ON AN INTERESTING CIRCLE IN A TRIANGLE

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Abstract. In this paper we provide a purely synthetic proof of the "Interesting circle" problem stated by P. Dolgirev in [1].

1. Introduction

Interesting circle theorem. *Let A', B', C' be the tangent points of sides of $\triangle ABC$ and its incircle ω. Draw the tangent lines from vertices A, B, C to the incircle γ of $\triangle A'B'C'$ and denote by A_1, A_2, B_1, B_2, C_1, C_2 the intersections of these tangent lines with sides of $\triangle ABC$. Then points A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle belonging to the pencil generated by ω and γ.*

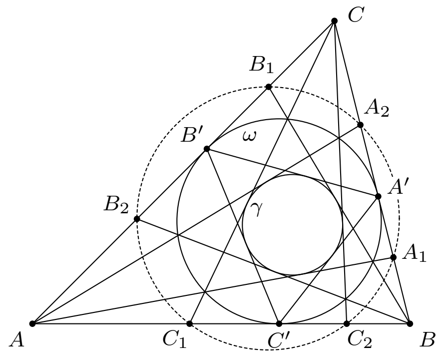

Fig. 1. Interesting circle theorem

Proof outline. It would suffice to show that for each of six points A_1, A_2, B_1, B_2, C_1, C_2 the ratio of lengths of tangents to circles ω and γ is constant. Indeed, from the *Generalized Apollonius's theorem* (see [2], §2.4.), we obtain that the locus of points satisfying this property is a circle belonging to the same pencil, from which the desired result will follow.

We will be using the basic properties from projective geometry and the geometry of conics. These properties may as well be found in [2].

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Proof of the Interesting circle theorem. Let us divide our proof into two lemmas.

In *Lemma 1*, we are going to show that the ratio of lengths of tangents is the same for a pair of adjacent points of a hexagon $A_1A_2B_1B_2C_1C_2$ lying on one side of $\triangle ABC$. In *Lemma 2* we are going to show that the same property holds for a pair of adjacent points of the hexagon belonging to different sides of $\triangle ABC$. Hence, a similar reasoning may be applied to any other pair of adjacent points of a hexagon $A_1A_2B_1B_2C_1C_2$.

We will be using the denotations as in Fig. 1.

Lemma 1. *Let $B'C'$ and $A'C'$ touch the circle γ at U and V and let CC_1 and CC_2 touch the circle γ at P and Q respectively, see Fig. 2. Then we have:*

$$\frac{PC_1}{C_1C'} = \frac{QC_2}{C_2C'} \tag{1}$$

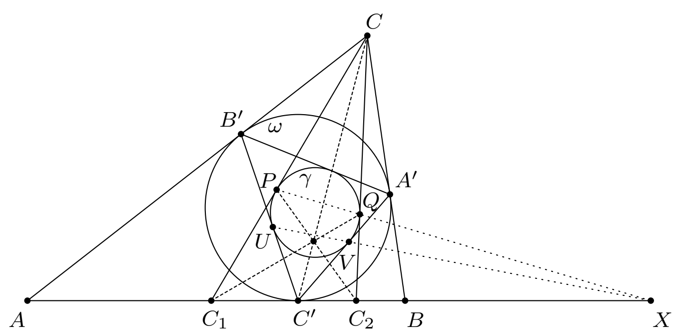

Fig. 2. Lemma 1

Proof. The property (1) is equivalent to the concurrency of C_2P, C_1Q and CC'. Indeed, applying the *Ceva's theorem* to $\triangle ABC$ and points P, Q and C' on its sides, we obtain:

$$\frac{PC_1}{C_1C'} \cdot \frac{C'C_2}{C_2Q} \cdot \frac{QC}{CP} = 1$$

The last factor is a ratio of lengths of tangents from C to γ and is therefore 1.

Lemma 1.1. *The concurrency of PQ, AB and UV implies the concurrency of C_2P, C_1Q and CC', see Fig. 2.*

Proof. Assume we have shown that PQ and UV intersect at X which lies on AB. Note that X is a pole of CC' with respect to γ, because PQ is a polar of C and UV is a polar of C'. Then we get that $(CC_1, CC_2, CC', CX) = -1$ holds. It follows that CC' passes through the intersection of C_2P and C_1Q.

