On the minimal sum of edges in an edge-dominated graph

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Пояснение. *Теорема 1 предлагалась на Санкт-Петербургской городской олимпиаде 2019 года. После дискуссий на кружочке появилась эта заметка. Все представленные результаты получены Павлом без моей помощи; записаны в большей части Павлом. Вероятно, мы будем подавать в журнал (и выкладывать на арXив) более обширную версию статьи за совместным авторством.*

Д. Черкашин

1 Introduction

A graph (finite, simple, undirected) is a pair (V, E), where V stands for a set of vertices, and E denotes a set of pairs of vertices, elements of E are called edges. Let G be a graph; for a given edge $e = (u, v)$ define its *closed edge-neighborhood* as an edge subset $N[e]$ formed by e and all adjacent to e edges of G. A weight function $f : E \to \{+1; -1\}$ is called a *signed edge domination function* of G if

$$\sum_{e' \in N[e]} f(e') \geqslant 1$$

for every $e \in E$; in this case we say that (G, f) is a *SED-pair* of order $|V|$. Let $s[(G, f)]$ be the sum of weights over all edges of a graph G equipped by a weight function f.

Denote by E_+ the set $\{(u, v) \in E \mid f(u, v) = 1\}$ and by E_- the set $\{(u, v) \in E | f(u, v) = -1\}$. Define

$$s_v = \sum_{e \in N(v)} f(e);$$

for each $v \in V$, where $N(v)$ stands for the set of edges containing v.

The following problem was posed by Xu in [2, 3].

Problem 1. *What is*

$$g(n) := \min\{s[(G, f)] \mid (G, f) \text{ is a SED-pair of order } n\}$$

for each positive integer n?

The only known result was provided by the following theorem.

Theorem 1 (Akbari – Bolouki – Hatami – Siami [1]). *(i) For every n*

$$g(n) \geqslant -\frac{n^2}{16}.$$

(ii) There is a sequence of SED-pairs of order n that satisfies[1]

$$s[G, f] \leqslant -\frac{n^2}{54} + 10n.$$

[1] In fact they claim the bound $-\frac{n^2}{72}$ but the provided example gives the bound $-\frac{n^2}{54} + 10n$.

We refine both items as follows.

Theorem 2. *(i) For every n*

$$g(n) \geqslant -\frac{n^2}{25}.$$

(ii) Suppose that $n = 4(p+q)p+1$, where p and q are positive integers satisfying $p^2 = 2q^2+1$. Then there is a SED-pair of order n that satisfies

$$s[G,f] < -\frac{n^2}{8(1+\sqrt{2})^2} + 4n.$$

Note that there are infinitely many p and q satisfying the condition in Theorem 2(ii) since $p^2 = 2q^2+1$ is a special case of Pell's equation; it is well known that the positive solutions are

$$p = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2}, \qquad q = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{2\sqrt{2}}, \qquad k \in \mathbb{N}.$$

Structure of the paper. Theorem 2(ii) is proved in Section 2. Section 3 is devoted to the proof of Theorem 2(i).

2 Examples

In this section we provide a sequence of SED-pairs that achieves the lower bound $-\frac{n^2}{8(1+\sqrt{2})^2} + 4n$.

We need several auxiliary definitions. Define graph $K_{X,Y,\frac{k}{l}} = (X \cup Y, E_{X,Y,\frac{k}{l}})$ for $|X| = al, |Y| = bl$ and integer $a, b, k \leqslant l$. Split X on a disjoint sets of size l: $X = X_1 \cup X_2 \cup \dots \cup X_a$, $|X_i| = l$; also split Y on the b disjoint sets of the same size: $Y = Y_1 \cup Y_2 \cup \dots \cup Y_b$, $|Y_i| = l$.

