ is well-defined.

Then

$$\sum_{e \in E} S_e \equiv \sum_{(e,e') \in E^2} lk(e,e') \equiv v(f) \equiv 1 \mod 2.$$

Here the first equality follows from definition of S_e.

The second equality holds because

• for any two segments $e, e' \in E$ we have $|f(e) \cap f(e')| \le 1$ because our six points are in general position;

• if the segments $e, e' \in E$ do not have common endpoints and $f(e) \cap f(e') \ne \emptyset$ then $lk(e,e') + lk(e', e) = 1$;

• if the segments $e, e' \in E$ have a common endpoint or $f(e) \cap f(e') = \emptyset$ then $lk(e,e') + lk(e', e) = 0$.

The third equality follows from Lemma 2.

Hence for some segment from E, say, A_2A_3, the number $S_{A_2A_3}$ is odd. Then Lemma 1 implies that triangles $A_1A_2A_3$ and $A_4A_5A_6$ are linked. **QED.**

Proof of the Conway-Gordon-Sachs Theorem.

In the proof we use the following definitions and well-known lemmas whose proofs are presented by completeness.

Consider a finite set A in 3-dimensional space. A plane π is *in general position* with respect to A, if

• the set of images of points from A under the orthogonal projection to the plane π are in general position in the plane π;

• all points from A are in one half-space with respect to π.

Lemma 3. *For any finite set of points in 3-dimensional space there exists a plane in general position with respect to this set of points.*

This lemma follows from Sard's Theorem. We do not give the proof of this lemma here.

Consider two segments in 3-dimensional space and a plane in general position with respect to the set of endpoints of these segments. Define what it means for the first segment to be *higher* than the second segment analogously to the definition of notion *higher* that is given above but replacing 'central projection' with 'orthogonal projection'. Let A, B be two broken lines (not necessarily closed) in 3-dimensional space. Consider a plane in general position with respect to the set of vertices of these broken lines. Denote by

$$lk(A, B)$$

the number of ordered pairs (a, b) of sides a of A and b of B such that a is higher than b. The number $lk(A, B)$ depends on the choice of plane but omit this in the notation.

Two broken lines in the plane are in *general position*, if the union of vertices of these broken lines is in general position.

For any graph G denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G.

Proof of the Conway-Gordon-Sachs Theorem. Edges of the graph K_6 piecewise-linearly embedded in $\mathbb{R}^3$ are broken lines in 3-dimensional space. No two of these broken lines have a common interior point. By Lemma 3 there exists a plane π in general position with respect to the set of vertices of broken lines that are edges of the graph K_6. Denote by a one of the vertices of the graph K_6. Consider the graph $K_5 = K_6 - a$ and the orthogonal projection $f : K_5 \to \pi$. By the choice of the plane π for any two edges e, e' of the graph K_5 broken lines $f(e), f(e')$ are in general position in π. Define $v(f)$ to be the sum of numbers of intersection points of broken lines $f(e)$ and $f(e')$ for all unordered pairs $\{e, e'\}$ of disjoint edges $e, e' \in E(K_5)$. By the choice of π the number $v(f)$ is well-defined. For any two distinct vertices a, b of the graph K_6 denote by ab the edge of the graph K_6 that contains vertices a, b. For any three distinct vertices a, b, c of the graph K_6 denote by abc the cycle of length 3 in the graph K_6 that contains vertices a, b, c. Denote by C_{ij} the cycle in the graph K_5 on three vertices other than i and j. We have

$$\sum_{bc \in E(K_5)} lk(abc, C_{bc}) \equiv \sum_{bc \in E(K_5)} (lk(ab, C_{bc}) + lk(ac, C_{bc})) + \sum_{bc \in E(K_5)} lk(bc, C_{bc}) \equiv$$

$$\equiv \sum_{bc \in E(K_5)} lk(bc, C_{bc}) \equiv v(f) \equiv 1 \mod 2$$

• The first equality is clear.

• Let us prove the second equality. We have $lk(ab, C_{bc}) = \sum_{e \in E(C_{bc})} lk(ab, e)$. For each vertex b of the graph K_5 and for each edge e of the graph $K_5 - b$ there exist exactly two those cycles of length 3 in the graph $K_5 - b$ that contain the edge e. Then each of two numbers $lk(ab, e), lk(ac, e)$ appears twice in the sum $\sum_{bc \in E(K_5)} (lk(ab, C_{bc}) + lk(ac, C_{bc}))$. Therefore this sum is even.