It would suffice to show that X lies on AB. Line CC' intersects γ, therefore we can consider a projective transformation which preserves γ and takes CC' to its diameter. Consider the following known theorem:

Reverse Blanchet's theorem. *Let AA', BB', CC' be cevians in $\triangle ABC$, intersecting at one point S. If $C'C$ is a bisector of $\angle B'C'A'$, then CC' is an altitude of $\triangle ABC$.*

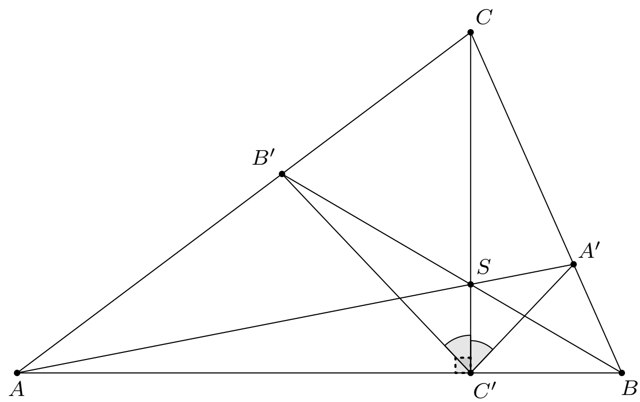

Fig. 3. The Reverse Blanchet's theorem

After the transformation we have: $\angle UC'C = \angle VC'C$ (see Fig. 4.), therefore from the *Reverse Blanchet's theorem* we get that CC' becomes an altitude in $\triangle ABC$. Because the center of γ' lies on CC', we get that $PQ \perp CC'$, $UV \perp CC'$ and therefore $PQ \parallel UV \parallel AB$, i.e. these lines intersect at infinity. Therefore their preimages intersect at one point, which completes the proof of *Lemma 1.*

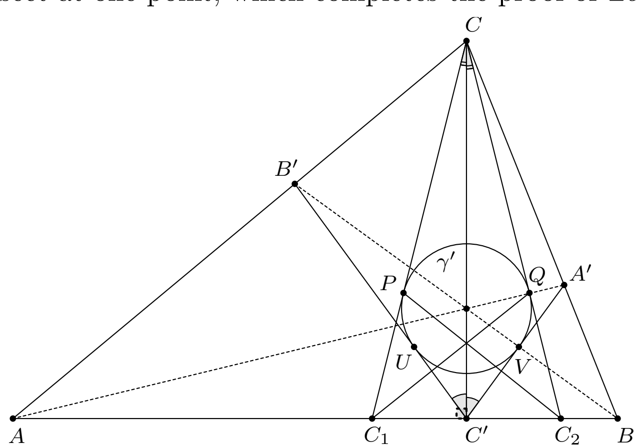

Fig. 4.

Lemma 2. *Let CC_2 and AA_1 touch the circle γ at E respectively and denote by L the intersection of CC_2 and AA_1, see Fig. 5. Then we have:*

$$\frac{A_1A'}{C_2C'} = \frac{A_1D}{EC_2}. \tag{2}$$

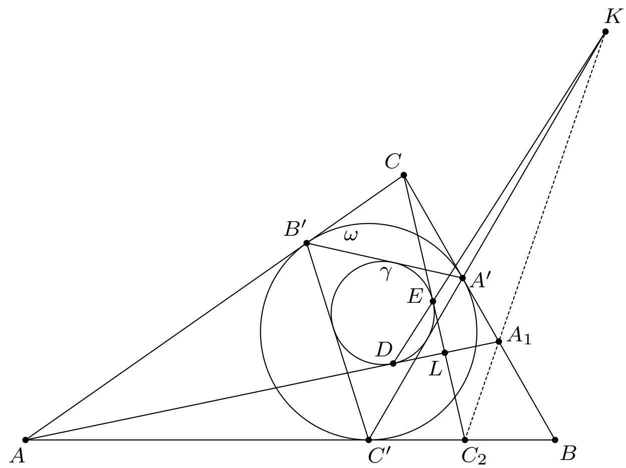

Fig. 5. Lemma 2

Proof. In *Lemma 1* we have shown that the property concerning the ratios of lengths of tangents follows from the concurrency of three lines. Analogously, we can show that (2) follows from the concurrency of DE, $C'A'$ and C_2A_1.