Now for each pair $1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b$ consider the following bipartite graph $G_{ij} = (X_i \cup Y_j, E_{ij})$ with parts X_i and Y_j (all graphs G_{ij} are isomorphic). Enumerate vertices as follows $X_i = \{v_1, v_2, \dots v_l\}$, $Y_j = \{u_1, u_2, \dots u_l\}$. Define E_{ij} as the set of all pairs (v_g, u_h), for which $g-h \mod l$ lies in $\{1, 2, \dots k\}$. Put

$$E_{X,Y,\frac{k}{l}} = \bigcup_{1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b} E_{ij}.$$

Obviously the degree of every vertex in G_{ij} equals to k, so the degree of a vertex in $K_{X,Y,\frac{k}{l}}$ is $bk = |Y|\frac{k}{l}$ for vertices in X, and $ak = |X|\frac{k}{l}$ for vertices in Y.

Now define graph $K_{X,\frac{k}{l}} = (X, E_{X,\frac{k}{l}})$ for $|X| = 2al$ and integer $a, k < l$. Split X on $2l$ disjoint sets of size a: $X = X_1 \cup X_2 \cup \dots \cup X_{2l}$. The edge between vertices u and v exists if and only if $i - j \mod 2l$ lies in

$$\{-k, -(k-1), \dots, -2, -1, 1, 2, \dots, k-1, k\},$$

where $v \in V_i$, $u \in V_j$. Then the degree of every vertex in $K_{X,\frac{k}{l}}$ equals to $2ak = |X|\frac{k}{l}$.

Let K_X be a complete graph (i.e. every pair of vertices forms an edge) on the vertex set X. Degree of each vertex in K_X equals to $|X| - 1$.

Now we are ready to provide the desired construction. Let p and q be a positive solution of $p^2 = 2q^2+1$. Put

$$A = \{a_1, a_2, \dots, a_{2p^2}\}, \quad B = \{b_1, b_2, \dots b_{2pq}\}, \quad C = \{c_1, c_2, \dots c_{2(p+q)p}\}.$$

Now define the vertex set

$$V = A \cup B \cup C \cup \{x\}.$$

The edge set E and weight function f are defined by explicit expressions for E_+ and E_-:

$$E_+ = E_{A,B,\frac{1}{1}} \cup E_B \cup E_{A\cup B\cup C,\{x\},\frac{1}{1}}; \qquad E_- = E_{A,\frac{q}{p}} \cup E_{B,C,\frac{q}{p}}.$$

Obviously,

$$s_{a_i} = 1, \quad s_{b_i} = 2p^2 - 2q^2, \quad s_{c_i} = -2q^2 + 1, \quad s_x = 4p(p+q).$$

Since there are no edges between A and C, and $p^2 > 2q^2$ our construction is a SED-pair.

Finally we count

$$s[G,f] = \frac{1}{2}\sum_{v\in V} s_v = \frac{2p^2 + 4(p^2-q^2)pq - (2q^2-1)(2p^2+2pq) + 4(p+q)p}{2} =$$

$$-2p^2q^2 + 2p^3q - 4pq^3 + 4p^2 + 3pq = -2p^2q^2 + 2pq(p^2 - 2q^2) + 4p^2 + 3pq = -2q^2p^2 + 4p^2 + 5pq <$$

$$-2p^2q^2 + 2n = -n^2\frac{2p^2q^2}{(4(p+q)p+1)^2} + 2n < -n(n-2)\frac{2p^2q^2}{(4(p+q)p)^2} + 2n =$$

$$= -n(n-2)\frac{q^2}{8(p+q)^2} + 2n < -n(n-2)\frac{1}{8(\sqrt{2}+1+\frac{1}{q^2})^2} + 2n <$$

$$< -n(n-2)\frac{(1+\sqrt{2}-\frac{1}{q^2})^2}{8(1+\sqrt{2})^4} + 2n < -n(n-2)\left(\frac{1}{8(1+\sqrt{2})^2} - \frac{1}{4(1+\sqrt{2})^3q^2}\right) + 2n <$$

$$< -\frac{n^2}{8(1+\sqrt{2})^2} + 2n\frac{1}{8(1+\sqrt{2})^2} + n^2\frac{1}{4(1+\sqrt{2})^3q^2} + 2n < -\frac{n^2}{8(1+\sqrt{2})^2} + 2n\frac{1}{8(1+\sqrt{2})^2} + n^2\frac{1}{n} + 2n =$$

$$= -\frac{n^2}{8(1+\sqrt{2})^2} + n\left(3 + \frac{1}{4(1+\sqrt{2})^2}\right) < -\frac{n^2}{8(1+\sqrt{2})^2} + 4n.$$