• The proof of the third equality is the same as the *proof of the second equality* in the proof of the linear Conway-Gordon-Sachs Theorem.

• The last equality follows from Lemma 5 below because for any two edges e, e' of the graph K_5 broken lines $f(e), f(e')$ are in general position in π.

Hence for some edge bc of the graph K_5 the number $lk(abc, C_{bc})$ is equal to 1 and Lemma 4 below implies that cycles abc and C_{bc} are linked. **QED.**

Lemma 4. *Let A and B be two closed broken lines in 3-dimensional space. Consider a plane in general position with respect to the set of vertices of these broken lines. If the number $lk(A, B)$ is odd then broken lines A and B are linked.*

This lemma a generalization of Lemma 1. We do not prove it here.

Lemma 5. *[Sk] Consider a plane π and a piecewise-linear map $f : K_5 \to \pi$. Assume that for any two edges e, e' of the graph K_5 broken lines $f(e), f(e')$ are in general position. Then the sum of numbers of intersection points of broken lines $f(e)$ and $f(e')$ for all unordered pairs $\{e, e'\}$ of disjoint edges $e, e' \in E(K_5)$ is odd.*

This lemma is a generalization of Lemma 2.

Proposition (BE82, §1). *Any two closed broken lines that are in general position in the plane intersect each other in an even number of points.*

Deduction of Lemma 5 from Proposition. For any piecewise-linear map $f : K_5 \to \pi$ that satisfies the hypothesis define $v(f)$ to be the sum of numbers of intersection points of broken lines $f(e)$ and $f(e')$ for all unordered pairs $\{e, e'\}$ of disjoint edges $e, e' \in E(K_5)$. Since for any two edges e, e' of the graph K_5 broken lines $f(e), f(e')$ are in general position, the number of intersection points of these broken lines is finite. So the number $v(f)$ is well-defined. This lemma is implied by the following two assertions.

(a) For the collection f_0 of five vertices of a regular pentagon we have $v(f_0) \equiv 1 \mod 2$.

(b) The parity of the number $v(f)$ does not depend on f.

Assertion (a) is clear. Let us prove (b). Assume that $f', f : K_5 \to \pi$ are piecewise-linear maps that satisfy the hypothesis. It suffices to prove that if these maps are equal at some subgraph K_4 of our graph then $v(f) \equiv v(f') \mod 2$. Indeed, if we prove this assertion it means that if we change the map f at one edge or one vertex and edges incident to this vertex so that the map f would still satisfy the hypohesis the number $v(f)$ does not change. Suppose that piecewise-linear maps $f, f' : K_5 \to \pi$ satisfy the hypothesis and they are equal at a subgraph K_4 of the graph K_5. Suppose that the vertex v is not from K_4. For any vertex i of the graph K_4 denote by $\triangle_i$ the cycle in K_4 on three vertices other than i. For any two vertices a, b of the graph K_5 denote by ab the edge of K_5 containing vertices a, b. There exists a map $f'' : K_5 \to \pi$ such that

• $f'' : K_5 \to \pi$ differs with $f : K_5 \to \pi$ exactly at interiors of those edges vi, where i is a vertex of K_4, that $f(vi) \neq f'(vi)$;

• if $f''(vi) \neq f(vi)$ then any intersection point of broken lines $f''(e)$ and $f'(e)$ does not lie on $f(\triangle_i)$.

For any vertex i of the graph K_4 broken lines $f(vi) \cup f''(vi)$ and $f(\triangle_i)$ are in general position and then Proposition implies that these broken lines intersect in an even number of points.

There exists broken line with endpoints $f(v)$ and $f'(v)$ that is in general position with $f(ab)$ and $f'(ab)$ for any edge ab of the graph K_5. Now it suffices to prove that $v(f) \equiv v(f'') \mod 2$ and $v(f') \equiv v(f'') \mod 2$. The proof of this fact is the same as the proof of Lemma 2. But one should replace 'segment' with 'broken line' and use Proposition instead of the fact that any two triangles whose vertices are in general position in the plane intersect in an even number of points. *QED.*

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