Indeed, assume that we have shown that DE and $C'A'$ intersect at a point K on C_2A_1, see Fig. 5. Apply the *Menelaus' theorem* to $\triangle BC_2A_1$ and line $A'C'$:

$$\frac{BC'}{C'C_2} \cdot \frac{C_2K}{KA_1} \cdot \frac{A_1A'}{A'B} = 1, \tag{3}$$

as well as to $\triangle LC_2A_1$ and line DE:

$$\frac{LE}{EC_2} \cdot \frac{C_2K}{KA_1} \cdot \frac{A_1D}{DL} = 1. \tag{4}$$

Dividing (3) by (4) and using $BC' = BA'$ and $LE = LD$ we get exactly (2):

$$\frac{A_1A'}{C_2C'} = \frac{A_1D}{EC_2}.$$

It would suffice to show that DE, $C'A'$ and C_2A_1 intersect at one point. Once again, the property has become purely projective. The arguments that follow will be based on the projective properties of conics.

Consider the following famous theorem, which may be found in [2]:

Pascal theorem. *Let A, B, C, D, E, F be points on a plane. Let $AB \cap DE = R$, $BC \cap EF = Q$, $CD \cap AF = P$. Then P, Q, R are collinear if and only if A, B, C, D, E, F lie on a conic.*

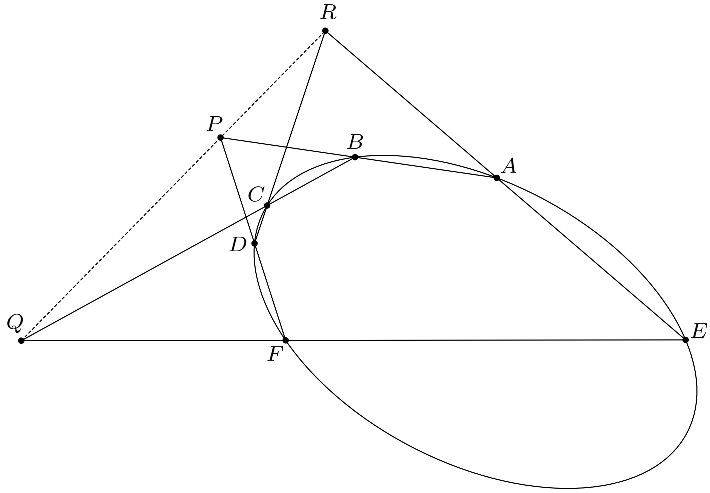

Fig. 6. Pascal theorem

It would suffice to show that points C_2, A_1 and K, defined as the intersection of DE and $C'A'$, are collinear, see Fig. 7.

Consider a hexagon $ADECA'C'$. We have: $AD \cap CA' = A_1$, $CE \cap AC' = C_2$ and $C'A' \cap DE = K$. Thus we get from *Pascal theorem* that the desired cocurrency is equivalent to points A, D, E, C, A', C' lying on a conic.

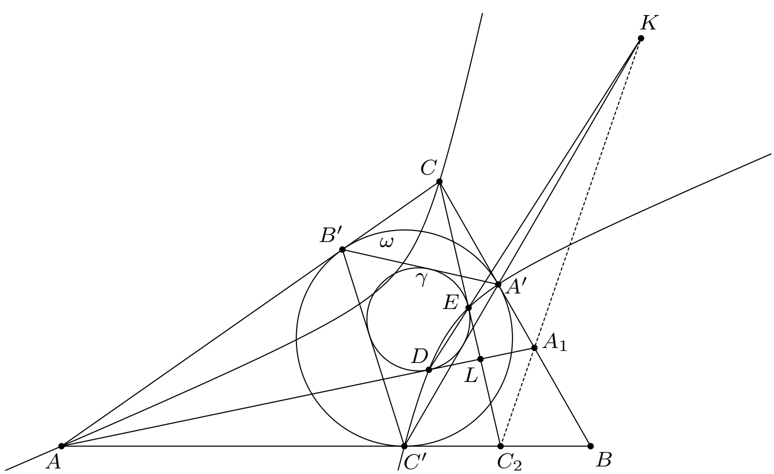

Fig. 7.