Here we use $n = 1 + 4p^2 + 4pq = 5 + 8q^2 + 4q\sqrt{2q^2+1} < 5 + 8q^2 + 4q(\sqrt{2}q + \frac{1}{q}) = 9 + (8+4\sqrt{2})q^2 < (17+4\sqrt{2})q^2 < 4(1+\sqrt{2})^3q^2$.

3 Lower bound on $-n^2/25$

Consider an arbitrary SED-pair (G,f), $G = (V,E)$.

It is known that for each $v,u \in V$ if $(v,u) \in E_- \cup E_+$, then $\deg(v) + \deg(u) \geqslant 0$ (check it by hands or see Lemma 1 in [1]). Let V_+ be $\{v \in V | s_v \geqslant 0\}$ and V_- be $\{v \in V | s_v < 0\}$. If V_- is empty, then $s[G,f] \geqslant 0$. Let x be

$$-\min_{v\in V_-} s_v$$

and consider an arbitrary vertex a such that $s_a = -x$. Let $N_-(a)$ be $\{v \in V | (a,v) \in E_-\}$. Then $|N_-(a)| \geqslant x$ and $s_v \geqslant x$ for each $v \in N_-(a)$, so $N_-(a) \subset V_+$. Then

$$x^2 \leqslant \sum_{v\in N_-(a)} s_v \leqslant \sum_{v\in V_+} s_v.$$

Clearly, V_- is an independent set (i.e. has no edges inside) so

$$\sum_{v\in V_+} s_v = \sum_{v\in V_-} s_v + 2\left(\sum_{(u,v)\in E_+|u,v\in V_+} 1 - \sum_{(u,v)\in E_-|u,v\in V_+} 1\right)$$

$$\leqslant \sum_{v\in V_-} s_v + 2\frac{|V_+|\cdot(|V_+|-1)}{2} \leqslant \sum_{v\in V_-} s_v + |V_+|^2.$$

So

$$\sum_{v\in V_-} s_v \geqslant x^2 - |V_+|^2;$$

recall that

$$\sum_{v\in V_+} s_v \geqslant x|N_-(a)| \geqslant x^2.$$

On the other hand

$$s[(G,f)] = \sum_{(x,y)\in E_+} 1 - \sum_{(x,y)\in E_-} 1 = \frac{\sum_{v\in V} s_v}{2},$$

and

$$\sum_{v\in V} s_v = \sum_{v\in V_+} s_v + \sum_{v\in V_-} s_v \geqslant 2x^2 - |V_+|^2.$$

Also

$$\sum_{v\in V} s_v = \sum_{v\in V_+} s_v + \sum_{v\in V_-} s_v \geqslant x^2 - x|V_-| = -x(|V_-|-x) = -x(|V|-|V_+|-x).$$

Put $y = \frac{x}{|V|}$, $k = \frac{|V_+|}{|V|}$. Then we have the following system of inequalities:

$$\begin{cases} s[(G,f)] \geqslant (y^2 - \frac{k^2}{2})|V|^2 \\ s[(G,f)] \geqslant \frac{-y(1-k-y)}{2}|V|^2. \end{cases}$$

So

$$g(n) \geqslant \min_{0\leqslant y\leqslant 1, 0\leqslant k\leqslant 1}\left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) n^2.$$

One may check by computer (or read explicit calculus in Appendix) that the minimum is $-\frac{1}{25}$ and is reached at $y = \frac{1}{5}$, $k = \frac{2}{5}$.

