An interesting sum of Goldbach

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In this note we present a short proof of the following formula of Goldbach [Se].

Theorem. *Let*

$$A := \{x^m \ : \ x, m \geq 2 \text{ are integers}\}.$$

Then

$$\sum_{a\in A} \frac{1}{a-1} = 1.$$

The following Lemma is evident.

Lemma. *For any $a \in A$ there exist unique and $b, z \in \mathbb{Z}$ such that $b \notin A$, $b, z \geq 2$ and $b^z = a$.*

Proof of Theorem.

$$\sum_{a\in A} \frac{1}{a-1} \overset{(1)}{=} \sum_{b,k\geq 2, b\notin A} \frac{1}{b^k - 1} \overset{(2)}{=} \sum_{k\geq 2}\sum_{n\geq 2} \frac{1}{n^k} \overset{(3)}{=} \sum_{n\geq 2}\left(\frac{1}{n-1} - \frac{1}{n}\right) = 1.$$

Here

- (1) holds by Lemma for $z = k$.
- (2) holds because for each $b, k \geq 2$, $b \notin A$

$$\frac{1}{b^k - 1} = \sum_{r\geq 1} \frac{1}{b^{rk}} = \frac{1}{b^k} + \sum_{r\geq 2} \frac{1}{b^{rk}}$$

and because by Lemma for $z = r$ we have for each k

$$\sum_{b\geq 2, b\notin A}\left(\frac{1}{b^k} + \sum_{r\geq 2}\frac{1}{b^{rk}}\right) = \sum_{b\geq 2, b\notin A}\frac{1}{b^k} + \sum_{a\in A}\frac{1}{a^k} = \sum_{n\geq 2}\frac{1}{n^k}.$$

- (3) holds because for each n

$$\sum_{k\geq 2}\frac{1}{n^k} = \left(\sum_{k\geq 1}\frac{1}{n^k}\right) - \frac{1}{n} = \frac{1}{n-1} - \frac{1}{n}. \qquad QED$$

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References

[Se] The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A001597`.