The following lemma will play a key role in the proof.

Lemma 2.1. *Let* Γ *be a conic and points* A *and* B *be points in its exterior. Denote by* AA_1, AA_2 *and* BB_1, BB_2 *the tangents to* Γ *from* A *and* B *respectively, see Fig. 8. Then we have that* A, B, A_1, A_2, B_1, B_2 *lie on a conic.*

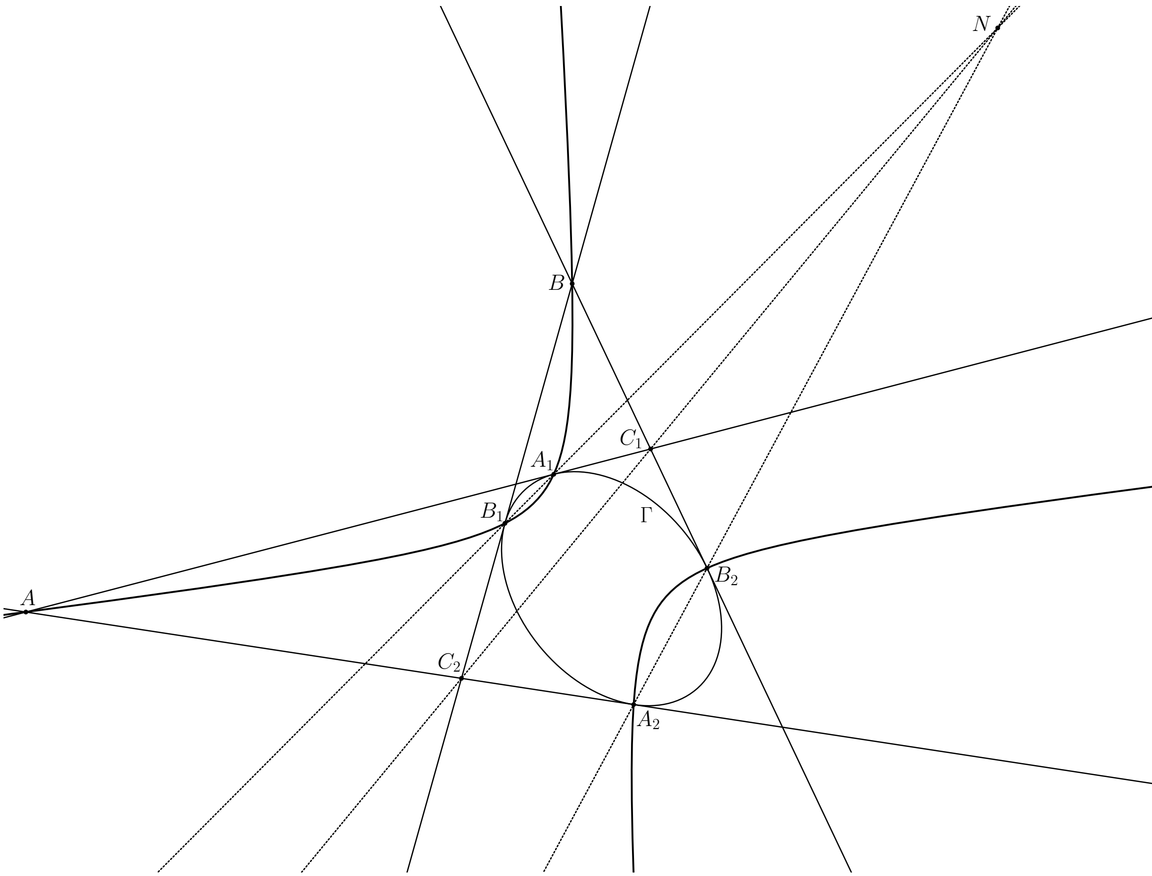

Fig. 8. Lemma 2.1

Proof. Consider a projective transformation which takes AB to infinity. Then $A'_1A'_2$ and $B'_1B'_2$ become the diameters of the ellipse. Consider an affine transformation which takes Γ' to a circle. Then $A'_2B'_2$, $C'_1C'_2$, $A'_1B'_1$ become parallel, as $A'_1A'_2$ and $B'_1B'_2$ become the diameters of a circle. Therefore their preimages intersect at one point.