Список литературы

[1] Saeed Akbari, Sadegh Bolouki, Pooya Hatami, and Milad Siami. On the signed edge domination number of graphs. *Discrete mathematics*, 309(3):587–594, 2009.

[2] Baogen Xu. On signed edge domination numbers of graphs. *Discrete Mathematics*, 239(1-3):179–189, 2001.

[3] Baogen Xu. On edge domination numbers of graphs. *Discrete Mathematics*, 294(3):311–316, 2005.

Appendix

We have to calculate

$$\min\left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) = -\frac{1}{2}\max\left(\min(k^2-2y^2, y-y^2-ky)\right).$$

Let $k_1, y_1 \in [0,1]$ be any values representing this maximum (the maximum is reached by compactness).

First, we show that $y_1^2 - \frac{k_1^2}{2} = -\frac{y_1(1-k_1-y_1)}{2}$. Indeed, this equality means that $k_1 = \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$. Suppose the contrary; if $k_1 > \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$ then

$$\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1 y_1) \leqslant y_1 - y_1^2 - k_1 y_1 <$$

$$< y_1 - y_1^2 - y_1 \frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2} = \left(\frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}\right)^2 - 2y_1^2 =$$

$$\min\left(y_1 - y_1^2 - y_1 \frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}\right)^2 - 2y_1^2\right)$$

and if $k_1 < \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$ then

$$\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1 y_1) \leqslant k_1^2 - 2y_1^2 <$$

$$< \left(\frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}\right)^2 - 2y_1^2 = y_1 - y_1^2 - y_1 \frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2} =$$

$$\min\left(y_1 - y_1^2 - y_1 \frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}\right)^2 - 2y_1^2\right).$$

In both cases

$$\min(k_1^2 - 2y_1^2, y_1 - y_1^2 - k_1 y_1) < \min\left(y_1 - y_1^2 - y_1 \frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}, \left(\frac{-y_1 + \sqrt{5y_1^2+4y_1}}{2}\right)^2 - 2y_1^2\right),$$

and $0 < \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2} < 1$ (because $y_1 < \sqrt{5y_1^2+4y_1} < y_1 + 2$), so (k_1, y_1) doesn't represent the maximum, a contradiction.

Since $y_1^2 - \frac{k_1^2}{2} = -\frac{y_1(1-k_1-y_1)}{2}$ for $k_1 = \frac{-y_1+\sqrt{5y_1^2+4y_1}}{2}$, one may search for $\max f(y)$ with $0 \leqslant y \leqslant 1$, where

$$f(y) = y - y^2 - y\frac{-y + \sqrt{5y^2+4y}}{2} = y - y\frac{y + \sqrt{5y^2+4y}}{2}.$$

Consider the derivative of f

$$f'(y) = \left(y - y\frac{y + \sqrt{5y^2+4y}}{2}\right)' = 1 - y - \frac{\sqrt{5y^2+4y}}{2} - y\frac{10y+4}{4\sqrt{5y^2+4y}} =$$

$$-\frac{(y + \sqrt{5y^2+4y})(5y-1)(y+1)}{\sqrt{5y^2+4y}(\sqrt{5y^2+4y}+1)}.$$

For $y > \frac{1}{5}$ one has $f'(y) < 0$, so $f(y) < f(\frac{1}{5})$ for each $y > \frac{1}{5}$. Analogously for $y < \frac{1}{5}$ one has $f'(y) > 0$, so $f(y) < f(\frac{1}{5})$ for each $y < \frac{1}{5}$. Then $f(y) \leqslant f(\frac{1}{5}) = \frac{2}{25}$ for each $y \in [0, 1]$. So

$$\min\left(\max\left(y^2 - \frac{k^2}{2}, -\frac{y(1-k-y)}{2}\right)\right) = -\frac{1}{2}\max\left(\min(k^2 - 2y^2, y - y^2 - ky)\right) = -\frac{1}{2}\max f(y) = -\frac{1}{25}.$$