Consider a hexagon $A_2AA_1B_1BB_2$. The intersections of its opposite lines are collinear. Therefore from the *Pascal theorem* it follows that A, B, A_1, A_2, B_1, B_2 lie on a conic.

Consider another famous theorem, which may be found in [2].

Three Conics theorem. *Let Γ_1, Γ_2, Γ_3 be three conics on the plane, each passing through points A and B and each pair having four common points. Then their three common chords intersect at one point, see Fig. 9.*

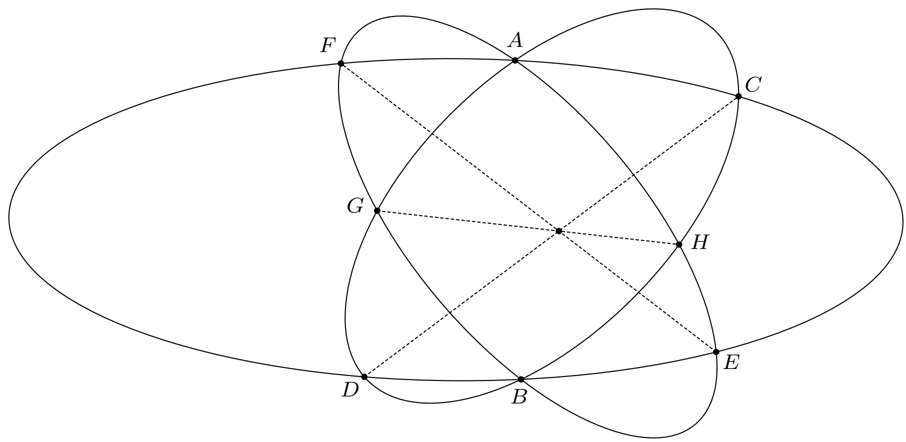

Fig. 9. The Three Conics theorem

Let us return to the proof of *Lemma 2*. We will be using the denotations as in Fig. 10.

In the problem statement we have that A', B', C' are tangency points of the incircle and the sides of $\triangle ABC$.

We are going to prove a generalized result, namely that the concurrency holds whenever the lines AA_1, BB_1, CC_1 intersect at one point. Consider the following theorem.

Interesting conic theorem. *Consider a triangle $\triangle ABC$ and a point F lying inside. Let $\triangle A_1B_1C_1$ be a cevian triangle of an arbitrary point F lying in the interior of $\triangle ABC$. Let γ be the incircle of $\triangle A_1B_1C_1$. Denote by AA' and AA'' the tangents from A to γ, and by BB' and BB'' the tangents from to γ. Then we have that eight points A, B'', A', B, B_1, A'', B', A_1 belong to a conic (shown in green color in Fig. 10.)*

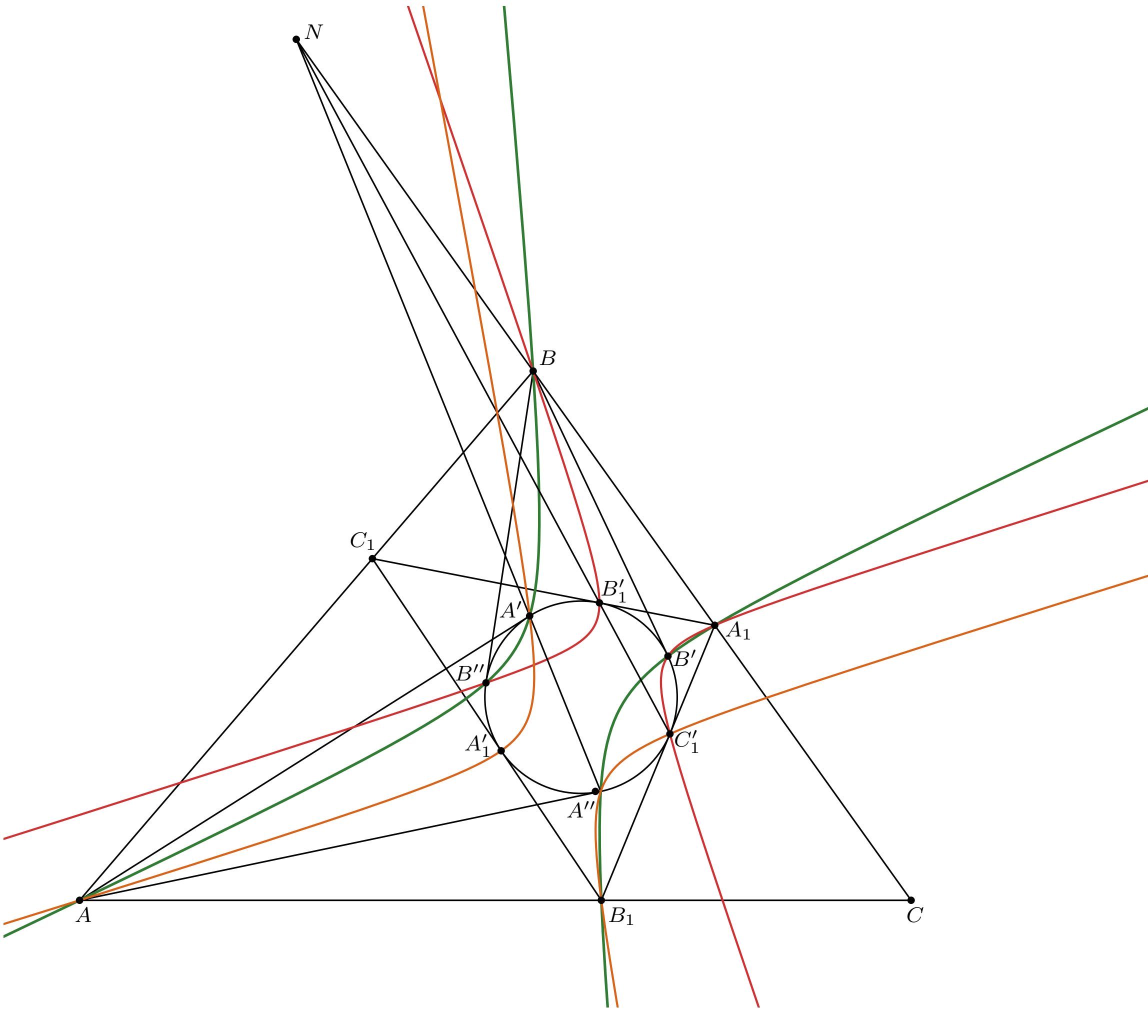

Fig. 10.

Proof. From *Lemma 2.1* applied to points A, B and the circle γ we have that points A, B'', A', B, A'', B' lie on a conic. It would suffice to show that A_1 belongs to the same conic, the same may be obtained for B_1.

First, note that from *Lemma 2.1* applied to points B, A_1 and the circle γ we get that C_1', B', A_1, B'', B_1', B lie on a conic, which is colored in red in Fig. 10. By a similar reasoning, points A, A_1', A', B_1, A'', C_1' belong to a conic colored in yellow.

Consider the three conics: the red conic, the circle γ and the green conic, which passes through points B'', A', B, A'', B'. Each oh these conics passes through two points: B' and B''. In the first section we proved that the lines $A''A'$, $C_1'B_1'$, CB are concurrent. But these lines are the common chords of our conics. If a green conic doesn't pass through A_1, we get a contradiction with the *Three conics theorem*, because a line intersects a conic at two points at most.

Therefore we get that B'', A', B, A'', B', A_1 belong to a conic. But this conic has five common points with the green conic, therefore we have shown that A_1 lies on the green conic. If we take a yellow conic instead of the red one, then by a similar reasoning it can be shown that B_1 lies on the green conic, which completes the proof of the theorem.

Applying the *Interesting conic theorem* and *Pascal theorem*, we get the desired concurrency of DE, $C'A'$ and C_2A_1 (in the denotations in Fig. 5.), which completes the proof of *Lemma 2*.

The *Interesting circle theorem* follows from two lemmas and the *Generalized Apollonius' theorem.*

References

[1] Journal of Classical Geometry, Vol. 3, p.53, Problems section.
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